Gradient bounds and a priori estimates for heat flows by Stochastic Analysis

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Heat equation on a manifold

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- How to estimate the gradient $|\nabla u|(x)$?

- **Li-Yau** Let M be complete and $\mathrm{Ric} \geq -K$, $K \geq 0$. Let u be a strictly positive solution of $\frac{\partial}{\partial t} u = \frac{1}{2}\Delta u$ on $M \times \mathbb{R}_+$ and let $a > 1$. Then

$$\left(\frac{|\nabla u|}{u}\right)^2 (x,t) - a\,\frac{\partial_t u}{u}(x,t) \leq c(n,a)\left[K + \frac{1}{t}\right]$$

[if $\mathrm{Ric} \geq 0$, i.e. $K = 0$, then $a = 1$ is possible].

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- **Hamilton** Let M be complete and $\mathrm{Ric} \geq -K$, $K \geq 0$. Let u be a strictly positive solution of $\frac{\partial}{\partial t} u = \frac{1}{2}\Delta u$ on $M \times \mathbb{R}_+$ and suppose $u \leq M$ where M is a real constant. Then

  $$\left(\frac{|\nabla u|}{u}\right)^2 (x,t) \leq 2\left[K + \frac{1}{t}\right] \log \frac{M}{u(x,t)}.$$

- **Li-Yau** (*Localized version*)
  Let $u > 0$ solve $\frac{\partial}{\partial t} u = \frac{1}{2}\Delta u$ on $B(x, R) \times [0, t]$. Then

$$\left(\frac{|\nabla u|}{u}\right)^2 (x,t) - a\,\frac{\partial_t u}{u}(x,t) \leq c(n,a)\left[K + \frac{1}{t} + \frac{1}{R^2}\right].$$

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  Let $0 < u \le M$ solve $\frac{\partial}{\partial t} u = \frac{1}{2}\Delta u$ on $B(x,R) \times [0,t]$. Then

$$\frac{|\nabla u|}{u}(x,t) \le c(n)\left[\sqrt{K} + \frac{1}{\sqrt{t}} + \frac{1}{R}\right]\left(1 + \log\frac{M}{u(x,t)}\right).$$

Harmonic functions

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- **Cheng-Yau**
  Let M be complete and $D \subset M$ be open, relatively compact. Let $u : D \to \mathbb{R}$ be harmonic and strictly positive. Then

$$\frac{|\nabla u|}{u}(x) \leq c(n) \left[\sqrt{K} + \frac{1}{r(x)} \right]$$

if $\quad \mathrm{Ric}|D \geq -K, \;\; K \geq 0 \;\; (\text{where } r(x) = \mathrm{dist}(x, \partial D)).$

Heat equation with respect moving Riemannian metrics

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- Find estimates for $\frac{|\nabla u|}{u}(x,t)$.

Non-linear heat flow and harmonic maps

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- If $\mathrm{Sect}^N \leq 0$, everything works well (Eells-Sampson).

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- In general, the heat flow for harmonic maps develops singularities (*blow up in finite time*), i.e. $\exists\, T > 0,\ x_0 \in M$:

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  i.e. $\exists\, T > 0$, $x_0 \in M$:

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- Try to understand how sectional curvature of the target N enters the estimates of $|du(t,\cdot)|$.

Damped parallel transport

- **Notation** Let X be a general semimartingale taking values in a Riemannian manifold M.
  Define the *damped parallel transport*

  $$\Theta_t\colon\ T_{X_0}M \to T_{X_t}M$$

  by the following covariant equation:

  $$\begin{cases} d\,//_t^{-1}\Theta_t = -\frac{1}{2}\,//_t^{-1}R^M(\Theta_t, dX_t)dX_t \\ \Theta_0 = \mathrm{id}_{T_{X_0}M} \end{cases}$$

  where $//_t$ is the usual parallel transport along X with respect to the Levi-Civita connection.

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- **Example** Let X be BM on (M, g). Then

$$d\,//_t^{-1}\Theta_t = -\frac{1}{2}\,//_t^{-1}\mathrm{Ric}^M(\Theta_t)\,dt.$$

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- Let $g(t)$ be a C^1 family of Riemannian metrics on M. Brownian motion X can be defined w/r to $g(t)$:

  $\forall f \in C^\infty(M)$,

$$d(f(X_t)) - \frac{1}{2}(\Delta_{g(t)} f)(X_t) = 0 \quad \text{(mod loc mart.)}$$

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- There is also a canonical notion of the damped parallel transport along such a BM.
- **Interesting Remark** This damped parallel transport is an isometry if and only if the metric evolves by forward Ricci flow.

Basic observation

- Consider the damped transports

$$\Theta_t^M \colon T_xM \to T_{X_t}M, \quad \text{resp. } \Theta_t^N \colon T_{Y_0}N \to T_{Y_t}N,$$

  along X on M, resp. along Y on N, where
  - X is BM on M with $X_0 = x$, and
  - $Y_t = u(T-t, X_t)$ is the image process on N for a solution u of the non-linear heat equation.

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  - X is BM on M with $X_0 = x$, and
  - $Y_t = u(T-t, X_t)$ is the image process on N for a solution u of the non-linear heat equation.
- Then

  $$\Big\{(\Theta_t^N)^{-1}\, du(T-t,\cdot)_{X_t(x)}\, \Theta_t^M :\ t < T\Big\}$$

  is a local martingale in $T_x^*M \otimes T_{u(T,x)}N$.

Integration by parts

- In particular, for each $v \in T_x M$,

$$m_t\, v := (\Theta_t^N)^{-1}\, du(T-t,\cdot)_{X_t(x)}\, \Theta_t^M\, v$$

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  is a local martingale in $T_{u(T,x)}N$.
- Now allow v to vary with time. For this purpose let $\ell(t)$ be any adapted process in $T_x M$ with absolutely continuous paths. Then

  $$m_t \ell(t) - \int_0^t m_r\, d\ell(r)$$

  is a local martingale, i.e.,

  $$(\Theta_t^N)^{-1} du(T-t,\cdot)_{X_t(x)}\, \Theta_t^M\, \ell(t) - \int_0^t (\Theta_r^N)^{-1} du(T-r,\cdot)_{X_r(x)}\, \Theta_r^M\, \dot\ell(r)\, dr$$

  is a local martingale.

Deformed anti-development

- Consider the process $\tilde{Y}$, taking values in $T_{u(T,x)}N$, defined by

$$d\tilde{Y}_t = (\Theta_t^N)^{-1}\, dY_t\,, \quad \text{where } Y_t = u(T-t, X_t).$$

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$$d\tilde{Y}_t = (\Theta_t^N)^{-1}\, dY_t\,, \quad \text{where } Y_t = u(T-t, X_t).$$

- Then

$$\tilde{Y}_t = \int_0 (\Theta_r^N)^{-1}\, du(T-r,\cdot)_{X_r(x)}\, //_r^M\, dB_r$$

  where B is the anti-development of the BM X in T_xM, and we see that

$$n_t := (\Theta_t^N)^{-1} du(T-t,\cdot)_{X_t(x)}\, \Theta_t^M\, \ell(t) - \tilde{Y}_t \int_0^t \langle \Theta_r^M\, \dot{\ell}(r), //_r^M\, dB_r \rangle$$

  is a local martingale as well.

- The idea is now to choose $\ell_t = \ell(t)$ such that first n_t is a true martingale and such that

  $$\ell_0 = v, \quad \ell_\tau = 0 \quad \text{and} \quad \Big(\int_0^\tau |\dot{\ell}_t|^2\, dt\Big)^{1/2} \in L^{1+\varepsilon}$$

  (some $\varepsilon > 0$) where, for instance, $\tau = \tau(x) \wedge T$ with $\tau(x)$ the first exit time from some rel. compact neighbourhood D of x.

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- Taking expectations we get

$$\mathbb{E}[n_0] = \mathbb{E}[n_\tau]$$

  which gives a gradient formula.

Theorem (Non-linear gradient formula)

Let $u : M \times [0,T] \to N$ be a solution of the non-linear heat equation. Then, for each $v \in T_xM$,

$$\boxed{du(T,\cdot)_x v = -\mathbb{E}\left[\tilde{Y}_\tau \int_0^\tau \langle \Theta_s^M \dot{\ell}_s, //_s \, dB_s \rangle\right]}$$

where:

- $Y_t = u(T-t, X_t)$ *and* $\tilde{Y}_t$ *its deformed anti-development*
- $\tau = \tau(x) \wedge T$ *with* $\tau(x) = \inf\{t > 0 : X_t(x) \notin D\}$
- ℓ_t *an adapted process with values in* T_xM *such that*

$$\Big(\int_0^\tau |\dot{\ell}_t|^2 \, dt\Big)^{1/2} \in L^{1+\varepsilon} \quad \text{and } \ell_0 = v, \ \ell_\tau = 0.$$

Theorem (derivative estimate)

$$\left|du(T,\cdot)_x v\right| \le \underbrace{\left\| \int_0^\tau \langle \Theta_s^M \, \dot{\ell}_s, //_s^M \, dB_s \rangle \right\|_p}_{=: \ \mathbf{(I)}} \cdot \underbrace{\left\| \tilde{Y}_\tau \right\|_q}_{=: \ \mathbf{(II)}}$$

where $1 \le p < \infty$ *and* $1/p + 1/q = 1$.

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- **To estimate (I):** Let $p = 2$ and $|v| \le 1$. Then

$$\textbf{(I)} \le c(n) \left(k + \frac{1}{\sqrt{T}} + \frac{1}{r(x)} \right), \quad r(x) := \operatorname{dist}(x, \partial D),$$

if $\operatorname{Ric}|D \ge -(n-1)k^2$ with $k \ge 0$

F.-Y. Wang & A. Th., *JFA* (1998)

- **To estimate (II) :**

  Let $-\kappa_1 \le \mathrm{Sect}^N \le \kappa_2$ with $\kappa_1, \kappa_2 \ge 0$.

  Then

  $$\exp\left(-\frac{\kappa_1}{2}\int_0^t h(dY, dY)\right) \le \left|(\Theta_t^N)^{-1}\right| \le \exp\left(\frac{\kappa_2}{2}\int_0^t h(dY, dY)\right)$$

  where h denotes the metric on N and $Y_t := u(T-t, X_t)$.

  M. Arnaudon, X.-M. Li, A. Th., *Ann. Inst. H. Poincaré* (1999)

Applications *Sharp a priori estimates for harmonic maps*

- Let $D \subsetneqq M$ be a relatively compact domain, and let $u\colon D \to N$ be a harmonic map. Then, for each $v \in T_xM$, $x \in D$,

$$\boxed{(du)_x v = -\mathbb{E}\left[\widetilde{u(X)}_\tau \int_0^\tau \langle \Theta^M_s\, \dot{\ell}_s, //_s\, dB_s \rangle\right]}$$

where $\tau = \inf\{t > 0 : X_t(x) \notin D\}$. In particular,

$$\left|(du)_x v\right| \le \left\| \int_0^\tau \langle \Theta^M_s\, \dot{\ell}_s, //^M_s\, dB_s \rangle \right\|_p \cdot \left\| \widetilde{u(X)}_\tau \right\|_q$$

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- *Liouville theorems for harmonic maps*

Example **Harmonic maps of bounded dilatation**

Let $\lambda_1(x) \geq \lambda_2(x) \geq \ldots \geq \lambda_n(x) \geq 0$ be the eigenvalues of $(du)_x^* \, (du)_x \colon T_xM \to T_xM$.

Example Harmonic maps of bounded dilatation

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A map $u\colon M \to N$ is said to be of *K-bounded dilatation* (for some constant $K > 0$) $\iff$ $\lambda_1 \leq K^2(\lambda_2 + \ldots + \lambda_n)$ on M.

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Theorem (non-linear a priori estimate)

Let M be compact with $\partial M \neq \emptyset$, $Ric^M \geq k \in \mathbb{R}$ and $\mathrm{Sect}^N \leq -\beta < 0$. Let $u \colon M \to N$ be a harmonic map of K-bounded dilatation. Then

$$|(du)_x|^2 \leq \frac{K^2}{\beta} C\big(dist(x, \partial M)\big), \quad x \in M \backslash \partial M,$$

where $C(r) = \frac{\pi^2}{4} \frac{n+3}{r^2} + \frac{\pi}{2} \frac{\sqrt{\alpha\,(n-1)}}{r} + \alpha$ *and* $\alpha = -(k \wedge 0)$.

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Theorem (Gradient formulas in the linear case)

- *Let* $\frac{\partial}{\partial t}u = \frac{1}{2}\Delta u$ *such that* $\begin{cases} u\big|_{t=0} = u_0 \\ u(t,\cdot)|\partial D = u_0|\partial D. \end{cases}$ *Then*

$$du(T,\cdot)_x v = -\mathbb{E}\left[u_0(X_\tau(x)) \int_0^\tau \left\langle Q_s\,\dot\ell_s, dB_s\right\rangle\right],\ \tau = \tau(x)\wedge T.$$

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Theorem (Gradient formulas for harmonic functions)

- *Let* $\Delta u = 0$ *on* D. *Then*

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- If $\quad \mathrm{Ric}|D \geq -(n-1)k^2$ for some $k \geq 0$,
it is easy to derive from here:

$$|\nabla u|(x) \leq \|u\|_D \left\| \int_0^\tau \langle Q_r\, \dot{\ell}_r, //_r\, dB_r\rangle \right\|_2 \leq \|u\|_D\, c(n) \left(k + \frac{1}{r(x)}\right)$$

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- But recall *Cheng-Yau*! For u **positive** we want:

$$|\nabla u|\,(x) \leq u(x) \cdot c(n) \left(k + \frac{1}{r(x)}\right).$$

Problem

$$\frac{|\nabla u|}{u}(x) = -\mathbb{E}\Bigg[\underbrace{\frac{u(X_\tau(x))}{u(x)}}_{L^1\ ?}\ \underbrace{\int_0^\tau \langle Q_s\,\dot{\ell}_s, dB_s\rangle}_{L^\infty\ ?}\Bigg]$$

Note: $Y_t := \dfrac{u(X_t(x))}{u(x)}$ is a strictly positive martingale; $Y_0 = 1$.

Lemma

Let Y be a strictly positive martingale; $Y_0 = y$

$$\Longrightarrow \forall \alpha \in \,]0,1[\,: \quad \mathbb{E}\big[\langle Y, Y\rangle_\infty^{\alpha/2}\big]^{1/\alpha} \le C_\alpha\, y\,.$$

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Proof. Without restriction $Y_0 = 1$

$$Y = 1 + \beta_{\langle Y,Y\rangle}, \quad \beta \text{ real BM} \quad (\text{with } \beta_0 = 0)$$

$\Longrightarrow \langle Y, Y\rangle \le S := \inf\{s > 0 : \beta_s = -1\}$

But $S \sim 1/\beta_1^2$

$$\Longrightarrow \mathbb{E}[\,S^{\alpha/2}]^{1/\alpha} = \mathbb{E}[\,|\beta_1|^{-\alpha}]^{1/\alpha} =: C_\alpha = \frac{1}{\sqrt{2}}\left(\frac{\Gamma(\frac{1-\alpha}{2})}{\Gamma(\frac{1}{2})}\right)^{1/\alpha}.$$

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Note Here,

$$\forall \alpha < 1: \quad \mathbb{E}\Big[\Big(\int_0^\tau |du|^2(X_r)\,dr\Big)^{\alpha/2}\Big]^{1/\alpha} \le C_\alpha\, u(x)$$

$$\text{et} \quad \mathbb{E}\big[u(X_\tau)^\alpha\big]^{1/\alpha} \le C_\alpha\, u(x).$$

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Theorem (M. Arnaudon, B. Driver, A. Th., 2006)

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- **Consequence** $N_s\ell_s - \int_0^s N_r d\ell_r$ submartingale if $\dot{\ell}_s \leq 0$
[Here: ℓ_s scalar, $\ell_0 = 1$, $\ell_\tau = 0$ with $\tau = \tau(x)$]

Corollary (Inequalities of Bismut type)

$$|du|^\alpha(x) \le -\mathbb{E}\left[\int_0^\tau N_r\,\dot\ell_r\,dr\right], \quad \alpha \ge \frac{n-2}{n-1}.$$

By Hölder, if $\alpha < 2$,

$$|du|^\alpha(x) \le \mathbb{E}\Bigg[\bigg(\int_0^\tau |du|^2(X_s)ds\bigg)^{\alpha/2} \times \bigg(\int_0^\tau \exp\Big\{\tfrac{\alpha}{\alpha-2}\int_0^s \mathrm{Ric}(n_r,n_r)\,dr\Big\}\,|\dot\ell_s|^{\frac{1}{2-\alpha}}\,ds\bigg)^{\frac{2-\alpha}{2}}\Bigg].$$

Let $\alpha \in \left[\frac{n-2}{n-1}, 1\right[$ if $n \geq 3$, and $\alpha \in \,]0,1[$ if $n = 2$,

$p > 1$ such that $\alpha p < 1$,

$q > 1$ conjugated exponent to p.

Suppose that $\mathrm{Ric} \geq -K$ for some $K \geq 0$. Then,

$$|\nabla u|(x) \leq C_{\alpha p}\, u(x)\, \mathbb{E}\left[\left(\int_0^\tau \exp\left\{\frac{\alpha}{2-\alpha} Ks\right\} |\dot{\ell}_s|^{\frac{2}{2-\alpha}}\, ds\right)^{\frac{(2-\alpha)q}{2}}\right]^{\frac{1}{\alpha q}}.$$

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Corollary

$$|\nabla \log u|(x) \leq c(n)\left[k + \frac{1}{r(x)}\right]$$

if $Ric^M \geq -(n-1)\, k^2$, $k \geq 0$.

From gradient estimates to Harnack inequalities

An obvious consequence of the Cheng-Yau estimate is:

Corollary (Harnack inequality)

Let u be harmonic on $B_r(x) \subset M$ where M is complete. Then

$$\sup_{B_{r/2}(x)} u \le c(n, r, k) \inf_{B_{r/2}(x)} u$$

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- **Elliptic case** Let $u\colon D \to \mathbb{R}$ be harmonic and $x_1, x_2 \in D$. Fix $B \subset D$ open, rel. compact, connected, such that $x_1, x_2 \in B$.

  Then $\frac{u(x_1)}{u(x_2)} \leq C$ where the constant C depends only on
  - the lower bound for Ric on B;
  - $\mathrm{dist}_B(x_1, x_2)$ and $\mathrm{dist}(x_i, \partial B)$, $i = 1, 2$;
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- **Parabolic case** Want to compare $P_T f(x_1)$ and $P_T f(x_2)$

- **Harnack inequality with power** $\alpha > 1$ (F.-Y. Wang)

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- *Idea* Estimate the derivative of

$$[0,1] \ni r \mapsto \log(P_T f^{\beta(r)})^{\alpha(r)}(\gamma(r))$$

$$\begin{aligned} \text{where } \gamma(0) &= x,\ \gamma(1) = y \\ \alpha(0) &= \alpha,\ \alpha(1) = 1 \\ \beta(0) &= 1,\ \beta(1) = \alpha \end{aligned}$$

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- If B is an open relatively compact connected neighbourhood of x, y, the constant should depend only on a lower bound of *Ric* on B.

- *Need local gradient estimates of the type:*

$$\frac{|\nabla P_T f(x)|}{P_T f(x)} \leq \frac{C_1}{T \wedge 1} + C_2\, P_T \left(\frac{f}{P_T f(x)} \log \frac{f}{P_T f(x)} \right)$$

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- For $u = P_T f > 0$ use
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  - $(\frac{1}{2}\Delta - \partial_t) \, u \log u = \frac{1}{2} \frac{|\nabla u|^2}{u}$
- Then, if $\operatorname{Ric} \geq -K$,

$$N_t := \frac{1}{2} \, \frac{T-t}{1+K(T-t)} \, \frac{|\nabla P_{T-t} f|^2}{P_{T-t} f}(X_t) + (P_{T-t} f \log P_{T-t} f)(X_t)$$

is a local submartingale and $\mathbb{E}[N_0] \leq \mathbb{E}[N_T]$ gives an inequality of the wanted type (→ *localization*).

Corollary (Wang-Arnaudon-A.Th. 2007)

Let M be an arbitrary complete Riemannian manifold. Then, for any $\delta > 2$ there exists a positive function $C_\delta \in C([0,\infty[\, \times M)$ such that the transition density $p_t(x,y)$ of P_t with respect to the volume measure satisfies

$$p_t(x,y) \le \frac{\exp\left\{-\mathrm{dist}(x,y)^2/(2\delta t) + C_\delta(t,x) + C_\delta(t,y)\right\}}{\sqrt{\mathrm{vol}(B(x,\sqrt{2t}))\mathrm{vol}(B(y,\sqrt{2t}))}},$$

$$x,y \in M,\ \ t \in\,]0,1[\,.$$