Applications of Ricci flow singularity analysis

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- **Theorem (Hamilton-Shi)**

  If $(M^n, g(t))$ is a solution of the Ricci flow on a closed manifold on maximum time interval $[0, T)$, where $T < \infty$, then

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- In other words, the solution exists as long as the curvature stay bounded.

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- Theorem

  If $\{(M_i^n, g_i(t), x_i)\}$ *with* $t \in (\alpha, \omega) \in 0$ *is a sequence of complete pointed solutions of the Ricci flow satisfying that*

  $$|Rm(g_i(t))|_{g_i(t)} \leq C$$

  for some C *independent of* i *and*

  $$\mathrm{inj}_{g_i(0)}(x_i) \geq \delta > 0$$

  for some δ *independent of* i*, then there exists a subsequence which converges as* $i \to \infty$ *to a pointed complete solution to Ricci flow* $(M_\infty, g_\infty(t), x_\infty)$ *with* $t \in (\alpha, \omega)$*.*

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- Definition
  A Riemannian manifold (M^n, g) is said to be locally κ-noncollapsed on the scale ρ if for any $r \leq \rho$, and for any $x \in M$ and the ball $B(x, r)$, such that if $|\operatorname{Rm}(y)| \leq \frac{1}{r^2}$ for all $y \in B(x, r)$, then

$$\frac{V(x, r)}{r^n} \geq \kappa.$$

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Theorem (Perelman)

Let $g(t)$, $t \in [0, T)$, be a smooth solution to the Ricci flow on a closed manifold M^n. If $T < \infty$, then $g(t)$ is not locally collapsed for any scale $\rho > 0$.

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Theorem (Cheeger-Gromov-Taylor)

. For any $c > 0$*,* $r_0 > 0$ *and dimension* n*, there exists a constant* $\delta_0 > 0$ *such that if* (M^n, g) *is a complete Riemannian manifold with* $|K_{\text{sec}}| \leq 1$ *and if*

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There exists a localized/stronger version.

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- Theorem (Hamilton-Perelman)

  For any maximum solution to the Ricci flow which satisfies the type I or II (a). There exists a sequence of dilations of the solution which converges in the limit to a singularity model of the corresponding type.

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  Type I: The solution exists for $-\infty < t < \Omega$ with $0 < \Omega < \infty$ and $|\operatorname{Rm}| \le \frac{\Omega}{\Omega - t}$, everywhere and with equality somewhere at $t = 0$.

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  - In the case of Type II and Type III, if the solution has nonnegative curvature operator (or Kähler with bisectional curvature), it must be a gradient/expanding steady soliton. (Applications of Li-Yau-Hamilton inequality.)
  - Note that both Type I and Type II solution are defined on some interval of $(-\infty, \Omega)$. We call such solutions (of Ricci flow) ancient solutions.

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  2) Compactness on the space of 3-dimensional κ-noncollapsed ancient solutions with bounded curvature.

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  - Similarly one can obtain similar curvature estimate, as well as the compactness theorem for the Kähler ancient solutions with nonnegative bisectional curvature satisfying differential Harnack inequality.
  - We shall look into the proof of Perelman's theorem later on.

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  Notation: $\mathcal{O}_P(M)$ is the ring of the holomorphic functions of polynomial growth.
  $\kappa(M, \mathcal{O}_P(M))$ is the Kodaira dimension. It is the same as the transcendental degree of the quotient field of the ring $\mathcal{O}_P(M)$.

  - Another consequence is a conjecture of Yau:
    Assume that M satisfies $0 \leq R_{i\bar{i}j\bar{j}} \leq A$ for some A and of maximum volume growth then the curvature of M has average quadratic decay.

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- Theorem (N-Wu)

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  $$R \geq \delta S \,\mathrm{id} > 0$$

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  - It can be viewed as a Bonnet-Meyer type theorem.

Compact ancient solutions

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Theorem (Hamilton)

The only solutions to Ricci flow on a surface which are complete with bounded curvature on an ancient time interval $-\infty < t < T$ *and where the curvature* S *satisfies*

$$\limsup_{t\to-\infty}(T-t)|S| < \infty$$

are round sphere and the flat plane, and their quotients.

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- Then 2) follows from Hamilton's LYH estimate $S_t \geq 0$.

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- Classify all compact ancient solutions?

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  $$g(t) = u(t)h, \quad u = \frac{\sinh(-t)}{\cosh x + \cosh t}.$$

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- Perelman showed the existence of a 3-dimensional example. Moreover the example is κ-noncollapsed with positive curvature.
- The example is obtained by taking the limit of a sequence of solutions on $(-t_i, 0)$.
- Physicists are interested in the ancient solutions due to the connection between the Ricci flow and the so-called RG flow in string theory. We are informed that they have explicit examples as Perelman's for three dimension.

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- Theorem (N, to appear)

  Assume that $(M, g(t))$ *is a closed type I,* κ*-non-collapsed (for some* $\kappa > 0$*) ancient solution to the Ricci flow with positive curvature operator. Then* $(M, g(t))$ *must be the quotients* $\mathbb{S}^n$*.*

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  - This provide a high dimensional analogue of Hamilton's surface result, at least for the compact case.

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$$\mathcal{W}(g, f, \tau) \doteq \int_M \left(\tau(|\nabla f|^2 + S) + f - n\right) u$$

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- $$\nu(g) \doteqdot \inf_{\tau} \mu(g, \tau) > 0$$

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with $\mu(g, \tau) \to 0$ as $\tau \to 0$ and $\mu(g, \tau) \to \infty$ as $\tau \to \infty$, if $S > 0$.
- $$\nu(g) \doteqdot \inf_{\tau} \mu(g, \tau) > 0$$
is meaningful for $S > 0$.
- Note that $\nu(g)$ is invariant under the scaling.

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More precisely, for suitable chosen $x_i \in M, t_i \to -\infty$, the re-scaled solutions $(M, x_i, g_i(s))$ with $g_i(s) = \frac{1}{-t_i} g((-t_i)s)$ converges to a gradient shrinking soliton.

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In the case of Type-I ancient solution, x_i can be taken as $o \in M$ fixed.

Convergence result

- Theorem (Bohm-Wilking)

  On a compact manifold the normalized Ricci flow evolves a Riemannian metric with positive curvature operator to a limit metric with constant sectional curvature.

Diameter estimate

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- Lemma

  Assume that $(M, g(t))$ *is an type I ancient solution of Ricci flow. Assume that the diameter of* M $D_1 \doteqdot \mathrm{Diam}(M, g(-1)) < \infty$*. Then there exists a* $C = C(n, A)$ *such that for any* $t \leq -1$*, the diameter of* $(M, g(t)$ *satisfies*

$$\mathrm{Diam}(M, g(t)) \leq (2C + \max\{D_1, 1\} + 1)\sqrt{-t}.$$

Diameter estimate

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  - The similar result holds for the 2-positive curvature operator (due to Böhm-Wilking's general convergence result) and positive complex sectional curvature (due to Brendle-Schoen's result on positive complex sectional curvature is preserved under Ricci-flow and Bohm-Wilking's general convergence result).

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Dimension reduction

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- Theorem (Dimension reduction)

  Let $(M^n, g(t))$ *with* $t \in (-\infty, \omega)$, $\omega > 0$, *be a complete noncompact ancient solution of the Ricci flow with bounded nonnegative curvature operator. Suppose there exist sequences* $x_i \in M$, $r_i \to \infty$, *and* $A_i \to \infty$ *such that* $\frac{r_0(p,x_i)}{r_i} \geq A_i$ *and*

  $$R(y,0) \leq r_i^{-2} \quad \textit{for all } y \in B_0(x_i, A_i r_i).$$

  Assume further that there exists an injectivity radius lower bound at $(x_i, 0)$*; namely,* $\mathrm{inj}(x_i, g(0)) \geq \delta r_i$ *for some* $\delta > 0$. *Then a subsequence of solutions* $(M^n, r_i^{-2} g(r_i^2 t), x_i)$ *converges to a complete limit solution* $(M^n_\infty, g_\infty(t), x_\infty)$ *which is the product of an* $(n-1)$*-dimensional solution (with bounded nonnegative curvature operator) with a line.*

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  - Dimension reduction was introduced by Hamilton to study the steady gradient solitons. Perelman is the one who first introduce this version of dimension reduction.

A helping lemma

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- Lemma (Hamilton, Infinite ASCR point picking)

  Let (M^n, g) *be a complete noncompact Riemannian manifold with* $\mathrm{ASCR}(g) = \infty$ *and let* $O \in M^n$. *There exist a sequence of points* $\{x_i\}_{i=1}^{\infty}$ *with* $d(x_i, O) \to \infty$ *and sequences* $\varepsilon_i \to 0$ *and* $r_i > 0$ *with* $R(x_i)\, r_i^2 \to \infty$ *such that the balls* $B(x_i, r_i)$ *are disjoint,* $d(x_i, O)/r_i \to \infty$, *and*

  $$\sup_{B(x_i, r_i)} R \leq (1 + \varepsilon_i)\, R(x_i)\,. \qquad (0.1)$$

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This was proved by Hamilton in his '*formation of singularities*' paper.

Proof of Perelman's theorem -following Hamilton

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- First, for any Type I ancient solution with bounded nonnegative curvature operator and $ASCR < \infty$, Hamilton proved (in '*Formation of singularity*' paper) that $AVR > 0$.

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Perelman's proof

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- Perelman used his dimension reduction result to treat the case $\mathrm{ASCR} = \infty$. For $\mathrm{ASCR} = 0$ he reduced it to the gap theorem for manifold with nonnegative sectional curvature. The case of $0 < \mathrm{ASCR} < \infty$, he argues that one can take limit to converge to a cone and apply the maximum principal on the smooth part of the cone.

The Kähler case

The Kähler case

- We used a different scheme of argument since the dimension reduction uses Toponogov triangle comparison theorem.
- Our argument in fact also works for the real case and provides a unified proof to both the real and Kähler case.
- Step1-reduce to gradient shrinking solitons:

Proposition

Let $(M, g(t))$ *for* $t \in (-\infty, 0)$ *be an* κ *noncollapsed ancient solution with bounded nonnegative curvature operator. Let* $\tau = -t$*. Then for suitable* $x_i(\tau_i)$*, the re-scaled pointed solution* $(M, x_i(\tau_i), \frac{1}{\tau_i} g(t\tau_i)$ *converges to s gradient shrinking soliton.*

Shrinking solitons

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- Step2-A result on the shrinking solitons.

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Proposition

Let (M, g) be a non-flat gradient shrinking soliton. Assume that the Ricci curvature of M is nonnegative. Then there exists $\delta > 0$ such that $S(x) \geq \delta$ for all $x \in M$.

- Easy to show if curvature is bounded. Some cleverness is needed for the case without curvature bound.
- Step3-Dimension reduction for the real case; For the complex case, use a classification result instead.

A classification result on solitons

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- Theorem

  (N) Let (M^m, g) *be a non-flat gradient shrinking soliton to Kähler-Ricci flow.*
  (i) If the bisectional curvature of M *is positive then* M *must be compact and isometric-biholomorphic to* $\mathbb{P}^m$*.*
  (ii) If M *has nonnegative bisectional curvature but not positive then the universal cover* $\tilde{M}$ *splits as* $\tilde{M} = N_1 \times N_2 \times \cdots \times N_l \times \mathbb{C}^k$ *isometric-biholomorphically, where* N_i *are compact irreducible Hermitian Symmetric Spaces.*

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  - Made essential use of earlier joint work with L.-F. Tam on the structure of nonnegative curved Kähler manifolds.