Collapsing and geometrization in dimension 3

Joint work with G. Besson, L. Bessières, S. Maillot, J. Porti

Lyon, May 2008

Thurston's Geometrization Conjecture

Theorem [Perelman 2003] *Every closed orientable irreducible 3-manifold M splits into two (possibly empty or not connected) compact 3-submanifolds: $M = H \cup G$ such that:*

1. *$int(H)$ admits a complete* **hyperbolic** *structure.*

2. *G is a* **graph** *3-manifold.*

3. *$\partial H = \partial G$ is a collection (maybe empty) of essential tori.*

Moreover this splitting is unique up to isotopy.

A *graph 3-manifold* is obtained by gluing along some boundary components finitely many elementary pieces homeomorphic to a **solid torus** $S^1 \times D^2$ or a **composite space** $S^1 \times \{$**punctured disk**$\}$.

Therefore a graph manifold is obtained by gluing together geometric pieces which are not hyperbolic.

The Geometrization Conjecture can be viewed as a canonical Thick/Thin decomposition of a closed, orientable, irreducible 3-manifold.

$H =$ the thick part of M: there is no Riemannian metric on $int(H)$ with a lower bound on the curvature and arbitrarily small volume.

$G =$ the thin part of M: there are Riemannian metrics on G with bounded curvature and arbitrarily small volume, by Cheeger and Gromov.

According to Perelman for a Riemannian metric g on M and for $\varepsilon > 0$, we call :

ε-**thin part** of M :
$M^-(\varepsilon) = \{x \in M : \exists\, 0 < \rho \leq 1$ such that: $vol(B(x,\rho)) \leq \varepsilon\rho^3$ and $\sec(g_n) \geq -\frac{1}{\rho^2}$ on $B(x,\rho)$ $\}$

$M \setminus M^-(\varepsilon) = M^+(\varepsilon) = \varepsilon$-**thick part**.

We say that a sequence g_n of Riemannian metrics **collapses** if there is a sequence $\varepsilon_n \to 0$ such that for all n, $M = M^-(\varepsilon)(g_n)$ is ε_n-thin with respect to g_n.

Theorem [Besson-Courtois-Gallot 1995], [Souto 2000]

Any sequence g_n of Riemannian metrics on M, with sectional curvature $K_{g_n} \geq -1$ and which is volume minimizing, converges to a complete hyperbolic metric with constant curvature -1 on the thick part H and collapses on G.

Positive solutions of many fundamental problems in 3-manifold topology, including the Poincaré Conjecture and the Universal Cover Conjecture follow from the solution of the Geometrization Conjecture.

More generally, the solution of the Geometrization Conjecture implies that there is an algorithm to decide whether two 3-manifolds are diffeomorphic.

In the most difficult case of a closed and **atoroidal** 3-manifold it shows that the topology predicts the geometry:

Theorem[Perelman] *Let M be a closed orientable 3-manifold, with $\pi_2(M) = \{0\}$.*

1. *$\pi_1 M$ finite $\Leftrightarrow$ M is spherical, i.e. $M = S^3/\Gamma$ with $\Gamma \subset S(O, 4)$ a finite orthonal group.*

2. *$\pi_1 M$ infinite and $\mathbb{Z} \oplus \mathbb{Z} \not\subset \pi_1(M)$ $\Leftrightarrow$ M is hyperbolic.*

Deformations of metrics on a 3-manifold

Let M be a closed, orientable 3-manifold. In early 1980, R. Hamilton introduced the following evolution equation:

$$\frac{dg}{dt} = -2\,\mathrm{Ric}(g),$$

where the unknown $g = g(t)$ is a family of riemannian metrics on M depending on a time parameter $t \in \mathbf{R}$.

A **Ricci flow** is a solution to this equation.

If the initial metric has constant sectional curvature K, then the solution of the Ricci flow is $g(t) = (1 - 4Kt)g(0)$.

$K > 0 \Rightarrow$ the metrics $g(t)$ shrink to a point in finite time, when $T \to \frac{1}{4K}$.

$K < 0 \Rightarrow$ the metrics $g(t)$ expands indefinitely, while the rescaled metrics $\frac{1}{4t}g(t)$ converge to the metric of constant curvature -1.

In general the metrics tends to become more homogeneous along the flow, but the sectional curvature can blow up in finite time at some points of the manifold.

For any initial Riemannian metric $g(0)$ on M, R. Hamilton proved the following fresults:

- *The existence and uniqueness of a short time solution for the Ricci flow.*
- *The solution can be extended as long as the norm of the curvature tensor does not blow up at some point.*
- *If the flow* $g(t)$ *is defined on* $[0, \infty[$ *and the rescaled solution* $\frac{1}{4t}g(t)$ *has uniformly bounded curvature, then it produces the canonical thick/thin decomposition.*

The following is a ground result of the Ricci flow theory :

Theorem [R. Hamilton]
Let M be a closed orientable 3-manifold which carries a Riemannian metric $g(0)$ with positive Ricci curvature everywhere.
Then the metrics along the Ricci flow shrink to a point in a finite time.
After rescaling the metric and the time so that the volume stays constant, the metrics (along the rescaled flow) converge on M to a metric with constant, positive sectional curvature.

G. Perelman has defined a monotonic quantity along the Ricci flow, called **entropy**, which allows to control the way the singularities appear and to classify them.

Then Perelman was able to carry out Hamilton's program using a surgery process that clips off the regions of the manifold where the curvature is **too high**.

Let (M, g) be a Riemannian manifold.

For $x \in M$, $\text{Rm}(x)$ denote the curvature tensor and $R(x)$ the scalar curvature.

Define $R_{min}(g) =$ minimum of the scalar curvature.

The Riemannian metric g on M is **normalised** if:

- $|\,\text{Rm}\,| \leq 1$

- $\text{vol}(B(x,1)) \geq \frac{1}{2}\text{vol}(B_{euclid}(x,1))$

For any initial normalised Riemannian metric, Perelman defined a Ricci flow with surgery on the manifold M.

A **a solution with surgery** of the Ricci flow on an interval $I \subset \mathbb{R}$ is a piecewise $\mathcal{C}^1$ 1-parameter family of metrics $\{g(t)\}_I$ such that:

1) $I_{\text{sing}} := \{t \in I\}$, where the function $t \to g(t)$ is not continuous, is a discrete subset called **singular times** such that:

(i) $\forall t_s \in I_{\text{sing}}$, $g(t)$ is continuous from the left and has a limit $g_+(t_s)$ from the right.

(ii) $t \to \bar{g}(t)$ defined by $\bar{g}(t_s) = g_+(t_s)$ and $\bar{g}(t) = g(t)$ on $]t_s, t_s + \varepsilon]$ is $\mathcal{C}^1$ on $[t_s, t_s + \varepsilon]$

(iii) $R_{min}(g_+(t_s)) \geq R_{min}(g(t_s))$

(iv) $g_+(t_s) \leq g(t_s)$

2) At all regular time the Hamilton-Ricci equation is satisfied.

Theorem[Perelman]
Let M be a closed, orientable, irreducible 3-manifold. For every initial normalised Riemannian metric g_0 on M, then :

A) $\pi_1 M$ **finite** $\Rightarrow \exists\, t_0 > 0$ *and a solution with surgery on* $[0, t_0]$*, such that each point in* $(M, g(t_0))$ *has a canonical neighborhood with high curvature* $\Rightarrow$ **spherical**.

B) $\pi_1 M$ **infinite** $\Rightarrow$ *there is a solution with surgery on* $[0, +\infty[$ *with initial condition* g_0*, such that:*

(i) $\mathrm{vol}(g(t)) \leq C t^{3/2}$ *for a constant* $C > 0$.

(ii) For $0 < \varepsilon << 1$, $x_n \in (M^+(\varepsilon), \frac{1}{4t_n} g(t_n))$ *and* $t_n \to +\infty$*, the pointed sequence* $(M, \frac{1}{4t_n} g(t_n), x_n)$ $\mathcal{C}^2$*-converges to a complete hyperbolic 3-manifold with finite volume.*

(iii) The sequence $(M, \frac{1}{4t_n} g(t_n)$ *has local curvature bounds.*

According to Perelman, a sequence (M_n, g_n) has *local curvature bounds* if:

$\forall \delta > 0\ \exists \bar{r} = \bar{r}(\delta) > 0$, $K_0 = K_0(\delta)$ and $K_1 = K_1(\delta)$ such that:

If $0 < r < \bar{r}(\delta)$, $vol(B(x,r)) > \delta r^3$ and sec $\geq -\frac{1}{r^2}$ on $B(x,r)$

$\Rightarrow$

$|Rm| < K_0 r^{-2}$ and $|\nabla Rm| < K_1 r^{-3}$

• The constants $\bar{r}(\delta)$, $K_0(\delta)$ and $K_1(\delta)$ are independent of n.

• Scale invariant inequalities.

• When $r \to 0$, $\frac{vol(B(x,r))}{r^3} \to \frac{4\pi}{3}$ and $\frac{-1}{r^2} \to -\infty$

Thus such an r always exist, but $r \ll \rho_n(x) =$ collapsing scale

A detailed account about Perelman's construction of the Ricci flow with surgery can be found in:

B. Kleiner and J. Lott: Notes on Perelman's papers.
ArXiv: Math.DG/0605667.

J. Morgan and G. Tian: Ricci flow and the Poincaré conjecture.
Clay Mathematics Monograph, vol. 3, 2007.

H.D. Cao and X.P. Zhu :
Hamilton-Perelman's Proof of the Poincaré Conjecture and the Geometrization Conjecture.
Asian J. Math. 2006.

From now on M will be aspherical, that is to say a **closed irreduccible 3-manifold with infinite fundamental group**

For a riemannian. metric g on M :

- $\widehat{R}(g) = R_{min}(g)\,\mathrm{vol}(g)^{2/3}$.

- M aspherical $\Rightarrow \widehat{R}(g) \leq 0$ [Gromov-Lawson; Schoen-Yau; Perelman].

- $t \to \widehat{R}(g(t)) \nearrow$ along the Ricci flow with surgery.

- $\widehat{R}(M) = \sup \widehat{R}(M, g)$.

- $V_0(M) = \min\{Vol(M^3 \setminus L) \mid$
L hyperbolic link in $M,$ possibly empty$\}$

R. Myers $\Rightarrow$ there is always a hyperbolic link in a closed 3-manifold.

The set of volumes of hyperbolic 3-manifolds is well ordered $\Rightarrow$ $V_0(M) > 0$ is realized by the complement of a hyperbolic link $L_0 \subset M$.

Theorem

Let M be a closed, orientable, aspherical 3-manifold. Then :

(1) $\hat{R}(M) \leq -6V_0(M)^{2/3} \Rightarrow M$ **hyperbolic**.

(2) $-6V_0(M)^{2/3} < \hat{R}(M) \leq 0 \Rightarrow M$ **toroidal** *(i.e. contains an incompressible torus or admits a Seifert fibration).*

• Perelman's Ricci flow with surgeries is crucial for the proof of case (2).

Corollary

Let M be a closed, orientable, aspherical 3-manifold, then $-6V_0(M)^{2/3} \leq \hat{R}(M) \leq 0$.

Moreover:

(1) M **hyperbolic** $\Leftrightarrow \hat{R}(M) = -6V_0(M)^{2/3}$**.**

(2) M **graph manifold** $\Leftrightarrow \hat{R}(M) = 0$**.**

Hence the hyperbolic metric has a minimal volume among all metrics with scalar curvature $\geq R_{hyp} = -6$.

Sufficient condition for hyperbolicity

$$\boxed{\hat{R}(M) \leq -6V_0(M)^{2/3} \Rightarrow M \text{ hyperbolic.}}$$

Claim[M. Anderson] *Let* $L_0 \subset M$ *be a hyperbolic link with* $\mathrm{vol}(M \setminus L_0) = V_0(M)$*. If* $L_0 \neq \emptyset$*, then there is a riemannian metric* g *on* M *with*
$\hat{R}(g) > -6\,\mathrm{vol}(M \setminus L_0)^{2/3} = -6V_0(M)^{2/3}$*.*

Consider the orbifold $\mathcal{O}$ with underlying space M, singular locus L_0 and branching indices $n >> 1$ along L_0.

$\mathcal{O}$ is hyperbolic by Thurston' s hyperbolic surgery theorem.

In a small tubular neighborhood of L_0 one can smooth the orbifold metric in such a way that:

$\forall \varepsilon > 0 \Rightarrow$ metric g_ε with $K_{g_\varepsilon} \geq -1$ and $\mathrm{vol}(M, g_\varepsilon) < (1 + u(\varepsilon))\,\mathrm{vol}(\mathcal{O})$, $u(\varepsilon) \to 0$ as $\varepsilon \to 0$.

$\mathrm{vol}(\mathcal{O}) < \mathrm{vol}(M \setminus L_0)$ (Schläfli), hence for $\varepsilon << 1$

$\Rightarrow \mathrm{vol}(M, g_\varepsilon) < \mathrm{vol}(M \setminus L_0)$ and $R_{min}(g_\varepsilon) \geq -6$.

$\Rightarrow \hat{R}(g_\varepsilon) > -6\,\mathrm{vol}(M \setminus L_0)^{2/3} = -6V_0(M)^{2/3}$.

Weak Collapsing Theorem

Let M be a non-simply connected, closed, orientable, irreducible 3-manifold. Suppose that there exists a sequence g_n of Riemannian metrics on M satisfying:

(1) The sequence $\mathrm{vol}(g_n)$ is bounded.

(2) Let $\varepsilon > 0$ and $x_n \in (M^+(\varepsilon), g_n)$. Then the pointed sequence (M, g_n, x_n) $\mathcal{C}^2$-subconverges towards a complete hyperbolic 3-manifold with volume $< V_0(M)$.

(3) The sequence g_n has local curvature bounds in the sense of Perelman.

Then M is a graph manifold or contains an incompressible torus.

The proof leads to the the following weaker version of Perelman's collapsing theorem:

Corollary *Let M be a closed, orientable, irreducible, non-simply connected* 3-*manifold. If M admits a sequence of Riemannian metrics that collapses with local curvature bounds, then M is a graph manifold.*

For the **collapsing case** T. Shioya and T. Yamagushi (Math. Ann. 333, 2005) have a stronger result.

Namely they do not assume M to be non-simply connected and do not need the L.C.B. condition.

When the thick part is **not empty**, their arguments cannot be applied anymore.

Sufficient condition for toroidality.

$\hat{R}(M) > -6V_0(M)^{2/3} \Rightarrow M$ contains an incompressible torus or is Seifert fibered.

Let g_0 be a normalized Riem. metric on M with $0 > \hat{R}(g_0) > -6V_0(M)^{2/3}$.

Perelman's long term Ricci flow $\Rightarrow$ sequence of metrics $g_n = \frac{1}{4t_n} g(t_n)$ on M, with initial term g_0.

The sequence $\hat{R}(g_n) \leq 0$ is non decreasing $\Rightarrow \liminf \hat{R}(g_n) > -6V_0(M)^{2/3}$.

To apply the weak collapsing theorem, we need only to check Hypothesis (2)

Claim *If H appears as a pointed $\mathcal{C}^2$-limit of some subsequence of (M, g_n), then* $\text{vol}(H) < V_0(M)$.

Assume $\text{vol}(H) \geq V_0(M)$.

Monotonicity of $\widehat{R}(g_n)$ and choice of $g_0 \Rightarrow$ $\lim \widehat{R}(g_n) \geq \widehat{R}(g_0) > -6V_0(M)^{2/3}$.

Let $\xi > 0$. Choose a compact core $\bar{H}(\xi)$ of H such that $\text{vol}(\bar{H}(\xi)) \geq (1 - \xi)\,\text{vol}(H)$.

$\mathcal{C}^2$ convergence $\Rightarrow$ for $n >> 1$ $\exists$ $\bar{H}_n(\xi) \subset M_n$ with volume $\geq (1 - \xi)^2\,\text{vol}(H)$ and scalar curvature $\leq -6(1 - \xi)$.

$\Rightarrow R_{\text{min}}(g_n) \leq -6(1 - \xi)$ and $\text{vol}(M_n) \geq \text{vol}(\bar{H}_n(\xi)) \geq (1 - \xi)^2\,\text{vol}(H)$.

Letting $\xi \to 0 \Rightarrow$ $\lim \widehat{R}(g_n) \leq -6\,\text{vol}(H)^{2/3} \leq -6V_0(M)^{2/3}$.

Contradiction!

Let $H^1, \ldots, H^m$ be hyperbolic limits of the thick parts of the sequence $M_n = (M, g_n)$.

For each i we fix a compact core $\bar{H}^i \subset H^i$ and for each $n >> 1$ a submanifolds $\bar{H}^i_n \subset M_n$ which approximates $\bar{H}^i$.

Proposition *Either M contains an incompressible torus or for each $i \in \{1, \ldots, m\}$, $\forall n$, $\bar{H}^i_n$ is abelian in M. (i.e. Image of $\pi_1(\bar{H}^i_n) \to \pi_1(M)$ is abelian).*

Proof

If M does not contain an incompressible torus $\forall i \in 1, \ldots, m$ and $\forall n$, each component of $\partial \bar{H}_n^i$ is compressible in M.

$\Rightarrow$ $\bar{H}_n^i$ is abelian in M or homeomorphic to a link exterior in M.

But vol$(H^i) < V_0(M) \Rightarrow \bar{H}_n^i$ **not homeomorphic** to a link exterior in M.

For $n >> 1$ the thick components are abelian in M.

Now we cover the thin part by **local models** with virtual abelian fundamental groups:

Let $\varepsilon_n \to 0$, $\forall x \in M_n^-(\epsilon_n)$, we choose a radius $0 < \rho(x) \leq 1$ such that :

- $\sec(g_n) \geq -\rho^{-2}(x)$ on $B(x, \rho(x))$

- $\mathrm{vol}(B(x, \rho(x))) < \varepsilon_n \rho^3(x)$.

Proposition [local models]
$\forall D > 1$, $\exists n_0(D)$ *such that for* $n > n_0(D)$ *:*

(a) Either M_n *is* $\frac{1}{D}$*-close from* X^3 *compact with* $sec \geq 0$*.*

(b) Or $\forall x \in M_n^-(\epsilon_n)$ $\exists\, \nu(x) \in]0, \rho(x)[$ *such that:*

(1) $B(x, \nu_x)$ $\frac{1}{D}$*-close to* $N_{\nu_x}(S) \subset X^3$*.*

(2) X^3 *non-compact with* $sec \geq 0$ *and* **soul** S*,* $diam(S) \leq \frac{\nu_x}{D}$

(3) $\mathrm{vol}(B(x, \nu(x))) \leq \frac{1}{D}\nu^3(x)$ *and* $sec|_{B(x,\nu_x)} \geq -\frac{1}{\nu_x^2}$

Compact manifolds with $sec \geq 0$ are geometric. We assume that only non compact models $B(x, \nu_x)$ occur.

Cheeger-Gromoll:

• $X^3 \cong$ normal bundle of its soul $S = *$, S^1, S^2, T^2, K^2.

$\Rightarrow B(x, \nu_x) \cong B^3$, $S^1 \times D^2$, $S^2 \times I$, $T^2 \times I$, $K^2 \tilde{\times} I$.

The proof is by contradiction like in Cheeger-Gromov "local soul theorem".

L.C.B. hypothesis is used to show that, after rescaling by the radius , the Gromov-Hausdorff limit of a ball is in fact smooth, using Petersen's $\mathcal{C}^2$-convergence theorem.

A path-connected subset $A \subset M$ is **null-homotopic** if the image $\pi_1(A) \to \pi_1(M)$ is trivial.

Proposition

$\exists\, D_0 > 0$ such that if $D > D_0$ and $n \geq n_0(D)$ then :

(a) Either a thick component of M_n is not null-homotopic in M_n.

(b) Or there is a point $x \in M_n^-(\varepsilon_n)$ such that $B(x, \nu(x))$ is not null-homotopic in M_n.

By contradiction assume that for
$D >> 1 \ \exists \ n \geq n_0(D)$: all the thick components of M_n and the ball $B(x, \nu(x))$ $(\forall x \in M_n^-(\varepsilon_n))$ are null-homotopic in M_n

By using a construction of Gromov build a 2-dimensional covering of M_n by open subsets which are null-homotopic in M_n.

$\Rightarrow$ Contradiction! with the following result:

Proposition [Gómez-Larrañaga and González-Acuña] *If a closed, connected, orientable, irreducible 3-manifold X has a covering of dimension 2 by open subsets which are homotopically trivial in X, then X is simply connected.*

Let $\mathcal{V}$ be a thick component or one of the local models $B(x, \nu_x)$ which is not null-homotopic in M_n.

Then $M_n \setminus \mathcal{V}$ is a Haken 3-manifold, which admits a geometric decomposition, by Thurston.

A relative version of Gromov's construction (like in the Covering Proposition) $\Rightarrow$ covering of dimension 2 of $M_n \setminus \mathcal{V}$ by virtual abelian local models and thick components such that the covering is 0-dimensional on $\partial(M_n \setminus \mathcal{V})$.

Gromov' vanishing theorem $\Rightarrow$ the simplicial volume $\|M_n \setminus \mathcal{V}\| = 0$.

Thus $M_n \setminus \mathcal{V}$ is a graph manifold, and so is M_n.

In the late 70's W. Thurston proved the Geometrization Conjecture for **Haken 3-manifolds**:

Theorem [W. Thurston]
The Geometrization Conjecture is true for a compact, orientable and irreducible 3-manifold which has a non-empty boundary or contains a π_1-injective closed embedded surface of genus ≥ 1.

In the early 90's, A. Casson-D. Jungreis, D. Gabai, G. Mess and P. Tukia settled the case where $\pi_1 M$ contains a subgroup $\mathbb{Z} \oplus \mathbb{Z}$

Theorem[Torus Theorem]
Let M be a closed, orientable and irreducible 3-manifold. If M is **toroidal** *(i.e. $\mathbb{Z} \oplus \mathbb{Z} \subset \pi_1(M)$), then M contains a π_1-injective embedded torus or M admits a foliation by circles with finite holonomy (***Seifert fibration***).*

For $\epsilon > 0$, a **standard ϵ-neck** is the Riemannian product $N_\epsilon = S^2 \times] - \epsilon^{-1}, \epsilon^{-1}[$, where S^2 is the round 2-sphere with scalar curvature 1.

Let $\epsilon > 0$, (M, g) a Riemannian 3-manifold, $x \in M$ and $U \subset M$ a neighborhood of x.

1. U is a ϵ-**neck at** x if $R(x) > 0$ and (U, x) with the rescaled metric $R(x)g$ is $\mathcal{C}^2$ ϵ-near from a standard ϵ-neck $(N_\epsilon, *)$ with $* \in S^2 \times \{0\}$.

2. U is a ϵ-**cap** at x if $U = V \cup W$, where $x \in V$, V is a closed 3-ball, $\bar{W} \cap V = \partial V$ and W is a ϵ-neck.

If M has a positive constant curvature or if every point $x \in (M, g)$ is the center of a $\frac{1}{1000}$-neck or a $\frac{1}{1000}$-cap, (M, g) is said **locally canonical**.

Theorem

A locally canonical, closed, orientable Riemannian 3-manifold is spherical or $S^2 \times S^1$*.*