Singular Spaces with Generalized Lower Ricci Bounds

—

Geometric and Analytic Aspects

.

Karl-Theodor Sturm, Bonn University

Lyon, Glimpses of Geometry, May 15-17, 2008

Curvature Bounds for Singular Spaces

Metric measure space (M, d, m):
(M, d) complete separable metric space, m locally finite measure on M.

Generalizations of Riemannian manifolds,
including singularities, rich geometric structure

Alexandrov '51: gen. sectional curvature bounds for metric spaces
Gromov '81: GH-metric on the space of compact metric spaces

Key Results.
$\{(M,d)$ with sect. curv. $\geq K\}$ is closed under GH-conv.

$\{(M,d)$ with sect. curv. $\geq K,$ dim $\leq N,$ diam $\leq L\}$ is compact under GH-conv.

Ricci Curvature

Crucial for analysis and stochastics: Ricci curv. $\geq K$
S.T.Yau, Cheeger, Colding, Perelman et al. '85-'08

Aim: Gener. Ricci Curvature bound $\mathrm{Ric}(M, \mathsf{d}, m) \geq K$

- equivalent to $\mathrm{Ric}_p(\xi,\xi) \geq K\,\|\xi\|^2$ if M is Riem. mfd.
- stable under convergence
- intrinsic, synthetic

based on the concept of mass transportation

Brenier, McCann, Otto, Otto/Villani, Cordero-Erausquin/McCann/Schmuckenschläger,

St., Lott/Villani, v.Renesse, Ohta, Ohta/St.

Part I.

The L^2-Wasserstein space on a Riemannian manifold

L^2-Wasserstein Space

Let (M, d) cpl. Riemannian manifold, dx Riemannian volume

$$\mathcal{P}_2(M) \quad = \quad \left\{ \text{prob. meas. } \mu \text{ on } M \text{ with } \int_M d^2(x, x_0)\, \mu(dx) < \infty \right\}$$

$$d_W(\mu, \nu) \quad = \quad \inf_q \left[\int_{M \times M} d^2(x, y)\, d\, q(x, y) \right]^{1/2}$$

where $\inf_q$ is taken over all couplings q of μ and ν,
i.e. $q(A \times M) = \mu(A),\ q(M \times B) = \nu(B)$ for all $A, B \subset M$

L^2-Wasserstein Space

Let (M,d) cpl. Riemannian manifold, dx Riemannian volume

$$\mathcal{P}_2(M) \quad = \quad \left\{ \text{prob. meas. } \mu \text{ on } M \text{ with } \int_M d^2(x,x_0)\,\mu(dx) < \infty \right\}$$

$$\begin{aligned} d_W(\mu,\nu) \quad &= \quad \inf_q \left[\int_{M\times M} d^2(x,y)\,d\,q(x,y) \right]^{1/2} \\ &\le \quad \inf_F \left[\int_M d^2(x,F(x))\,d\,\mu(x) \right]^{1/2} \end{aligned}$$

where $\inf_F$ is taken over all maps $F: M \to M$ with $F_*\mu = \nu$.

Brenier, McCann: Monge = Kantorovich

$\forall \mu_0, \mu_1 \in \mathcal{P}_2(M)$ with $\mu_0(dx) \ll dx$: $\exists!$ optimal transport map $F_1 : M \to M$ such that

$$dq = (Id, F_1)_* d\mu_0$$

$\exists!$ $d^2/2$-convex function $\varphi : M \to \mathbb{R}$ s.t. geodesic μ_t in $\mathcal{P}_2(M)$ connecting μ_0 and μ_1 is given by

$$\mu_t := (F_t)_* \mu_0,$$

with

$$F_t(x) = \exp_x(t \nabla \varphi(x)).$$

Riemannian Structure of $\mathcal{P}_2(M)$

Tangent space:

$$T_{\mu_0}\mathcal{P}_2 \quad = \text{closure of} \quad \{\Phi = \nabla\varphi \quad \text{for smooth } \varphi : M \to \mathbb{R}\}$$

Riemannian tensor:

$$\langle \Phi, \Psi \rangle_{T_{\mu_0}\mathcal{P}_2} \quad = \quad \int_M \Phi \cdot \Psi d\mu_0$$

Exponential map:

$$\exp_{\mu_0}(t\,\Phi) \quad = \quad [\exp(t\Phi)]_* \mu_0$$

Heat Equation as Gradient Flow

The gradient flow

$$\frac{\partial \nu}{\partial t} = -\nabla S(\nu) \qquad \text{on } \mathcal{P}_2(\mathbb{R}^n)$$

for the relative entropy $S(\rho\, dx) = \int \rho \cdot \log \rho\, dx$ is given by $\nu_t(dx) = \rho_t(x)dx$ where ρ solves the heat equation

$$\frac{\partial}{\partial t}\rho = \triangle \rho \qquad \text{on } M.$$

$\mathbb{R}^n$: *Jordan/Kinderlehrer/Otto, Otto;*

Riemann (M, g): *Ohta, Savare, Erbar;*

Finsler (M, F, m): *Ohta/Sturm (⇝ last part of talk)*

Wiener space: Fang/Shao/Sturm

Heat Equation and Ricci Curvature

- Let $S(\rho\, dx) = \int \rho \cdot \log \rho \, dx$. Then $\frac{\partial}{\partial t}\rho = \triangle \rho$ and

$$\text{Hess } S \geq K \qquad \Leftrightarrow \qquad \text{Ric} \geq K$$

- Let $S(\rho\, dx) = \frac{1}{m-1} \int_M \rho^m(x)\, dx$. Then $\frac{\partial}{\partial t}\rho = \triangle(\rho^m)$ and

$$\text{Hess } S \geq 0 \quad \Leftrightarrow \quad \begin{cases} m & \geq & 1 - \frac{1}{n} \\ \text{Ric} & \geq & 0 \end{cases}$$

Part II.

Ricci Bounds for Metric Measure Spaces

Ricci Bounds for Metric Measure Spaces

Relative entropy $\mathrm{Ent}(\cdot, m): \quad \mathcal{P}_2(M) \to [-\infty, \infty]$

$$\mathrm{Ent}(\nu, m) = \begin{cases} \int_M \rho \log \rho \, dm & \text{, if } \nu = \rho \cdot m \\ +\infty & \text{, if } \nu \not\ll m \end{cases}$$

Definition. $\mathrm{Ric}(M, \mathsf{d}, m) \geq 0$

$\Leftrightarrow \ \forall \mu_0, \mu_1 \in \mathcal{P}_2(M): \ \exists$ geodesic μ_t s.t. $\forall t \in [0,1]$:

$$\mathrm{Ent}(\mu_t | m) \quad \leq \quad (1-t)\mathrm{Ent}(\mu_0 | m) + t\,\mathrm{Ent}(\mu_1 | m)$$

Ricci Bounds for Metric Measure Spaces

Relative entropy $\mathrm{Ent}(\cdot, m): \quad \mathcal{P}_2(M) \to [-\infty, \infty]$

$$\mathrm{Ent}(\nu, m) = \begin{cases} \int_M \rho \log \rho \, dm & \text{, if } \nu = \rho \cdot m \\ +\infty & \text{, if } \nu \not\ll m \end{cases}$$

Definition. $\mathrm{Ric}(M, \mathsf{d}, m) \geq K$

$\Leftrightarrow \ \forall \mu_0, \mu_1 \in \mathcal{P}_2(M): \ \exists$ geodesic μ_t s.t. $\forall t \in [0,1]$:

$$\mathrm{Ent}(\mu_t|m) \quad \leq \quad (1-t)\mathrm{Ent}(\mu_0|m) + t\,\mathrm{Ent}(\mu_1|m) - \frac{K}{2}\, t(1-t)\, \mathrm{d}_{\mathrm{w}}^2(\mu_0, \mu_1)$$

Basic Examples

Riemannian manifolds: (Otto/Villani, Cord/McCann/Schmuck, vRenesse/Sturm)

$$\mathrm{Ric}(M, \mathsf{d}, m) \geq K \quad \Longleftrightarrow \quad \mathrm{Ric}(\xi,\xi) \geq K \quad \text{for all } \xi \in TM, \ |\xi| = 1$$

Weighted spaces:

$$\mathrm{Ric}(M, \mathsf{d}, m) \geq K \quad \text{and} \quad \mathrm{Hess}\, V \geq L \quad \Longrightarrow \quad \mathrm{Ric}(M, \mathsf{d}, e^{-V} m) \geq K + L$$

Finsler spaces: (Ohta/Sturm) $\rightsquigarrow$ *last part of this talk*

Wiener space: (Fang/Shao/Sturm) $\mathrm{Ric}(M, \mathsf{d}, m) \geq 1$

Alexandrov spaces: *Conjecture!*

Discrete spaces: (Bonciocat/Sturm) h*-rough curvature,* h*-midpoints*

Globalization, Stability

Theorem ('Globalization')

Let $M = \cup_{i \in I} M_i$ be compact. Then

$$\mathrm{Ric}(M_i, \mathsf{d}, m) \geq K \quad (\forall i) \quad \Longrightarrow \quad \mathrm{Ric}(M, \mathsf{d}, m) \geq K.$$

Theorem ('Stability')

Let $\mathrm{Ric}(M_n, \mathsf{d}_n, m_n) \geq K$ and $(M_n, \mathsf{d}_n, m_n) \stackrel{mGH}{\longrightarrow} (M, \mathsf{d}, m)$.

Then $\mathrm{Ric}(M, \mathsf{d}, m) \geq K$.

Proof of the Stability Theorem

For all n exists an optimal coupling q_n of m and m_n and Markov kernels $Q_n(x,dy),\ Q'_n(y,dx)$ such that

$$Q_n(x,dy)m(dx) = q_n(dx,dy) = Q'_n(y,dx)m_n(dy)$$

Map $\nu = \rho \cdot m$ on M to $\nu_n = \rho_n \cdot m$ on M_n with

$$\rho_n(y) = \int_M \rho(x) Q'_n(y,dx)$$

$\Rightarrow$ $\quad \mathrm{Ent}(\nu_n, m_n) \leq \mathrm{Ent}(\nu, m)$ and $d_W(\nu,\nu_n) \leq c \cdot [-\log d_W(m,m_n)]^{-1}$

Proof of the Stability Theorem

$$\begin{aligned}
\mathrm{Ent}(\nu_n, m_n) &= \int \rho_n \log \rho_n \, dm_n \\
&= \int \left(\int \rho(x) Q'_n(y, dx) \right) \cdot \log \left(\int \rho(x) Q'_n(y, dx) \right) \, dm_n(y) \\
&\overset{(*)}{\leq} \int \rho(x) \log \rho(x) Q'_n(y, dx) \, dm_n(y) \\
&= \int \rho(x) \log \rho(x) \, dm(x) \\
&= \mathrm{Ent}(\nu, m)
\end{aligned}$$

□

$(*)$ *Jensen's inequality for convex* $r \mapsto r \log r$

Part III.

The Curvature-Dimension Condition

Jacobi Field Calculus

$F_t = \exp(t\nabla\varphi): \ M \to M,$
$\mu_t = (F_t)_* \, \mu_0$
$J_t := \text{det } dF_t \,, \quad y_t := \log J_t$

$$-\ddot{y}_t \quad \geq \quad \frac{1}{n}\dot{y}_t^2 \, + \, \mathrm{Ric}(\dot{F}_t, \dot{F}_t) \quad \geq \quad \frac{1}{N}\dot{y}_t^2 \, + \, K \cdot |\dot{F}_t|^2 \qquad (**)$$

if dim $\leq N, \quad \mathrm{Ric} \geq K.$

Two main cases

1. Ignore n (i.e. $N = \infty$). Then

$$(**) \quad \Leftrightarrow \quad -y \text{ is } K \cdot |\dot{F}_t|^2 \text{ - convex}$$

$$\Leftrightarrow \quad \mathrm{Ent}(\cdot \mid m) \text{ is } K \text{ - convex}$$

2. Assume $\mathrm{Ric} \geq 0$ (i.e. $K = 0$). Then

$$(**) \quad \Leftrightarrow \quad J^{1/N} \text{ is concave}$$

$$\Leftrightarrow \quad S_N(\cdot \mid m) \text{ is convex}$$

Rényi entropy functional for $\mu = \rho \cdot m + \mu^{\text{sing}}$

$$S_N(\mu \mid m) := -\int \rho^{1-1/N}\, dm$$

K-Convexity of the Entropy

Consider $\quad t \mapsto (F_t)_* \mu_0 = \mu_t = \rho_t \, m \quad$ geodesic in $\mathcal{P}_2(M)$.

$$\begin{aligned} \mathrm{Ent}(\mu_t \mid m) &= \int \rho_t \log \rho_t \, dm \\ &\overset{(i)}{=} \int \rho_t(F_t) \cdot \log \rho_t(F_t) \, J_t \, dm \\ &\overset{(ii)}{=} \int \rho_0 \cdot (\log \rho_0 - y_t) \, dm \\ &= \mathrm{Ent}(\mu_0 \mid m) - \int y_t \, d\mu_0 \end{aligned}$$

$$\Rightarrow \quad \partial_t^2 \, \mathrm{Ent}(\mu_t \mid m) = - \int \ddot{y}_t \, d\,\mu_0 \overset{(iii)}{\geq} K \cdot \int |\dot{F}_t|^2 \, d\,\mu_0 = K \cdot \mathrm{d}_{\mathrm{w}}^2(\mu_0, \mu_1)$$

(i) Change of variables $J_t = \det dF_t$ *(ii) Transport property* $\rho_t(F_t) \cdot J_t = \rho_0$

(iii) Basic estimate for $y_t = \log J_t$.

General K, N

The Curvature-Dimension Condition $CD(K,N)$ for $K, N \in \mathbb{R},\ N \geq 1$:

$\forall\ \rho_0 m,\ \rho_1 m:\ \exists$ geodesic $\rho_t m$
(represented as push forward Θ_t of a measure Θ on the set of geodesics $\gamma : [0,1] \to M$ under the projection map $\gamma \mapsto \gamma_t$):

$$\rho_t^{-1/N}(\gamma_t) \geq \tau_{K,N}^{(1-t)}(\gamma_0,\gamma_1) \cdot \rho_0^{-1/N}(\gamma_0) + \tau_{K,N}^{(t)}(\gamma_0,\gamma_1) \cdot \rho_1^{-1/N}(\gamma_1)$$

for Θ-a.e. γ.

Here $\tau_{K,N}^{(t)}(x,y) = t^{1/N} \left(\frac{\sin(\sqrt{\frac{K}{N-1}}\, t\, \mathrm{d}(x,y))}{\sin(\sqrt{\frac{K}{N-1}}\, \mathrm{d}(x,y))} \right)^{\frac{(N-1)}{N}}$, e.g. $\tau_{0,N}^{(t)}(x,y) = t$

Curvature-Dimension Condition

Theorem. For Riemannian manifolds:

$$CD(K,N) \quad \Longleftrightarrow \quad \mathrm{Ric}_M \geq K \quad \text{and} \quad \dim_M \leq N$$

Example.

$$M = [0, L]\,, \quad \mathrm{d} = |\cdot|\,, \quad m(dx) = V(x)\,dx, \quad V(x) = \left(\sin(\sqrt{\tfrac{K}{N-1}}x)\right)^{N-1}$$

(any $K > 0\,, \quad N > 1\,, \quad L = \sqrt{\frac{N-1}{K}} \cdot \pi$)

Consequences of $CD(K,N)$

Theorem. ("Bishop-Gromov Theorem") For $v(r) = m(\overline{B}_r(x_0))$

$$\frac{v(r)}{v(R)} \geq \frac{\int_0^r \sin(\sqrt{\frac{K}{N-1}}t)^{N-1}\, dt}{\int_0^R \sin(\sqrt{\frac{K}{N-1}}t)^{N-1}\, dt}$$

Corollary. ("Myers Theorem") $L \leq \sqrt{\frac{N-1}{K}} \cdot \pi$

Theorem. ("Poincaré / Lichnerowicz Inequality")

$$K\frac{N}{N-1} \cdot \int_M f^2 dm \leq \int_M |\nabla f|^2 dm$$

where $|\nabla f|(x) = \limsup_{y \to x} \frac{|f(y)-f(x)|}{d(y,x)}$.

Brunn-Minkowski Inequality

Theorem. ('Brunn-Minkowski Inequality') $CD(K,N)$ implies

$\forall\, A_0, A_1 \subset M,\ A_t = \{\gamma_t :\ \gamma \in \mathcal{G}(M), \gamma_0 \in A_0, \gamma_1 \in A_1\}$:

$$m(A_t)^{1/N} \ \geq\ \tau^{(1-t)}(\vartheta) \cdot m(A_0)^{1/N} + \tau^{(t)}(\vartheta) \cdot m(A_1)^{1/N}$$

$\vartheta = \min / \max\{\mathrm{d}(\gamma_0, \gamma_1) :\ \gamma_0 \in A_0, \gamma_1 \in A_1\}$.

Example $K = 0$.

$$m(A_t)^{1/N} \geq (1-t) m(A_0)^{1/N} + t\, m(A_1)^{1/N}$$

Stability of CD(K,N)

Theorem. The curvature-dimension condition is stable under convergence.

Theorem. For all $K, N, L \in \mathbf{R}$ the space of all (M, d, m) with $CD(K, N)$ and with diameter $\leq L$ is compact.

Part IV.

Heat Flow on Finsler Spaces

Finsler Spaces (M, F, m)

- M smooth n-dimensional manifold
- $F : TM \to \mathbb{R}$ with
  - smooth on $TM \setminus \{0\}$
  - $F(x, .) : T_x M \to \mathbb{R}$ norm (for each $x \in M$)

  .
- m measure on M with smooth density in local coordinates

Finsler Spaces (M, F, m)

Basic notions:

- Dual norm $F^*(x,.) : T_x^*M \to \mathbb{R}$
- Legendre transform $J^*(x,.) : T_x^*M \to T_xM$
  In local coordinates: $J^*(x,\alpha)_i = \frac{1}{2}\frac{\partial}{\partial \alpha^i} F^{*2}(x,\alpha)$ for $i = 1, \ldots, n$
- Differential $Du(x) \in T_x^*M$ of smooth function $u : M \to \mathbf{R}$
- Gradient $\nabla u(x) = J^*(x, Du(x)) \in T_xM$ nonlinear in u!
- Divergence of vector field Φ defined via $\int_M u \, \mathrm{div}\Phi \, dm = -\int_M \Phi \, u \, dm$
- Laplacian $\Delta u = \mathrm{div}(J^*(Du))$ cf. Chern, Shen

Heat Flow on Finsler Spaces

- either as gradient flow on $L^2(M,m)$ for the energy

$$\mathcal{E}(u) = \frac{1}{2}\int_M F^2(\nabla u)\,dm = \frac{1}{2}\int_M F^{*2}(Du)\,dm$$

- or as gradient flow on the L^2-Wasserstein space $\mathcal{P}_2(M)$ of probability measures on M for the relative entropy

$$\text{Ent}(u) = \int_M u \log u\,dm.$$

Theorem. Assume M is compact. Then both approaches coincide.

$$\forall u_0 \in L^2 : \quad \exists!\text{ weak solution to } \Delta u = \partial_t u.$$

Example. $M = \mathbb{R}^n, \quad F(x,.) = \|.\|, \quad u(t,x) = t^{-n/2}\exp(-\|x\|^2/4t).$

Heat Flow on Finsler Spaces

Contraction.

$\forall u_0, v_0 \in L^2$:

$$\|u_t - v_t\|_{L^2} \leq e^{-\lambda \cdot \text{ß} \cdot t} \cdot \|u_0 - v_0\|_{L^2}$$

where $1/\lambda =$ Poincaré constant for $\mathcal{E}$, ß $=$ uniform convexity bound for F^{*2}.

Regularity.

All local solutions to the heat equation are $\mathcal{C}^{1,\alpha}$.

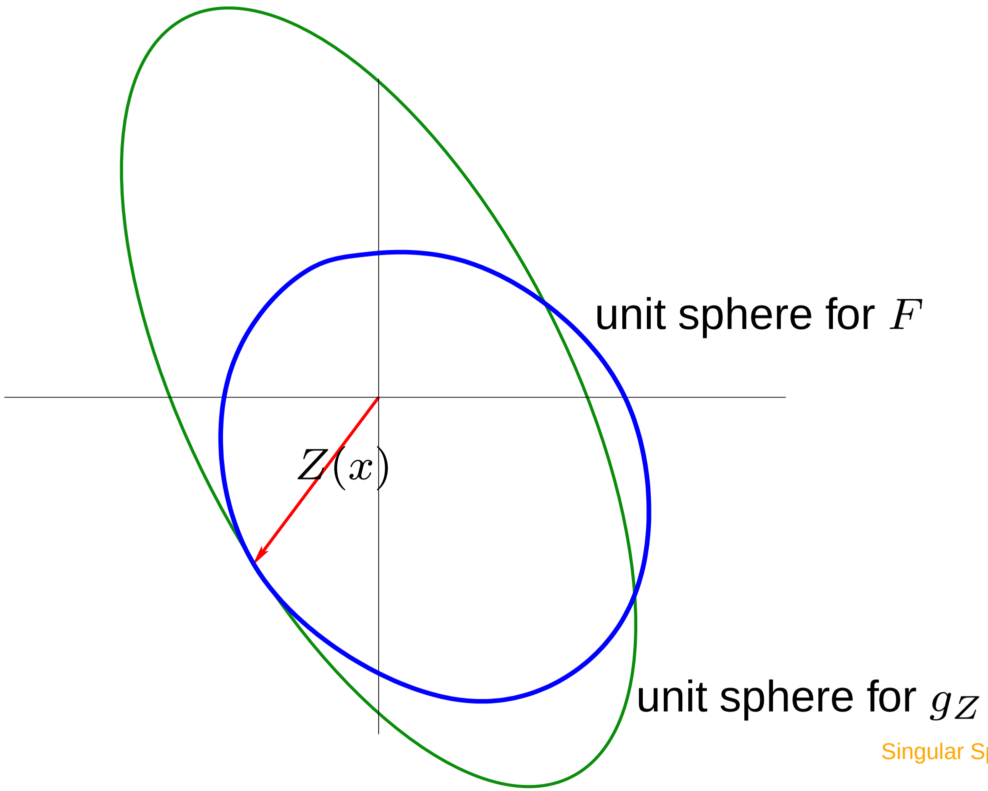
Remark. The following are equivalent:

- All solutions to the heat equation are $\mathcal{C}^2$.
- The heat equation is linear.
- M is Riemannian.

Riemannian Structures on Finsler Spaces

Given a non-vanishing vector field $Z: M \to TM$ we define a Riemannian structure g_Z on M by $g_Z(x) := g(x, Z(x))$ where (in local coordinates):

$$g_{ij}(x,\xi) := \frac{\partial^2}{\partial\xi^i\,\partial\xi^j}\left(\frac{1}{2}F^2(x,\xi)\right).$$



Ricci Bounds on Finsler Spaces

Theorem (Ohta).
(M, F, m) satisfies the curvature-dimension condition $\mathrm{CD}(K, N)$ if and only if $\mathrm{Ric}_{N,g_Z,m}(Z, Z) \geq K \cdot |Z|^2$ for all Jacobi fields Z.

Here for a Riemannian metric $g = g_Z$ and for any number $N \geq n$

$$\mathrm{Ric}_{N,g,m} = \mathrm{Ric}_g + \mathrm{Hess}V - \frac{1}{N-n}(DV \otimes DV)$$

where $V = V_g$ is chosen s.t. $dm = e^{-V} d\mathrm{vol}_g$.