Symplectic homogenization

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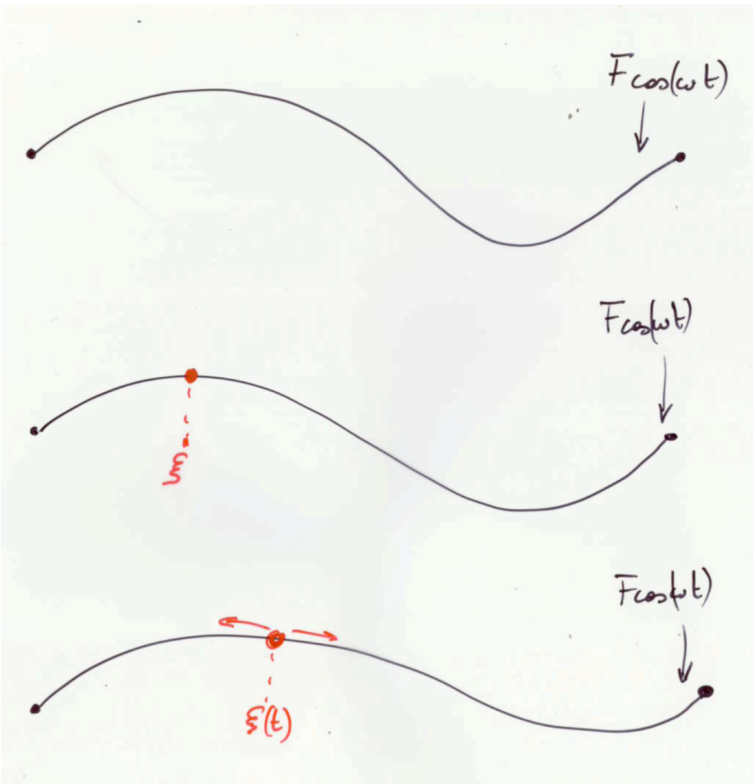
A. Origin of the problem

1) Self adaptative resonance (Ben Amar-Boudaoud-Couder 1999)

Vibrating string with a sliding bead : one observes self-adaptative resonance



Resonance frequency

$$\omega = \omega_j$$

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$$\omega = \omega_j(\xi)$$

Equilibrum position of the bead ξ “ solves” $\omega = \omega_j(\xi_\infty)$

Lagrangian :

$$E(\xi, u, u_x) =$$

$$\int_0^T \int_0^L \left[|\dot{\xi}|^2 + (1 + \mu\delta_\xi(x))|u_t(t,x)|^2 - \frac{1}{\varepsilon}|u_x(t,x)|^2 + f\cos(\omega t)u(t,x) \right] dtdx$$

or its non linear version

How does $\xi(t)$ behaves when ε goes to 0 : $\xi(t)$ approaches the solution of $\ddot{\xi}(t) + \nabla W(\xi) = 0$ where $W(\xi)$ is the minimum of the Lagrangian of the system, with fixed ξ. For almost quadratic energy , this will converge to the value of ξ yielding the maximal energy of the system (with fixed ξ), that is, resonance.

2) Geodesics of a homogenized metric. (Federer ? ; Acerbi-Buttazzo, 1984)

Let $\langle g(x)p, p\rangle$ be a metric on T^n.

The renormalized metric is : $g_k(x) = g(kx)$ ($k \in \mathbb{N}$)

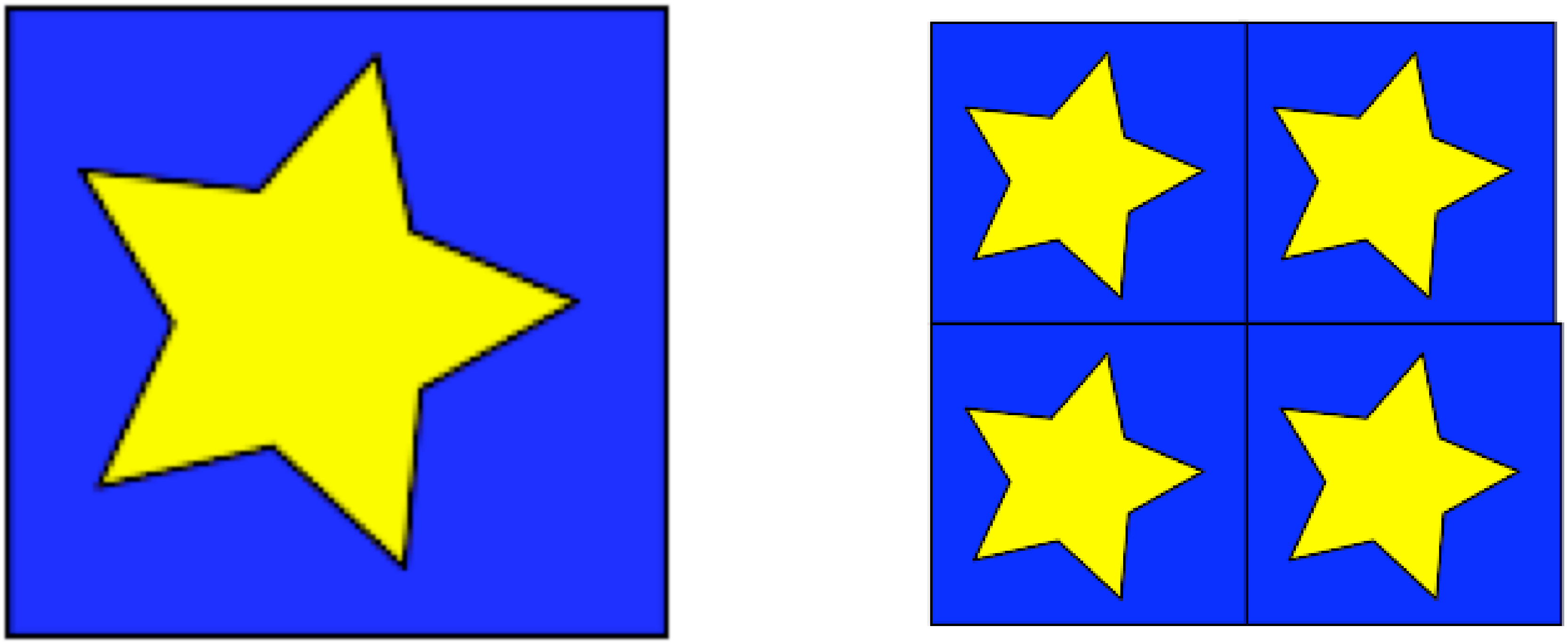
The distance d_k associated to the metric g_k, converges to $\overline{d}$, associated to a Finsler metric $\overline{g}$.

Do the geodesic flows of g_k converge to the geodesic flow of $\overline{g}$?

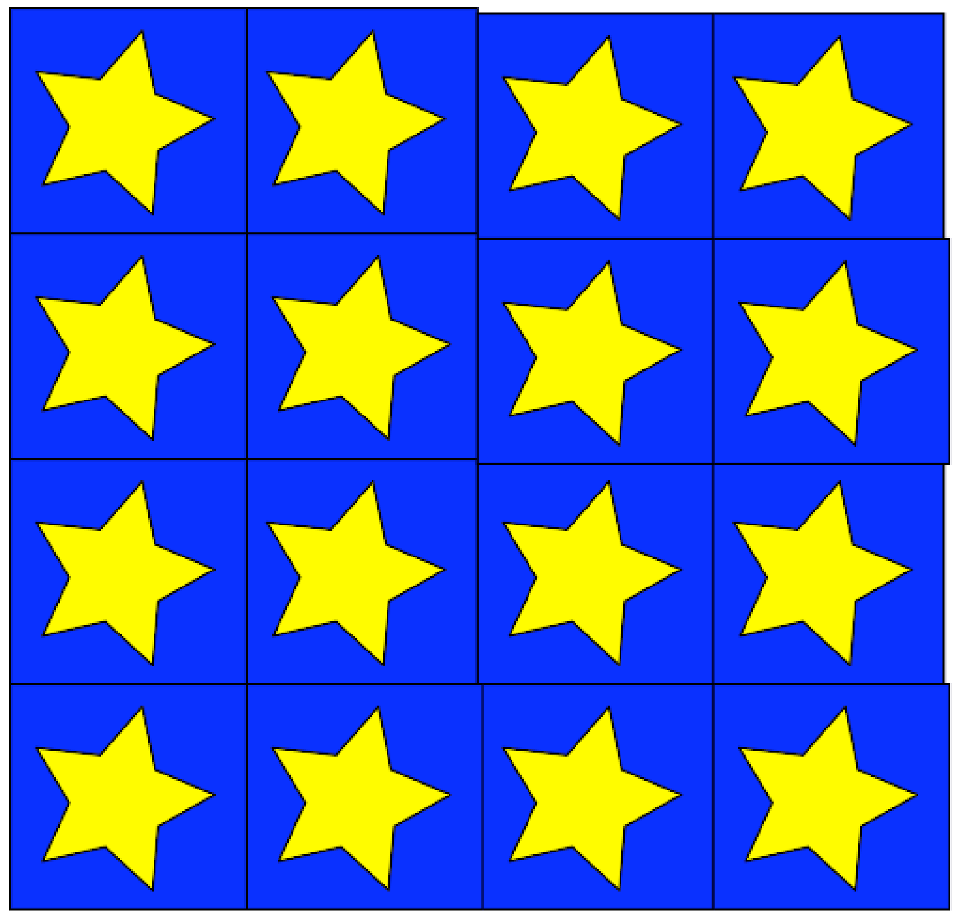
Does the lenght of the minimizing geodesic converge ?

Does the lenght of the non-minimizing (but obtained by minmax) geodesic converge ?

The metric g and the metric g_2



The metric g_4



3) Homogenization of Hamilton-Jacobi evolution equations (Lions-Papanicolaou-Varadhan, 1986)

$$(HJ_k) \qquad \frac{\partial}{\partial t}u(t,x) + H(k\cdot x, \frac{\partial u}{\partial x}(t,x)) = 0; u(0,x) = f(x)$$

Let u_k be "the" solution of (HJ_k). Then

$$\lim_{k\to\infty} u_k(t,x) = v(x)$$

is a solution of

$$\frac{\partial}{\partial t}v(t,x) + \overline{H}\left(\frac{\partial v}{\partial x}(x)\right) = 0$$

$\overline{H}$ is called the "effective Hamiltonian".

In what sense does $H(k\cdot x, p)$ converge to $\overline{H}(p)$? What happens by canonical (i.e. symplectic) change of variables ? Is there some type of dynamical convergence ?

B. General problem

$H(q,p)$ Hamiltonian (could be $\frac{1}{2}p^2 + V(q)$) with flow $\varphi^t(q,p)$

What happens when we look at $H(k\cdot q,p)$:

the flow becomes φ_k^t

$$\dot{p} = k\cdot\frac{\partial}{\partial q}H(k\cdot q,p), \dot{q} = -\frac{\partial}{\partial q}H(k\cdot q,p)$$

If $\rho_k(q,p) = (k\cdot q,p)$, we have $\varphi_k^t = \rho_k^{-1}\varphi^{kt}\rho_k$

Remark : In Hamiltonian dynamics,
homogenization = singular perturbation = long time behaviour of the flow

C. Main result

Theorem There is a projection operator

$$\mathcal{A} : C_0^2([0,1] \times T^*T^n, \mathbb{R}) \longrightarrow C_0^0(\mathbb{R}^n, \mathbb{R})$$

such that

(1) the sequence $H(k \cdot q, p)$ c-converges to $\overline{H}(p)$ (where $\mathcal{A}(H) = \overline{H})$. The same for the associated flows $\varphi_k^t = \rho_k^{-1}\varphi^{kt}\rho_k$ c-converges to $\overline{\varphi}^t$ (and $\overline{\varphi}^t$ is the flow of $\overline{H}$)

(2) $\mathcal{A}$ extends by continuity to a map

$$C_0^0([0,1] \times T^*T^n, \mathbb{R}) \longrightarrow C_0^0(\mathbb{R}^n, \mathbb{R})$$

(3) $\mathcal{A}$ is symplectically invariant (i.e.$\mathcal{A} \circ \psi = \mathcal{A}$)

(4) $\mathcal{A}(H)$ only depends on φ^1.

(5) $\mathcal{A}(H + K) = \mathcal{A}(H) + \mathcal{A}(K)$ if $\{H, K\} = 0$

(6) $H_1 \leq H_2$ implies $\mathcal{A}(H_1) \leq \mathcal{A}(H_2)$

D. Remarks

a) Almost all results quoted here can be dealt with using Γ-convergence if one adds suitable convexity assumption (cf. Braides)

b)The phrase "$\overline{\varphi}$ is the flow of $\overline{H}$ " has a well defined meaning, even though $\overline{H}$ is only C^0....

c) We can do homogenization with respect to only some variables (as in the vibrating string with sliding bead)

d) The map $\mathcal{A}$ can be extended to a symplectic invariant of sets (using (6)), by defining

$$\mathcal{A}(U) = \bigcup_{\text{supp}(H) \subset U} \text{supp}\mathcal{A}(H)$$

This also extends to Lagrangian submanifolds, and allows us to make (some) computations.

e) The solutions we consider for HJ are not viscosity solutions unless H is convex in p. However they share many of the same properties, except for semi-groups. In this case (3) has been proved by P. Bernard.

E. Applications

1) Homogenization for variational solutions of HJ equations, for non coercive Hamiltonians (one actually works with) and symplectic invariance of the effective Hamiltonian.

2) Convergence of (almost)) all “variational invariants” of the Hamiltonians of H_k to the invariants of H.

3) Compatibility with the time one map of the Hamiltonian

F. Some ingredients of the proof

The crucial step in the proof of the theorem is the construction (existence of) $\overline{H}$. This also allows to prove a number of properties of $\overline{H}$, and sometimes to compute it.

The proof of existence of $\overline{H}$ is related to

– the method of ‘passing obstacles one at the time”
– a result in symplectic topology, bounding the size of a Lagrangian contained in a unit disc cotagent bundle.

G. The metric c (analogue to Hofer's metric)

Principle : "All variational formulations are topologically equivalent"

- To L Hamiltonianly isotopic to the zero section in T^*T^n we associate $c(L)$
- If L is the graph of φ, $c(\varphi) = c(L)$

Properties : $H_\nu \xrightarrow{C^0} H$ implies $H_\nu \xrightarrow{c} H$ and $\varphi_\nu^t \xrightarrow{C^0} \varphi^t$ implies $H_\nu \xrightarrow{c} H$

Theorem[Crucial result (uses Floer homology] If L is a Lagrangian in DT^*T^n then $c(L) \leq C_n$

The set of Hamiltonians has a completion for c, containing all continuous Hamiltonians and their flows (see Humilière's work).
This is the proper setting for symplectic homogenization.

More questions :

What are other compact sets for the metric c ?

Connection with Aubry-Mather theory ?

What is the quantum version of all this ? (work in progress with T. Paul)

Thanks to the organizers

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