APPENDIX C

Radiant affine 3-manifolds with boundary, and certain radiant affine 3-manifolds

By Thierry Barbot[1] [2] and Suhyoung Choi

Let G be a group acting on an analytic manifold X. An (X, G)-manifold is a manifold admitting an atlas with charts with value in X and whose coordinate change mappings are restrictions of elements of G. It is well-known that equipping a manifold M with an (X, G)-structure is equivalent to giving a pair $(\mathbf{dev}, \rho)$, where $\mathbf{dev}$ is an immersion from the universal covering $\tilde{M}$ of M into X, and where ρ is a homomorphism from the fundamental group Γ of M into G, such that

$$\forall \gamma \in \Gamma \qquad \mathbf{dev} \circ \gamma = \rho(\gamma) \circ \mathbf{dev}.$$

Here, the action of Γ on $\tilde{M}$ is of course the action by deck transformations. The map $\mathbf{dev} : \tilde{M} \to X$ is the *developing map* of the structure, and $\rho : \Gamma \to G$ is the *holonomy homomorphism.*

A radiant affine n-manifold is an (X, G)-manifold, where X is the vector space $\mathbf{R}^n$, and G is the group $\mathrm{GL}(n, \mathbf{R})$ of linear transformations (cf. [**11**]). Such a manifold is naturally equipped with a transversely projective flow, the so-called *radial flow,* defined as follows: if $(x_1, \ldots, x_n)$ are local coordinates, the vector field generating the radial flow is:

$$X(x_1, \ldots, x_n) = \Sigma_{i=1}^n x_i \partial_{x_i}$$

Observe that this vector field does not depend on the coordinate systems as long as the origins are the same, and thus induces well-defined vector fields on $\tilde{M}$ and on M, which are said to be the *radiant vector field.* The flow generated by the vector fields are said to be *radial flow.* The radial flow has unique singularity at the origin O of $\mathbf{R}^n$ (cf. [**11**]) but nowhere on M, since $\mathbf{dev}$ misses O.

Let N be a closed real projective $(n-1)$-manifold, i.e. a $(\mathbf{R}P^{n-1}, \mathrm{PGL}(n, \mathbf{R})))$-manifold, where $\mathbf{R}P^{n-1}$ is the real projective space of dimension $n-1$, and $\mathrm{PGL}(n, \mathbf{R})$ is the group of projective transformations. Let φ be a projective automorphism of N. We can associate to the pair (N, φ) a family of radiant affine closed n-manifold: i.e., generalized affine suspensions, homeomorphic to a topological suspension of N by φ (see Chapter 4 and [**8, 42, 11, 4**]). They can be characterized by the following property: *a closed radiant affine manifold is a generalized affine suspension if and only if its radial flow admits a total cross-section, i.e. there is a closed embedded submanifold transverse everywhere to the flow and which meets every orbit of the flow.* (See Proposition 3.2.) (Note that the term "affine suspension is reserved

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for the case when N and φ are both affine.) A *Benzécri suspension* is an affine suspension so that φ is the identity or a finite-order automorphism of N. In this case the all orbits are closed, and N is Seifert-fibered.

In this appendix, we study the following particular case which was left from Chapter 12, i.e., Lemma 12.6, which is implied by the following theorem since developing maps of a universal cover is always obtainable from developing maps of the holonomy cover composed with the covering map to the holonomy cover from the universal cover.

THEOREM C.1. *There is no closed radiant affine* 3*-manifold whose developing map from the universal cover is an infinite cyclic covering over* $\mathbf{R}^3$ *minus a line.*

A subsurface S in an affine 3-manifold is *totally-geodesic* if every point of S has a neighborhood $\mathcal{O}$ affinely diffeomorphic to an open subset of $\mathbf{R}^3$ or of an affine half-space of $\mathbf{R}^3$ so that $S \cap \mathcal{O}$ corresponds to a closed affine subspace of codimension-one intersected with the open set. A totally geodesic subsurface has a natural induced affine structure as a two-dimensional manifold. A boundary component of an affine 3-manifold is *totally geodesic* if each boundary point has a neighborhood affinely diffeomorphic to an open subset of a half-space in $\mathbf{R}^3$.

THEOREM C.2 (Theorem B of Barbot [**3**]). *Let M be a closed radiant affine* 3*-manifold. If M includes a totally geodesic surface tangent to the radial flow, then M admits a total cross-section*; *i.e., M is a generalized affine suspension.*

The second result of this appendix (Theorem 4.4), we will prove is:

THEOREM C.3. *Let M be a compact radiant affine* 3*-manifold with a nonempty totally geodesic boundary. Suppose that each component is affinely homeomorphic to the quotient of a convex cone or* $\mathbf{R}^2 - \{O\}$ *by an affine action. Then M admits a total cross-section to the radial flow, and hence is a generalized affine suspension.*

We cannot prove this theorem by a method of "doubling": Some radiant affine 3-manifold N may not doubled, i.e., there may not be a radiant affine 3-manifold homeomorphic to the topological double of N in which N and a copy of N are affinely imbedded, meeting at boundary. An example comes from a convex real projective surface Σ with negative Euler characteristic and the boundary component with holonomy ϑ that has a nondiagonalizable 3×3-matrix with two distinct positive eigenvalues ([**13**] and [**15**].) These real projective surfaces exist, of course, as one can see that the construction of convex real projective surfaces in Goldman [**24**] can be modified to construct convex ones with this behavior. Such a holonomy ϑ does not commute with a projective automorphism in $\mathbf{R}P^2$ whose fixed points comprise a subspace that ϑ preserves. We can easily see that the Benzécri suspension of Σ cannot be doubled in the above sense.

The proof of the theorem is essentially that of Theorem B in [**3**] where the totally geodesic surface now is in the boundary.

We remark that the theorem is true without the assumption on boundary component which can be proved applying Barbot's method [**3**] for totally geodesic surface in the manifold tangent to the radial flow. For simplicity of proof, we prove this weaker but sufficient version here.

The proof of the first one goes as follows: we assume the existence of a radiant affine 3-manifold whose developing map is an infinite cyclic covering over $\mathbf{R}^3 \setminus \{x = y = 0\}$ where x, y, and z denote the standard coordinate functions of $\mathbf{R}^3$.

In the first section, we prove that the holonomy group is solvable: indeed, if not, it contains a hyperbolic element $\rho(\gamma)$ with one eigenvalue greater than 1, another less than 1 (and positive), and the last exactly equal to 1. The contradiction nearly arises from the fact that such a linear transformation does not act properly discontinuously on $\mathbf{R}^3 \setminus \{x = y = 0\}$, whereas γ must act properly discontinuously on $\tilde{M}$.

Since the holonomy group is solvable, the affine 3-manifold is a generalized affine suspension ([**4**]). Therefore, to achieve the proof, we will show that no projective surface has a developing map which is an infinite cyclic covering over the sphere minus two points.

For the second theorem, we will only prove for the cases when the fact that M has nonempty boundary makes any difference from the proof of Theorem B of [**3**].

If the holonomy of the boundary component contains a homothety, i.e., a linear transformation that is a positive multiple of the identity map, then all radial flow orbits are periodic and it follows easily that our manifold has a total cross-section. First, we show that if the boundary surface is not convex as an affine 2-manifold, then our affine manifold is a half-Hopf manifold. Then we look at the holonomy group of the fundamental group of the boundary component, which we may assume is an affine torus, and classify them into six cases as in [**3**]. Only four cases are applicable since the boundary torus is convex. We will show that in each case, our radiant affine manifold is finitely covered by torus times an interval which decomposes into submanifolds which are affinely isomorphic to domains in $\mathbf{R}^3$ quotient out by linear $\mathbf{Z} + \mathbf{Z}$-action. We show that the third case is a generalized affine suspension and forth cases are impossible and in the remaining two cases, the pieces must be generalized affine suspensions. (This proof is mostly a generalization of that of Theorem B in Barbot [**3**])

1. The nonexistence of certain radiant affine 3-manifolds

Let M be a closed radiant affine 3-manifold. We denote by Φ^t its radial flow. We denote by $p : \tilde{M} \to M$ the universal covering (we don't worry about the choice of a base point). Let Γ be the fundamental group of M: it acts naturally on $\tilde{M}$.

Let $\mathbf{dev} : \tilde{M} \to \mathbf{R}^3$ be the developing map, and $\rho : \Gamma \to \mathrm{GL}(3, \mathbf{R})$ be the holonomy homomorphism. They satisfy:

$$\forall \gamma \in \Gamma \qquad \mathbf{dev} \circ \gamma = \rho(\gamma) \circ \mathbf{dev}.$$

As above, the radial vector field induces radial flows in M and $\tilde{M}$ respectively. The orbits are said to be *rays* and **dev** restricted to each ray is a homeomorphism onto rays in $\mathbf{R}^3$ by Lemma C.1; i.e., an open half-line with endpoint at 0.

LEMMA C.1. *Let G be a Lie group acting on two manifolds X and Y. Let $f : X \to Y$ be a function equivariant for the actions of G. Let x be an element of X such that $f(x)$ is fixed by no element of G. Then, the restriction of f to the G-orbit of x is injective.*

PROOF. For every element g of G we have $f(gx) = g.f(x)$. □

We now assume that **dev** is an infinite cyclic covering map over $\mathbf{R}^3 \setminus \Delta$, where Δ is a line through the origin O. Our aim is to obtain a contradiction.

Since we want to show that such a M does not exist, we are free to replace M by any finite covering of itself. For example, we can consider only the case where M

is oriented, i.e. the case where every element of the holonomy group is of positive determinant.

Since **dev** is well-defined up to composition by a linear transformation, we can assume that Δ is the line $\{x = y = 0\}$. Then, since Δ has to be $\rho(\Gamma)$-invariant, every element $\rho(\gamma)$ of the holonomy group is of the form:

$$\rho(\gamma) = \begin{pmatrix} \multicolumn{2}{c}{\bar{\rho}(\gamma)} & \begin{matrix} 0 \\ 0 \end{matrix} \\ * & * & \lambda(\gamma) \end{pmatrix}$$

where $\lambda(\gamma)$ is a non-zero real number, and $\bar{\rho}(\gamma)$ an element of $\mathrm{GL}(2, \mathbf{R})$. Clearly, λ and $\bar{\rho}$ are homomorphisms.

We discuss more on generalized affine suspensions (see also Chapter 4): Let $\varphi : N \to N$ a projective diffeomorphism of a real-projective $(n-1)$-manifold N. Recall that $\mathbf{S}^{n-1}$ has a real projective structure induced from $\mathbf{R}P^{n-1}$ by the standard double cover, and the group $\mathrm{Aut}(\mathbf{S}^{n-1})$ of projective automorphisms of $\mathbf{S}^{n-1}$, which is isomorphic to the quotient group of $\mathrm{GL}(n, \mathbf{R})$ by homotheties. ($\mathbf{S}^{n-1}$ with this structure is said to be a *real projective sphere.*) We can always lift the chart of N to $\mathbf{R}P^{n-1}$ to $\mathbf{S}^{n-1}$ with respect to the standard double cover. Then the transition functions then lie in $\mathrm{Aut}(\mathbf{S}^{n-1})$ since a projective map defined on a small open subset of $\mathbf{S}^{n-1}$ extends to one defined on $\mathbf{S}^{n-1}$ always (see Chapter 2 of [**17**]). We gather that N has a natural $(\mathbf{S}^{n-1}, \mathrm{Aut}(\mathbf{S}^{n-1}))$-structure.

Let $f_i : V_i \to U_i \subset \mathbf{S}^{n-1}$ be a family of projective charts covering N. When V_i meets V_j, we have an element $\bar{g}_{ij}$ of $\mathrm{Aut}(\mathbf{S}^{n-1})$ such that on $V_i \cap V_j$:

$$f_i = \bar{g}_{ij} \circ f_j.$$

Let's choose representatives g_{ij} of the $\bar{g}_{ij}$ in $\mathrm{GL}(n, \mathbf{R})$. We impose the condition $g_{ij}g_{jk}g_{ki} = \mathrm{I}$ if $V_i \cap V_j \cap V_k$ is not empty. Such a choice is always possible: take for example the unique representative of $\bar{g}_{ij}$ with determinant ± 1. The set of the possible choices is parameterized by the cohomology group $H^1(N, \mathbf{R})$ of N. For every i, let W_i be the open cone in $\mathbf{R}^n$ with vertex at 0, the union of the half lines belonging to U_i. Let denote by W the quotient of the disjoint union of the W_i by the relation identifying each element x_j of W_j with the element $g_{ij}(x_j)$ of W_i (when $g_{ij}(x_j)$ belongs effectively to W_i, of course). This quotient is a noncompact radiant affine manifold, equipped with a complete radial flow $\hat{\Phi}^t$. The quotient of W by the relation 'being on the same orbit of $\hat{\Phi}^t$' is homeomorphic to N. The quotient map is a fibration by rays. Let N_0 be any section of this fibration. The manifold W is diffeomorphic to $N \times \mathbf{R}$.

Remember the projective transformation φ of N. It lifts to an affine diffeomorphism $\hat{\varphi}$ of W well-defined up to composition by $\hat{\Phi}^t$. If T is big enough, $\hat{\Phi}^T\hat{\varphi}(N_0)$ is a section of $\hat{\Phi}^t$ disjoint from N_0. Therefore, for every real positive t, $\hat{\Phi}^t\hat{\varphi}$ acts freely and properly discontinuously on W. The quotient of this action is a closed radiant affine manifold homeomorphic to the topological suspension N_φ of $\varphi : N \to N$.

Actually, such a lifting does not always exist for any choice of W, but for many of W above the given Σ, we can perform such liftings. The condition is: let $\bar{\rho} : \pi_1(W) \to \mathrm{GL}(n, \mathbf{R})$ be the holonomy homomorphism of W. Observe that $\pi_1(W)$ is isomorphic to $\pi_1(N)$. Let φ_* be the automorphism of $\pi_1(N)$ induced by

φ. Then, φ lifts if and only if $\det \circ \bar{\rho}$ is constant on the orbits of φ_*. For example, the choice of the g_{ij}'s of determinant ± 1 works.

Observe also that the construction is not uniquely defined: we made some choices. These choices are parameterized by an open subset of the first cohomology module $H^1(N_\varphi, R)$ satisfying the above requirement.

By construction, the radial flow of a generalized affine suspension admits a closed total cross-section homeomorphic to Σ. Note that this section, equipped with the projective structure induced by the transverse projective structure of the radial flow is isomorphic to the initial projective surface Σ. (See Proposition 3.2.)

Returning back to our radiant affine 3-manifold M:

PROPOSITION C.1. *The holonomy group $\rho(\Gamma)$ is solvable.*

PROOF. Denote by Γ' the first commutator subgroup of Γ. Since λ and $\bar{\rho}$ are homomorphisms, for every element of Γ' we have:

- $\bar{\rho}(\gamma)$ belongs to $\mathrm{SL}(2, \mathbf{R})$,
- $\lambda(\gamma) = 1$.

Observe that by definition $\rho(\Gamma)$ is solvable if and only if $\bar{\rho}(\Gamma')$ is solvable.

Let $\mathcal{F}^0$ be the foliation of $\mathbf{R}^3 \setminus \Delta$ whose leaves are the half-planes containing Δ in their boundaries. The leaf space of this foliation, i.e. the quotient of $\mathbf{R}^3 \setminus \Delta$ by the relation "being on the same leaf of $\mathcal{F}^0$", is naturally identified with the double covering of the real projective line $\mathbf{R}P^1$. Let $\mathcal{F}$ be the pull-back of $\mathcal{F}^0$ by $\mathcal{D}$. Since **dev** is an infinite cyclic covering, $\mathcal{F}$ is a foliation whose leaf space is naturally identified with the universal covering $\tilde{P}^1$ of $\mathbf{R}P^1$. The action of Γ' on the leaf space induced by the action of Γ on $\tilde{M}$ is a lifting of the projective action of $\bar{\rho}(\gamma') \in \mathrm{SL}(2, \mathbf{R})$ over $\mathbf{R}P^1$. According to Lemma C.2 below, if $\bar{\rho}(\Gamma')$ is not solvable, there is an element γ of Γ' preserving a leaf F of $\mathcal{F}$, and such that in an adequate coordinate system:

$$\rho(\gamma) = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for some real positive λ different from 1.

We fix this coordinate system with coordinate functions denoted by x, y, and z still. Since the required coordinate change sends the standard coordinate vectors to the eigenvectors of $\rho(\gamma)$, and the z-axis is in the eigendirection, Δ is still given by $x = 0$ and $y = 0$. Leaves of $\mathcal{F}^0$ are again given as zero sets of linear functions of the new coordinate functions x and y only. We may assume without loss of generality that F maps in the plane given by $x = 0$ under **dev**.

Let P be the inverse image by **dev** of the punctured plane $\{z = 0\} \setminus \{(0,0,0)\}$. Since **dev** is an infinite cyclic covering missing Δ, it follows that P is connected, and the restriction of **dev** to P is an infinite cyclic covering to the punctured plane. Moreover, γ preserves P. Since γ preserves the leaf F also, γ preserves each connected component of $\mathbf{dev}^{-1}(\{z = x = 0\})$ and, by same reason, $\mathbf{dev}^{-1}(\{z = y = 0\})$. Let C be a connected component of $\mathbf{dev}^{-1}(\{z = 0, x \geq 0, y \geq 0\})$. It is preserved by γ, and the restriction of **dev** to C is a homeomorphism over $\{z = 0, x \geq 0, y \geq 0\} \setminus \{(0,0,0)\}$. The action of $\rho(\gamma)$ on $\{z = 0, x \geq 0, y \geq 0\} \setminus \{(0,0,0)$ is given by $(x, y, 0) \mapsto (\lambda x, \lambda^{-1} y, 0)$. It is not properly discontinuous, since any path joining $\{z = 0, x = 0, y \geq 0\}$ to $\{z = 0, x \geq 0, y = 0\}$ intersects all its iterates by $\rho(\gamma)$. This is a contradiction since the action of γ on C has to be properly discontinuous. It follows that $\bar{\rho}(\Gamma)$, and therefore $\rho(\Gamma)$, is solvable. $\square$

For the proof of the following lemma C.2, we must first recall some facts about the actions of $\mathrm{PSL}(2,\mathbf{R})$ and its universal covering $\widetilde{\mathrm{SL}}(2,\mathbf{R})$ on $\mathbf{R}P^1$ and its universal covering, respectively. Let $q : \widetilde{\mathrm{SL}}(2,\mathbf{R}) \to \mathrm{PSL}(2,\mathbf{R})$ denote the covering map. Every element g of $\mathrm{PSL}(2,\mathbf{R})$ is either:

- *elliptic:* g has no fixed point in $\mathbf{R}P^1$. It is conjugate to a rotation,
- *parabolic:* g has one and only one fixed point. This fixed point is of saddle-node type, i.e. attractive on one side, and repulsive on the other side,
- *hyperbolic:* g has two fixed points: a repulsive one and an attractive one. It is conjugate to the element represented by:

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

An element of $\widetilde{\mathrm{SL}}(2,\mathbf{R})$ is said to be *elliptic, parabolic or hyperbolic* according to the nature of its projection $q(g)$. If this projection is trivial, g belongs to the center H of $\widetilde{\mathrm{SL}}(2,\mathbf{R})$. The group H is infinite cyclic. Let h be a generator of H. If g is not trivial and admits fixed points on the universal covering $\tilde{P}^1$ of $\mathbf{R}P^1$, it is parabolic or hyperbolic. In the first case, its fixed points are of saddle-node type; in the second case, they are attractive or repulsive.

LEMMA C.2. *Let Γ be a subgroup of $\widetilde{\mathrm{SL}}(2,\mathbf{R})$. We assume that $q(\Gamma)$ is not solvable. Then, it contains a hyperbolic element that fixes a point of $\tilde{P}^1$.*

PROOF. Note that Γ is not solvable since it is a cyclic extension of $q(\Gamma)$ which is not solvable. According to Hölder's theorem (see e.g. [**25**], IV.3.1), a group acting freely on the real line is Abelian. Therefore, the action of Γ on $\tilde{P}^1$ is not free: some element γ_0 of Γ admits a fixed point x_0 in $\tilde{P}^1$. If γ_0 is hyperbolic, we are done. If not, γ_0 is parabolic. Then, the fixed points of γ_0 are the H-iterates of x_0. We denote by x_i the image of x_0 by h^i. Observe that since $\tilde{P}^1$ is homeomorphic to the real line $\mathbf{R}$, orienting $\tilde{P}^1$ is equivalent to equip it with an archimedian total order. We orient $\tilde{P}^1$ in such a way that x_1 is greater than x_0. Taking the inverse of γ_0 if necessary, we can assume that all the γ_0-orbits in the open interval $]x_0, x_1[$ go from x_0 to x_1.

The stabilizer in $\mathrm{PSL}(2,\mathbf{R})$ of a point in $\mathbf{R}P^1$ is isomorphic to the group of affine transformations of the line. It is therefore solvable. It follows that there is an element γ of Γ such that $\gamma(x_0)$ is not one of the x_i's. Let γ_1 be the conjugate $\gamma\gamma_0\gamma^{-1}$. It is parabolic and fixes $\gamma(x_0)$. Therefore, it admits a fixed point x'_0 in $]x_0, x_1[$. Then, $\gamma_1^{-1}\gamma_0(x_0) = \gamma_1^{-1}(x_0)$ is less than x_0, and $\gamma_1^{-1}\gamma_0(x'_0)$ is greater than x'_0, since x'_0 is a fixed point of γ_1^{-1} and $\gamma_0(x'_0)$ is greater than x'_0. Therefore, the closed interval $[x_0, x'_0]$ is contained in its image by $\gamma_1^{-1}\gamma_0$. It follows that $\gamma_1^{-1}\gamma_0$ is a hyperbolic element admitting a repulsive fixed point in $]x_0, x'_0[$. $\square$

We know from Proposition C.1 that the holonomy group is solvable. It follows from Theorem A of [**4**] that M is affinely isomorphic to a generalized affine suspension. In particular, the radial flow admits a total cross-section.

Let Σ be such a total cross-section, and $\tilde{\Sigma}$ a lifting of Σ in $\tilde{M}$, i.e. a connected component of $p^{-1}(\Sigma)$. Let $\tilde{\Phi}^t$ be the lifting of Φ^t in $\tilde{M}$. Since Σ is a total cross-section, it is a fiber of some fibration of M over the circle. Hence, $\tilde{M} \setminus \tilde{\Sigma}$ is not connected. Every orbit of $\tilde{\Phi}^t$ meets $\tilde{\Sigma}$. This orbit remains in the past in one connected component of $\tilde{M} \setminus \tilde{\Sigma}$, and in the future, it remains in the other connected component. In other words, every orbit of $\tilde{\Phi}^t$ meets $\tilde{\Sigma}$ at one and only one point. The developing map sends injectively every orbit of $\tilde{\Phi}^t$ over a half-line in $\mathbf{R}^3$ (Lemma C.1 applied to the $\mathbf{R}$-actions). Therefore, it induces a local homeomorphism $\mathbf{dev}'$ from $\tilde{\Sigma}$ on the sphere $\mathbf{S}^2$ of half-lines. We denote by $\mathbf{S}_*$ the sphere $\mathbf{S}^2$ punctured at $(0,0,1)$ and $(0,0,-1)$. Since $\mathbf{dev}$ is an infinite cyclic covering over $\mathbf{R}^3 \setminus \Delta$, the map $\mathbf{dev}'$ is an infinite cyclic covering over S_*. Therefore, $\tilde{\Sigma}$ is the universal covering of Σ, and $\mathbf{dev}'$ is the developing map of a real projective structure on Σ. The holonomy homomorphism $\hat{\rho}$ of this structure is the composition of the restriction of ρ to $\hat{\Gamma}$ with the projection of $\mathrm{GL}(3, \mathbf{R})$ in $\mathrm{Aut}(\mathbf{S}^2)$, where $\hat{\Gamma}$ is the group of elements of Γ which preserve $\tilde{\Sigma}$.

In order to find a contradiction, i.e., in order to achieve the proof of Theorem C.1, it suffices to show:

PROPOSITION C.2. *Given an real projective structure on the closed surface* Σ, *its developing map* $\mathbf{dev}'$ *can not be an infinite cyclic covering over* $\mathbf{S}_*$.

PROOF. Suppose that Σ is a closed surface with such a structure. We first complete $\tilde{\Sigma}$ by the path-metric induced from the Riemannian metric μ on $\mathbf{S}^2$ by $\mathbf{dev}'$ obtaining the Kuiper completion $\check{\Sigma}$ of Σ. Recall that $\tilde{\Sigma}_\infty$ denotes the set of ideal points $\check{\Sigma} - \tilde{\Sigma}$. The developing map $\mathbf{dev}'$ also extends to an obvious distance decreasing map : $\check{\Sigma} \to \mathbf{S}^2$. In our situation, it is easy to see that there exist only two ideal points in $\check{\Sigma}$, mapping to $(0,0,1)$ and $(0,0,-1)$, and that $\check{\Sigma}$ is obtained from $\tilde{\Sigma}$ by adding these two points.

Our surface Σ is obviously not convex since $\tilde{\Sigma}$ is not convex. A 2-*crescent* in $\check{M}$ is a convex hemisphere or lune D in $\check{M}$ with interior in $\tilde{M}$ and the interior of a convex segment in the boundary δD of D includes the nonempty $\tilde{M} \cap \delta D$. Theorems 4.6 and 4.5 of [**17**] show that $\check{\Sigma}$ includes a 2-crescent (see also Section 5 of [**13**]). By definition of 2-crescents, there exists a nontrivial open arc in the boundary of the 2-crescents that is in $\tilde{\Sigma}_\infty$, and hence the set of ideal points in a 2-crescent is uncountable. However, $\check{\Sigma}$ contains only two ideal points. This is a contradiction. ☐

2. Radiant affine 3-manifolds with boundary have total cross-sections

Now we begin the proof of Theorem C.3. Let M be a compact radiant affine 3-manifold with totally geodesic boundary. Since we may prove the result for a finite cover of M, we assume without loss of generality that the boundary components of M are tori.

LEMMA C.3. *A radiant affine* 3-*manifold* M *admits a total cross-section if and only if it has a closed* 1-*form taking a positive value for each radiant vector.*

PROOF. The existence of a total cross-section and the flow is transversal to it shows that M is diffeomorphic to a bundle over a circle so that the radiant vector field corresponds to the vector field transversal to each fiber. The differential of the fiber map gives us the closed form.

Given a closed form with above property, we can approximate it by a non-vanishing closed form with rational period. Such a closed form obviously gives a fibration $M \to \mathbf{S}^1$ (see [**46**]). ☐

LEMMA C.4. *If M is a radiant affine 3-manifold, and a finite cover N of M admits a total cross-section to the radial flow, then M admits a total cross-section.*

PROOF. A total cross-section in N corresponds to a closed 1-form on N which is positive for radial vectors. Clearly such 1-form descends to one on M by averaging over the finite group action as the action should preserve the flow direction. $\square$

Let $\tilde{M}$ be the universal cover of M with a development pair $(\mathbf{dev}, h)$. The radiant flow lines induce a *radiant foliation* on $\tilde{M}$. Let Q be the space of leaves of the radiant foliation in $\tilde{M}$, which has a natural real projective structure. The group of deck transformations acts on Q as a group of projective automorphisms (see Barbot [**3**] for details). As $\tilde{M}$ is simply connected, Q is simply-connected also.

There is a quotient map $f : \tilde{M} \to Q$ which is a fibration whose fibers are rays. We see that the developing map $\mathbf{dev}$ induces an immersion $\mathbf{dev}' : Q \to \mathbf{S}^2$ where $\mathbf{S}^2$ is the space of rays in $\mathbf{R}^3$, and the deck transformation group acts on Q so that $h'(\vartheta) \circ \mathbf{dev}' = \mathbf{dev}' \circ \vartheta$ for a deck transformation ϑ and $h'(\vartheta)$ the induced projective map from $h(\vartheta)$.

Choose a boundary component K of M and a component $\tilde{K}$ of $p^{-1}(K)$. As K is tangent to the radial flow, K has Euler characteristic zero. By taking a finite cover of M, K is assumed to be a torus. A deck transformation group G which is a subgroup of $\pi_1(M)$ and isomorphic to $\mathbf{Z}$ or $\mathbf{Z}+\mathbf{Z}$, acts on $\tilde{K}$. Let c be the image of $\tilde{K}$ in Q, which is a simple geodesic in the boundary ∂Q of Q.

An affine automorphism $\varphi : \tilde{M} \to \tilde{M}$ is a *homothety* if each ray is preserved and $\mathbf{dev} \circ \varphi = s\mathrm{I} \circ \mathbf{dev}$ for a positive scalar s. Note that an affine automorphism φ of $\tilde{M}$ always induces a real projective automorphism of Q and φ acts trivially on Q if and only if φ is a homothety.

Let ϑ be an element of G. Then ϑ is not a homothety. If not, then the radial flow is periodic, and M is easily shown to be a Benzécri suspension by Proposition 3.3 of [**3**]. (Note that for these arguments, there is no difference when M is closed or has nonempty boundary.) We are done in this case.

As $\tilde{K}$ is totally geodesic, $c = f(\tilde{K})$ is a geodesic boundary component of Q. Suppose that K is compressible in M. Then $\tilde{K}$ is compressible in $\tilde{M}$. Let D be an imbedded compressing disk with boundary in $\tilde{M}$. Then the radial projection $f|D : D \to Q$ maps a disk D onto Q with boundary ∂D onto a boundary component $c = f(\tilde{K})$ of Q. This means that Q is a compact surface, and hence Q is a compact disk. As Q is bounded by a geodesic c, Q must be projectively diffeomorphic to a 2-hemisphere, and $\mathbf{dev}'$ is an embedding onto a 2-hemisphere in $\mathbf{S}^2$. (This can be seen by a doubling argument and the uniqueness of projective structure on $\mathbf{S}^2$.) This shows that $\tilde{M}$ is affinely diffeomorphic by $\mathbf{dev}$ to an affine half-space with boundary containing O with O removed.

LEMMA C.5. *Suppose that Q is real projectively diffeomorphic to a hemisphere in $\mathbf{S}^2$, or $\tilde{M}$ is affinely diffeomorphic to an affine half-space H_1 with boundary containing O with O removed. Then M is a half-Hopf manifold.*

PROOF. We will use the second hypothesis. The punctured half-plane $H_1 - \{O\}$ includes a compact disk D with boundary in $\partial H_1 - \{O\}$ which is transversal to every ray. There exists a deck transformation ϑ of $H_1 - \{O\}$ sending D to $\vartheta(D)$ disjoint from D as the deck transformation groups are properly discontinuous. Clearly $H_1 - \{O\}$ quotient out by ϑ has a total cross-section corresponding to D, and is a half-Hopf manifold. Hence, M is finitely covered by a generalized affine suspension, and so M is a generalized affine suspension. The total cross-section has positive Euler characteristic, and hence is a compact disk transversal to radial flow. The lemma now follows. ☐

Assume that K is incompressible from now on. First, suppose that K is affinely homeomorphic to a quotient of $\mathbf{R}^2 - \{O\}$ by an affine action. Now, we need to use the *holonomy cover* M_h of M, i.e., the cover of M corresponding to the kernel of the holonomy homomorphism h. As we described in [**17**], the developing map $\mathbf{dev}$ induces an immersion $\mathbf{dev}_h : M_h \to \mathbf{R}^3$ and the holonomy homomorphism induces a homomorphism $h_h : \pi_1(M)/\pi_1(M_h) \to \mathrm{GL}(3, \mathbf{R})$. Also f induces a fibration $f_h : M_h \to Q_h$ to a projective surface Q_h covered by Q. The immersion $\mathbf{dev}_h$ induces an immersion $\mathbf{dev}'_h : Q_h \to \mathbf{S}^2$ and h_h induces a homomorphism $h'_h : \pi_1(M)/\pi_1(M_h) \to \mathrm{Aut}(\mathbf{S}^2)$. The surface $\tilde{K}$ corresponds to a surface K_h covering K. We see easily that K_h is affinely diffeomorphic to $\mathbf{R}^2 - \{O\}$. A deck transformation group G_h acts on K_h so that K is affinely homeomorphic to K_h/G_h. Hence G_h is an infinite cycle group.

A *homothety* in M_h is an affine automorphism φ of M_h acting on each ray and satisfying $\mathbf{dev} \circ \varphi = s\mathrm{I} \circ \mathbf{dev}$ for a positive scalar s. As above, if an element of G_h is a homothety, then M is a generalized affine suspension.

Assume now that no element of G_h is a homothety. We claim that in this case M is a half-Hopf manifold. Let ϑ be a generator of G_h. Then $h_h(\vartheta)$ acts on the totally geodesic plane P including $\mathbf{dev}_h(K_h)$. We assume that P is the xy-plane for simplicity. As no element of G_h is a homothety, G_h acts effectively on Q_h. We divide our cases according to the conjugacy classes of $h'_h(\vartheta)$ in $\mathrm{Aut}(\mathbf{S}^2)$, depending on the conjugacy classes of the corresponding matrix $M(\vartheta)$. Let P' be the great circle in $\mathbf{S}^2$ corresponding to P, and H_1 and H_2 the two hemispheres bounded by them. We easily see that one of the following occurs:

(i) The only $h'_h(\vartheta)$-invariant subset of H_i including a neighborhood of P' in H_i is H_i itself.
(ii) The only subset of H_i with this property equals $H_i - \{x\}$ for a point $x \in H_i^o$.
(iii) $h'_h(\vartheta)$ under a suitable coordinates of the affine space H_i^o is of form

$$\begin{pmatrix} \cos 2\pi\theta & \sin 2\pi\theta \\ -\sin 2\pi\theta & \cos 2\pi\theta \end{pmatrix}, \theta \neq 0.$$

Note that $\mathbf{dev}'_h|c$ is an imbedding onto P', and hence a neighborhood U of c in Q_h imbeds onto a neighborhood of P' in say H_1. We see that $\mathbf{dev}'_h \circ \vartheta|U = h'_h(\vartheta) \circ \mathbf{dev}'_h|U$, and so $\mathbf{dev}'_h|\vartheta(U)$ is also an imbedding onto a neighborhood of P'. Therefore, by induction, we see that $\mathbf{dev}'_h| \bigcup_{i \in \mathbf{Z}} \vartheta^i(U)$ is an imbedding onto an $h'_h(\vartheta)$-invariant subset of H_1 including a neighborhood of P'.

In case (i), Q_h maps homeomorphic onto H_1, and by Lemma C.5, M is a half-Hopf manifold.

In case (ii), Q_h maps homeomorphic to $H_1 - \{x\}$. Therefore M_h under $\mathbf{dev}_h$ maps homeomorphic to $H'_1 - l$ for the upper half-space H'_1 corresponding to H_1 and a line l through O transversal to the xy-plane. Hence, we may identify M_h with its image, and each deck transformation acts on l and the upper-half space. We see that the group of deck transformations acting on K_h equals the entire deck transformation group of of M_h and hence the group of deck transformations acting on $\tilde{K}$ equals $\pi_1(M)$. This means that K is homotopy equivalent to M by the inclusion map where $K = \partial M$. We see easily by a topological doubling argument and $\mathbf{Z}_2$-homology computation that such a situation cannot happen.

In case (iii), θ must be irrational since otherwise $h'_h(\vartheta)$ is periodic and hence ϑ must be periodic near c_h and hence on Q_h; but this means that a power of ϑ is a homothety on M_h.

Under the suitable coordinates, for some large r, there is an open neighborhood U_r of c which under $\mathbf{dev}'_h$ maps homeomorphic to the set of form the union of P' and the complement of a ball of radius r.

Let r_0 be an infimum of possible values of r. Then we see easily that U_{r_0} exists. We claim that (1) Q_h equals this set U_{r_0} or (2) Q_h maps homeomorphic to H_1 under $\mathbf{dev}'_h$. Suppose that there exists a boundary point x of U_{r_0} in Q_h. also. As ϑ acts as an irrational rotation, we see that every point of the boundary of $\mathbf{dev}'(U_{r_0})$ is an image of a boundary point of U_{r_0} in Q_h. Assume that $r_0 > 0$. In this case, the boundary γ of U_{r_0} in Q_h maps homeomorphic to the closure γ' of the orbit of ϑ in H_1, which is homeomorphic to a circle. If there exists a point of ∂Q_h in the boundary γ, then every point of γ is in ∂Q_h as ∂Q_h is closed and ϑ acts on ∂Q_h. But this implies that a boundary component of M_h is not totally geodesic. Therefore, there exists a regular neighborhood of γ mapping to a regular neighborhood of γ'. This contradicts the minimality of r_0 if $r_0 > 0$. Hence, we obtain that $U_{r_0} = Q_h$. When $r_0 = 0$, the claim follows easily.

In case (1), we obtain that M_h maps homeomorphic under $\mathbf{dev}_h$ to the complement L of a closed convex cone in H'_1 not meeting the boundary $P - \{O\}$. We identify M_h with this set L. Then a contradiction as in (ii) occurs.

In case (2), we see that M is a half-Hopf manifold by Lemma C.5.

Now, we will be working on $\tilde{M}$ from now on (i.e., not on M_h). We assume that $\tilde{K}$ is affinely homeomorphic to a convex cone in $\mathbf{R}^2$ from now on.

Since $h(G)$ acts on a convex cone $\mathbf{dev}(\tilde{K})$, if $h(\varphi)$ for $\varphi \in G$ is a homothety, then φ is a homothety near $\tilde{K}$ in $\tilde{M}$, and hence φ is a homothety on $\tilde{M}$. Thus $q \circ h$ is injective. We identify G with $h(G)$ from now on.

Let $q : \mathrm{GL}(3, \mathbf{R}) \to \mathrm{SL}(3, \mathbf{R})$ be the homomorphism whose kernel consists of $s\mathrm{I}$ for $s \neq 0$. Since no element of G is a homothety in $\tilde{M}$, Barbot [**3**] shows that the identity component H of the Zariski closure of $q(G)$ is conjugate to the following groups:

Case D: H is the group of all diagonal matrices with positive eigenvalues.
Case P: H is the group of matrices of form:

$$\begin{pmatrix} e^u & t & 0 \\ 0 & e^u & 0 \\ 0 & 0 & e^v \end{pmatrix}, u, v \in \mathbf{R}, 2u + v = 0.$$

Case U: H is the group of matrices of form:

$$\begin{pmatrix} 1 & s & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, s, t \in \mathbf{R}.$$

Case C: H is the group of matrices of form:

$$\begin{pmatrix} 1 & s & t \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}, s, t \in \mathbf{R}.$$

Case S: H is the group of matrices of form:

$$\begin{pmatrix} e^u \cos\theta & e^u \sin\theta & 0 \\ -e^u \sin\theta & e^u \cos\theta & 0 \\ 0 & 0 & e^v \end{pmatrix}, u, t \in \mathbf{R}, 2u + v = 0.$$

Case T: H is the group of matrices of from:

$$\begin{pmatrix} 1 & 0 & s \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}, s, t \in \mathbf{R}.$$

Since $q(G) \cap H$ must be a finite index subgroup of $q(G)$, we assume that $q(G)$ is a lattice of H by taking a finite cover of M and choosing $(\mathbf{dev}, h)$ carefully, i.e., conjugating h by an element of $\mathrm{GL}(3, \mathbf{R})$.

Let ϑ be an element of G. Then since ϑ acts on $\mathbf{dev}(\tilde{K})$ a convex 2-dimensional subset, case S does not occur here. In case T, G acts on xy-plane as a homothety, and there are no other G-invariant subspaces of codimension one as $q(G)$ is a Zariski dense subgroup of H. But since $\mathbf{dev}(K)$ is a G-invariant subspace so that $\mathbf{dev}(K)/G$ is homeomorphic to the torus K/G, this is a contradiction. Hence case T does not occur.

LEMMA C.6. *Let D be a convex cone in $\mathbf{R}^2$, and G' an abelian group isomorphic to $\mathbf{Z} + \mathbf{Z}$. If D/G is homeomorphic to a torus, then an element of G' is not a homothety.*

PROOF. If each element of G' is a homothety, then each element of G' acts on each ray in D ending at O. Since $\mathbf{Z} + \mathbf{Z}$ cannot act properly discontinously and freely on a real line, this is absurd. □

We see that G is a lattice in a connected two-dimensional subgroup of group H of matrices in $\mathrm{GL}(3, \mathbf{R})$ as follows:

Case D: H is a group of all diagonal matrices of positive eigenvalues.

Case P: H is a group of matrices of form

$$\begin{pmatrix} e^u & t & 0 \\ 0 & e^u & 0 \\ 0 & 0 & e^v \end{pmatrix}, u, v, t \in \mathbf{R}.$$

Case U: H is a group of matrices of form:

$$\begin{pmatrix} e^u & s & t \\ 0 & e^u & s \\ 0 & 0 & e^u \end{pmatrix}, u, s, t \in \mathbf{R}.$$

Case C: H is a group of matrices of form:

$$\begin{pmatrix} e^u & s & t \\ 0 & e^u & 0 \\ 0 & 0 & e^u \end{pmatrix}, u, s, t \in \mathbf{R}.$$

We say that an affine 3-manifold M *decomposes* into affine 3-manifolds $N_1, \dots, N_n$ if each N_i is the closure of a component of M with two-sided separating totally geodesic surfaces in M removed.

PROPOSITION C.3. *Our manifold M or a finite cover of M decomposes into compact radiant affine 3-manifolds N_i each of which is affinely isomorphic to a quotient of a domain in $\mathbf{R}^3 - \{O\}$ by an action of G. Every piece N_i is homeomorphic to torus times an interval and has totally geodesic boundary which are tori isotopic to K or a finite cover of K.*

PROOF. The proof as follows is similar to what are in [**3**]:

The group H acts on the projective sphere $\mathbf{S}^2$ effective as a subgroup of the group $\mathrm{Aut}(\mathbf{S}^2)$ of projective automorphisms of $\mathbf{S}^2$. The space $\mathbf{S}^2$ decomposes into two-dimensional open orbits and one-dimensional orbits and zero-dimensional orbits under H.

Case D: The zero-dimensional orbits are six points in $\mathbf{S}^2$ comprising three pair of antipodal points, one-dimensional ones are twelve lines in $\mathbf{S}^2$ with endpoints in the three points, and two-dimensional orbits are eight open triangles bounded by the closures of the lines.

Case P: The zero-dimensional orbits comprise two pair of antipodal points $\{p, -p\}$ and $\{q, -q\}$, one-dimensional orbits are four lines with endpoints $\pm p$ and $\pm q$ and two lines that is a great circle containing $p, -p$ with p and $-p$ removed, and two-dimensional orbits are four open lunes bounded by the closure of the union of one-dimensional orbits.

Case U: Zero-dimensional orbits are two points p and $-p$. One-dimensional orbits are two components of a great circle containing p and $-p$ with $\{p, -p\}$ removed. Two-dimensional orbits are two open hemispheres bounded by the great circle.

Case C: There are two cases:

(i) Zero-dimensional orbits are two points p and $-p$. One-dimensional orbits are great circles containing p and $-p$ with p and $-p$ removed. Their union equals $\mathbf{S}^2 - \{p, -p\}$ and there are no two-dimensional orbits.

(ii) Zero-dimensional orbits are points of a great circle through two points $\{p, -p\}$. One dimensional orbits are the other great circles through p and $-p$ with p and $-p$ removed. There are no two-dimensional orbits.

We see that all one-dimensional orbits are simple geodesics in $\mathbf{S}^2$ and are not closed ones.

Recall that the developing map $\mathbf{dev} : \tilde{M} \to \mathbf{R}^3$ induces an immersion $\mathbf{dev}' : Q \to \mathbf{S}^2$ and the deck transformation group acts on Q as well so that $\mathbf{dev}' \circ \vartheta$ equals $h'(\vartheta) \circ \mathbf{dev}'$ where $h'(\vartheta) = q \circ h(\vartheta)$ for each deck transformation ϑ of $\tilde{M}$. (The action is not necessarily proper.)

LEMMA C.7 (Lemma 2.4.3 of Dupont [**19**]). *Let V be a (U, X)-manifold where U is a Lie group acting on a space X. Let L be a connected subgroup of U, and ω an*

L-orbit in X and $\hat{\omega}$ a connected component of $D^{-1}(\omega)$ where (D, j) is a development pair of V. Suppose that the action of L on ω is covered by the action of L on $\hat{\omega}$, and there exists a subgroup Γ_0 of deck transformation group so that $\hat{\omega}/\Gamma_0$ is compact and $j(\Gamma_0)$ is in L. Then the action of L on a neighborhood of ω is covered by an action of L on a neighborhood of $\hat{\omega}$ such that, for any element γ of Γ_0, the action of $j(\gamma)$ coincides with the action of γ as a deck transformation.

Since G acts on $f(\tilde{K}) \subset Q$ as in the premise of the above lemma, the extension argument (i.e., the proof of Theorem B in [**3**] using essentially Lemma C.7) shows that H acts on Q, Q is a union of orbits of dimension zero, one, or two, and each orbit maps homeomorphic to an orbit in $\mathbf{S}^2$ under $\mathbf{dev}'$. The proof of this claim, similar to what is in [**3**], goes as follows: Let $\mathcal{O}$ be the maximal connected subset of Q including c where the H-action is defined and whose restriction to G coincide with the deck transformation action. Our claim follows from the following lemma:

LEMMA C.8. *The H-action is defined everywhere, i.e. $\mathcal{O}$ equals Q.*

PROOF. We have to give a more precise definition of $\mathcal{O}$: this is the maximal connected subset including c of the set of points x in Q such that:

- There is a continuous action of H on Q so that for any element k of H on some element $k.x$, such that, for every k, $\mathbf{dev}'(k.x) = k\mathbf{dev}'(x)$.
- for every element g of G, we have the equality $g.x = gx$ (remember that G is a group of deck transformations, and therefore acts on Q).

The fundamental group of an orbit of the radial flow is trivial or cyclic. It follows that there are no zero-dimensional H-orbits in $\mathcal{O}$. (An Abelian group of rank 2 such as G cannot act on a connected one-dimensional space properly discontinuously and freely.)

By Lemma C.7, $\mathcal{O}$ is open. Since Q is connected, the lemma will be proven if we show that $\mathcal{O}$ is closed.

As H acts on $\mathcal{O}$, $\mathcal{O}$ is a union of one- or two-dimensional H-orbits in Q. The restriction of $\mathbf{dev}'$ to each H-orbit maps homeomorphic to a H-orbit in $\mathbf{S}^2$ by Lemma C.1. In particular, two-dimensional H-orbits in $\mathcal{O}$ are open surfaces, and G acts properly discontinuously and freely on each of them.

We note that each two-dimensional H-orbit in $\mathcal{O}$ has to have at least one adjacent one-dimensional H-orbit. If not, the two-dimensional orbit is disconnected from c, a contradiction. (This fact is needed at the end of this argument.)

We see that $\mathcal{O}$ is a union of one-dimensional orbits in case C, or two-dimensional orbits and adjacent one-dimensional orbits joined in a "chain-like" manner in cases D, P, and U.

Suppose that x is a boundary point of $\mathcal{O}$ in Q. We aim to obtain a contradiction. If $\mathbf{dev}'(x)$ lies in a zero-dimensional orbit of H, then x is a zero-dimensional orbit of H. This is a contradiction by above. The image $\mathbf{dev}'(x)$ does not lie in a two-dimensional orbit as each two-dimensional H-orbit of Q is open in Q and $\mathcal{O}$ is a union of H-orbits of Q. Let $\mathbf{dev}'(x)$ be in a one-dimensional orbit J, which is a geodesic, and let J' be a component of $\mathbf{dev}'(-1)(J)$ containing x. Then J' is a closed subset of Q. The restriction of $\mathbf{dev}'$ to J' is injective.

Suppose that the intervals J and $\mathbf{dev}'(J')$ have a common endpoint. Then, for any element g of G, the intervals $\mathbf{dev}'(J')$ and $g\mathbf{dev}'(J')$ are not disjoint. Actually, inverting g if necessary, we can assume that $\mathbf{dev}'(J')$ contains $g\mathbf{dev}'(J')$. Since the restriction of $\mathbf{dev}'$ to every H-orbit in $\mathcal{O}$ is injective, a union A of some

H-orbits near J' is an open set where H-acts, and the restriction of $\mathbf{dev}'$ to A is injective. (We can use a two-dimensional orbit for A unless we are in case C.) Since $\mathbf{dev}'$ restricted to $A \cup J'$ and $A \cup g(J')$ are both injective, $\mathbf{dev}'(J')$ contains $g\mathbf{dev}'(J')$. It follows that J' is G-invariant. We deduce that J' is contained in $\mathcal{O}$; a contradiction.

Therefore, the endpoints of $\mathbf{dev}(J')$ both belong to J.

Suppose that no element of G fixes a point of J. Then, all the G-orbits in J are dense. There is an element g of G such that $g(\mathbf{dev}'(J'))$ meets $\mathbf{dev}'(J')$. Since as above $\mathbf{dev}'$ restricted on $A \cup J'$ and $A \cup g(J')$ are both injective, J' is g-invariant, meaning that H acts on J' again.

This contradiction shows that some element g of G acts as an identity map on J. Therefore, the totally geodesic surface D in $\tilde{M}$ corresponding to J' is filled with rays on which g acts, and hence, maps to the union of closed orbits of radial flow in M.

We claim that, for any deck transformation ϑ, J' cannot meet $\vartheta(J')$ in a transversal manner. Suppose not. Then D and $\vartheta(D)$ meet at a ray l. This ray is fixed by g, hence g is a power of a deck transformation g' so that $l/\langle g' \rangle$ maps to the closed orbit in M. Similarly, $\vartheta \circ g \circ \vartheta^{-1}$ is a power of g'. Thus, a finite power of g acts trivially in J' and $g(J')$ since a power of $\vartheta \circ g \circ \vartheta^{-1}$ must equal a power of g. This means that g is a homothety, a contradiction.

Also, the collection of sets of form $\vartheta(J')$ are locally finite. If not, then there exists a sequence of points $p_i \in \varphi_i(D)$ converging to $p \in \tilde{M}$ for a sequence of deck transformations φ_i. As $\varphi_i \circ g \circ \varphi_i^{-1}$ acts on $\varphi_i(D)$ as homotheties by a fixed factor $s, s > 0$, we see that $\varphi_i \circ g \circ \varphi_i^{-1}$ moves p_i to a points in a compact subset of $\tilde{M}$. This implies that infinitely many of these deck-transformations must be the same. We may choose $\varphi_i(D)$ and $\varphi_j(D)$ sufficiently close so that $\varphi_i \circ g \circ \varphi_i^{-1}$ equals $\varphi_j \circ g \circ \varphi_j^{-1}$. However this means that $\varphi_i \circ g \circ \varphi_i^{-1}$ is a homothety since it acts trivially on $\varphi_i(J')$ and another geodesic $\varphi_j(J')$, a contradiction.

By above two conclusions, it follows that D covers a closed surface in M tangent to the radial flow, and a subgroup G' of rank 2 acts on D and on J'. All we did above applies once more: there is a connected Abelian group H' including G' as a lattice, and H' is again of type D, P, U, or C. We define similarly to $\mathcal{O}$ the locus $\mathcal{O}'$ of definition of the H'-action on Q. It contains J'. Since $\mathbf{dev}'(J')$ has both endpoints inside $\mathbf{dev}(J)$, H' cannot be in case U or C, since in these cases, $\mathbf{dev}'(J')$ should be a complete affine line. Hence, $\mathbf{dev}'(J')$ is contained in the boundary of a two-dimensional H'-orbit B. Moreover, we can choose so that it meets $\mathcal{O}$.

Suppose that H is in case D, P, or U. Then, J' is contained in the boundary of a two-dimensional H-orbit A. By looking at the image under $\mathbf{dev}'$ of A and the two-dimensional H'-orbit B meeting $\mathbf{dev}'(A)$, we obtain that there exists a geodesic c in A mapping into a one-dimensional H'-orbit adjacent to $\mathbf{dev}'(B)$ so that $\mathbf{dev}'(c)$ and $\mathbf{dev}'(J')$ share an endpoint. Since an endpoint of $\mathbf{dev}'(c)$ is a H'-orbit, c must be included in a H'-orbit by the same reason as the above paragraph with G replaced by G'. By the following lemma C.10, this is a contradiction.

If H is in case C, then as in the above paragraph a one-dimensional H'-orbit meets a one-dimensional H-orbit transversally. This is a contradiction by Lemma C.9.

This final contradiction achieves the proof of Lemma C.8. □

LEMMA C.9. *Let H' be a connected two-dimensional Abelian group including an Abelian group G' of rank 2 of deck transformations of $\tilde{M}$ so that $q \circ h|G'$ is injective. Then a one-dimensional orbit K' of H' does not meet a one-dimensional orbit K of H transversally.*

PROOF. Suppose that K and K' meet at a point x in Q. Then let S and S' be the corresponding totally geodesic surfaces in $\tilde{M}$. They meet at a ray l corresponding to x. As S/G and S'/G' correspond to immersed tori, l corresponds to a closed orbit of the radial flow in M. There is an element of g in $G \cap G'$ corresponding to this orbit. Then g acts as trivially on K and K' as we can see from the matrices of form D, P, U, and C; i.e., G acts as translations on one-dimensional orbits in $\mathbf{S}^2$. Hence, this orbit has two-directions in the transversal local cross-section where the return map looks like the identity map. Therefore, the holonomy of g must be a homothety. This contradicts our assumption. □

LEMMA C.10. *Let H' and G' be as in the preceding lemma. Let A be a two-dimensional H-orbit containing in its boundary a one-dimensional H-orbit C contained in $\mathcal{O}$. Then, no one-dimensional H'-orbit contained in $\mathcal{O}'$ meets A.*

PROOF. Since G is a lattice of H, and since H must be in cases D, P or U, for every element x of A, there is a sequence of elements h_n of H for which the $h_n x$ converge to some point of C by Lemma C.11. Assume that some one-dimensional H'-orbit B meets A. Then, some iterates $h_n b$, where b belongs to $B \cap A$, accumulates to a point in C. But B and C correspond to some immersed tori in M; therefore, such an accumulation is impossible. □

LEMMA C.11. *Let H'' be a connected two-dimensional Abelian group of projective transformations of the projective sphere $\mathbf{S}^2$. Let G'' be a lattice of H'', and let A be an open orbit of H'' in the projective plane. Let C be a one-dimensional orbit of H'' contained in the boundary of A. Then, the closure of the G''-orbit of any element of A contains a point of C.*

PROOF. Let F be a compact connected fundamental domain for the action of G'' on H'' by translations. Let a be any point of A, and x a point of C. The orbit $F.x$ of x by F is a compact part of C. Let U be an open neighborhood of $F.x$ in the projective plane. By continuity of the action of A on the real projective plane, and by compactness of F, there is an open neighborhood V near x such that for every element v of V, the orbit $F.v$ is contained in U. Now, since A is Abelian, and since F is a fundamental domain, $F.v$, and thus U, must contain an element of the orbit of a by G. Since this is true for any open neighborhood U of $F.x$, it follows that the orbit $G''.a$ accumulates at least on some point of $F.x$. □

We claim that the decomposition of Q into H-orbits are preserved under the action of the deck transformation group. A one-dimensional H-orbit l in Q does not meet with an image $\vartheta(m)$ of a one dimensional orbit m in Q transversally for a deck transformation ϑ by Lemma C.9. A one-dimensional H-orbit does not meet an image of two-dimensional H-orbit under a deck transformation by Lemma C.10. Thus, H-orbits map to H-orbits under deck transformations as there are no zero dimensional H-orbits in Q.

The H-orbits in $\mathbf{S}^2$ correspond to H-invariant submanifolds in $\mathbf{R}^3 - \{O\}$. Say these sets are *H-invariant sets* in $\mathbf{R}^3 - \{O\}$. We now choose domains in $\mathbf{R}^3 - \{O\}$ consisting of adjacent three- and two-dimensional H-invariant sets. In cases D, P, U, an *H-domain* is the union of an H-invariant open set and H-invariant two-dimensional sets in the boundary of the set. In case C, we choose two antipodal two-dimensional H-invariant sets so that $\mathbf{dev}(\tilde{K})$ is included in one of them. Their complement is the union of two convex H-invariant open sets. Call these sets *distinguished* 2-dimensional H-invariant sets. An *H-domain* in case C is the union of one convex open set and the two two-dimensional H-invariant sets included in its boundary.

We look at a component of the inverse image in Q of one-dimensional orbits under $\mathbf{dev}'$ or distinguished orbits in case C. Then they are one-dimensional H-orbits mapping homeomorphic to the orbits below. A component of the complement of these orbits are either two-dimensional orbits or a union of one-dimensional orbits in case C.

On $\tilde{M}$, these orbits correspond to a decomposition into H-invariant open sets and H-invariant two-dimensional sets mapping homeomorphic to their images in $\mathbf{R}^3$, which are also H-invariant.

Recall that H-invariant two-dimensional sets map (distinguished ones in case C) to imbedded closed surfaces in M under the covering map. By taking a finite cover of M if necessary, we assume that these are two-sided tori in M. Take one, say A, of the open H-invariant sets. Then since the deck transformation group acts on $\tilde{M}$ preserving the decomposition, it follows that A union with adjacent two-dimensional orbits map to a closed submanifold in M bounded by tori. Let us denote them by $N_1, \dots, N_h$. Then clearly, M decomposes into $N_1, \dots N_n$.

The manifolds N_i are obviously affinely homeomorphic to the quotients of 3-dimensional closed domains which are unions of H-invariant sets.

Take the H-invariant open set A adjacent to $\tilde{K}$, a two-dimensional H-invariant set. Then the closure of A is the union of A and adjacent one, two, or three two-dimensional H-invariant sets. The closure covers a compact radiant affine manifold N_1, i.e, it may be identified with a universal cover $\tilde{N}_1$ of N_1. The deck transformation group of M acting on A can be identified with the deck transformation group of $\tilde{N}_1$. Then since the deck transformation group acts on A nontrivially, it acts on $\tilde{K}$ also. It follows that $\pi_1(K) \to \pi_1(N_1)$ is an isomorphism. By three-manifold topology, N_1 is homeomorphic to a torus times an interval.

Removing N_1 from M and taking the closure in M, we get a new radiant affine 3-manifold which decomposes into $N_2, \dots, N_n$. By induction, we see that each N_i is homeomorphic to T^2 times an interval. The proof of Proposition C.3 is complete. ☐

COROLLARY C.1. *Let $p : \tilde{M} \to M$ be the universal covering. Then $p^{-1}(N_i)$ is connected and under p maps as a universal covering map onto N_i. Each $p^{-1}(N_i)$ is a three-dimensional H-invariant domain, and maps homeomorphic to a three-dimensional H-invariant set union with two adjacent two-dimensional H-invariant sets under* $\mathbf{dev}$.

We denote by $\tilde{N}_i$ the set $p^{-1}(N_i)$ for simplicity.

From now on, we assume that M satisfies the conclusion of Proposition C.3. We end this section by showing that M must be a generalized affine suspension in each of the cases D, P, U, and C.

In case C, choose the open H-invariant set $\tilde{A}$ adjacent to $\tilde{K}$. Then $\tilde{A}$ is bounded by $\tilde{K}$ and another orbit L so that the angle θ between $\tilde{K}$ and L is less than or equal to π.

If $\theta < \pi$, then L is another boundary component of $\tilde{M}$, since if not we could have enlarged $\tilde{A}$. Thus, $N_1 = M$ and M is a generalized affine suspension since we can use the 1-form dz/z for a linear function z to get a total cross-section where the plane $z = 0$ is disjoint from the image under **dev**.

Clearly, G acts on $\mathbf{dev}(A) \cup \mathbf{dev}(\tilde{K}) \cup \mathbf{dev}(L)$ properly discontinuously. If $\theta = \pi$, then this set is in the form of a half-space with a line in its boundary removed. By Lemma C.12, this is a contradiction.

In case U, let A be the open H-invariant set adjacent to $\tilde{K}$. The closure of A in $\tilde{M}$ equals $A \cup \tilde{K} \cup L$ for a two-dimensional H-invariant set L. The developing map **dev** sends homeomorphic this set to a radiant half-space with a line in the boundary passing through O removed. Again the following lemma gives us a contradiction.

LEMMA C.12. *Let N be a radiant affine manifold homeomorphic to $T^2 \times I$ with totally geodesic boundary. Let (D, j) be the development pair of N and $j(\pi_1(M))$ is in one of the above four cases D, P, U, or C. Let x, y, and z denote the standard coordinate functions of $\mathbf{R}^3$. Then a developing map of N can not be as follows: it maps $\tilde{M}$ into (but not necessarely onto) a half-space given by $z \geq 0$, and the two boundary components of $\tilde{M}$ respectively homeomorphic onto two components of $z = 0$ with the line given by $y = 0$ or $x = 0$ removed.*

PROOF. Suppose not. Then as above we assume that $G = j(\pi_1(N))$ lies in the identity component H of the Zariski closure of G so that H/G is homeomorphic to a torus.

As above, we see that H acts on $\tilde{N}$ without fixed points. Further H acts properly. We see that N is foliated by H-orbits quotient out by G, all homeomorphic to tori. Let H have a fixed orientation. Then each leaf of N has an induced orientation, and we see that two torus boundary components have ones. The orientation agrees with the boundary orientation induced from N at one component but disagrees at the other component.

Since H acts transitively on the two open lunes in the plane $z = 0$, and $-\mathrm{I}$ acts on the union A of the lunes and commutes with the action of H and is orientation-preserving, it follows that the H-orientation on A agrees or is the opposite of the boundary orientation of the plane $z = 0$ with respect to the half-space $z \geq 0$. Since the boundary-orientation should be the one induced from $\tilde{N}$ also, this is a contradiction. ☐

In case D, let A and L be as above. The open set $\mathbf{dev}(A)$ is a cone with three adjacent two-dimensional H-invariant sets. In its boundary, $\mathbf{dev}(\tilde{K} \cup L)$ includes only two of the adjacent two-dimensional H-invariant sets. Let m be the remaining H-invariant set. Then there exists a coordinate function z on $\mathbf{R}^3$ such that $z = 0$ is a plane through O including m. As above, dz/z induces a closed 1-form on N_1 taking positive values under the radial vector field. Thus N_1 admits a total cross-section by Lemma C.3.

Similarly, we can show that each N_i admits a total cross-section to the radial flow. We can patch the cross-sections together at the tori, and obtain a total cross-section for M. The reason is that each N_i is affinely homeomorphic to each other by maps induced by reflections along 2-dimensional H-invariant sets. Hence, we may assume that the homotopy classes of the total cross-sections are the same. This shows that M is a generalized affine suspension. There is a more detailed description of this case in section 3.2 of [**3**].

We now study the last but most complicated case P: Here, we are given standard coordinate functions x, y, and z so that our group H takes the form P.

As the xz-plane and xy-plane are H-invariant, the interior of $\mathbf{dev}(\tilde{N}_1)$ may be given by $y > 0$ and $z > 0$ (up to sign changes of coordinate functions). The boundary parts $\mathbf{dev}(\tilde{K})$ and $\mathbf{dev}(L)$ are obtained from intersecting the planes given by $y = 0$ and $z = 0$ with $\mathbf{dev}(\tilde{N}_1)$. As G acts on these sets so that the quotients are tori, we may assume without loss of generality that $\mathbf{dev}(\tilde{K})$ is one of the following form: $y = 0, z > 0, x > 0$; $y = 0, z > 0, x < 0$; or $y > 0, z = 0$; and similarly, $\mathbf{dev}(L)$ is of form: $y > 0, z = 0$; $y = 0, z > 0, x > 0$; or $y = 0, z > 0, x < 0$.

As there are exactly two adjacent two-dimensional H-orbits, we have two cases:

(i) $\mathbf{dev}(\tilde{K})$ and $\mathbf{dev}(L)$ are both triangles given by $y = 0, z > 0, x > 0$ and $y = 0, z > 0, x < 0$ respectively.

(ii) If $\mathbf{dev}(\tilde{K})$ is the open lune given by $y > 0, z = 0$, then $\mathbf{dev}(L)$ must be a triangle given by $y = 0, z > 0, x > 0$ or $y = 0, z > 0, x < 0$ and vice versa.

We define a Euclidean reflection R about xy-plane. Then the group F generated by $-\mathrm{I}$ and R is of order four. As elements of H commutes with F, we see that the closure in $\tilde{M}$ of each 3-dimensional H-invariant set in $\tilde{M}$ under $\mathbf{dev}$ maps homeomorphic to $\mathbf{dev}(A) \cup \mathbf{dev}(\tilde{K}) \cup \mathbf{dev}(L)$ as in (i) or (ii) or an image under an element of F.

We claim that types (i) and (ii) cannot occur for the same M. Suppose not. Let N_i be of type (i) and N_j be of type (ii). Then $\mathbf{dev}|N_i$ and $\mathbf{dev}|N_j$ are imbeddings onto the sets A_i and A_j in cases (i) and (ii) respectively. There exists an element of F sending A_i^o to A_j^o. The piece N_i is affinely homeomorphic to the quotient of A_i by G and N_j is affinely homeomorphic to that of A_j. Since each element of G commutes with elements of F, we may assume without loss of generality that A_i^o equals A_j^o and the domain D given by $y > 0, z > 0$ (i.e., we may use two different developing maps). Therefore, we identify N_i to the quotient A_i/G and N_j to A_j/G.

Note that H acts freely on the domain D, and G acts on the H-orbit Λ in D so that Λ/G is a torus homotopy equivalent to K. Hence Λ considered to be in A_i^o separates the boundary components of A_i. By same reason Λ separates the boundary components of A_j. This is clearly absurd for a single disk Λ (use $\mathbf{Z}_2$-intersection theory to see this). Therefore N_i in M are all of one type (i) or the other type (ii).

If every N_i is of type (i), we easily see that $\mathbf{dev}(\tilde{N}_i)$ lies in the set $z > 0$ by induction. Thus $\mathbf{dev}(\tilde{M})$ lies in the same set. Hence, the closed 1-form dz/z is G-invariant, and induces a closed 1-form on N_1 which takes a non-zero value under radiant vectors. This means that N_1 has a total cross-section by Lemma C.3. Hence M is a generalized affine suspension.

From now on we assume that each N_i is of type (ii). Thus, N_i and N_{i+1} meet at a torus which is a quotient of a triangle or a lune alternatively according to i.

Suppose that N_j and N_{j+1} meet at a torus which is a quotient of a triangle. Then we see that the remaining boundary components of $\tilde{N}_j \cup \tilde{N}_{j+1}$ map homeomorphic under $\mathbf{dev}$ to open lunes which are components of a plane with a line removed. This is a contradiction by Lemma C.12. Therefore M equals N_1 or the union of only two N_1 and N_2 meeting a torus which is a quotient of a lune. (We remark that this corresponds to a generalized affine suspension of π-annulus of type C.)

We will show that N_1 admits a total cross-section. Since N_2 can be obtained by the reflection R commuting with elements of G, this will show that M has a total cross-section.

We can easily show that the connected two-dimensional subgroup H of matrices of form P is of form:

$$\begin{pmatrix} e^a & be^a & 0 \\ 0 & e^a & 0 \\ 0 & 0 & e^c \end{pmatrix}$$

where a, b, c lies in a two-dimensional subspace of $\mathbf{R}^3$ with coordinate functions a, b, and c. A lattice L in P determines a subspace of $\mathbf{R}^3$, to be denoted by $P(L)$. Let the ac-plane have the orientation given by $\{e_a, e_c\}$ and the ab-plane have the orientation by $\{e_a, e_b\}$ where e_a, e_b, and e_c are unit vectors in the positive a-, b-, and c-axis respectively.

LEMMA C.13. *Let L be a lattice in a connected two-dimensional subgroup of group of matrices of form P. Let U be the domain given by $y > 0, z > 0$ union with the set U_{xz} given by $x > 0, y = 0, z > 0$ and the set U_{yz} given by $y > 0, z = 0$. Then L acts on U so that U/L is a manifold if and only if for the projections $p_{ac} : \mathbf{R}^3 \to \mathbf{R}^2$ to the ac-plane and $p_{ab} : \mathbf{R}^3 \to \mathbf{R}^2$ to the ab-plane, $g = p_{ac} \circ (p_{ab}|P(L))^{-1}$ is orientation-reversing.*

PROOF. Suppose that U/L is a manifold. Then U/L is homeomorphic to $T^2 \times I$. Let L' be a lattice in $P(L)$ corresponding to L by the above description. Then a connected two-dimensional group $\tilde{H}$ of elements of the above form with $a, b, c \in P(L)$ acts on U properly and without fixed points. We see that U is foliated by $\tilde{H}$-orbits. Thus, U/L is foliated by leaves that are $\tilde{H}$-orbits quotient out by L, homeomorphic to tori. Using the orientation on $P(L)$, each leaf has an induced orientation. We see easily that the leaf-space is homeomorphic to an interval and the boundary of U/L corresponds to the endpoints of the interval.

Consider the map $\mathcal{F} : \mathbf{R}^3 \to U^o$ given by

$$(a, b, c) \mapsto \begin{pmatrix} e^a & be^a & 0 \\ 0 & e^a & 0 \\ 0 & 0 & e^c \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = ((b+1)e^a, e^a, e^c)$$

which is a homeomorphism. Also, define $\mathcal{F}_{xz} : \mathbf{R}^2_{xz} \to U_{xz}$, where $\mathbf{R}^2_{xz}$ denotes the xz-plane, by

$$(a, 0, c) \mapsto \begin{pmatrix} e^a & be^a & 0 \\ 0 & e^a & 0 \\ 0 & 0 & e^c \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = (e^a, 0, e^c)$$

Define $\mathcal{F}_{xy} : \mathbf{R}^2_{xy} \to U_{xy}$, where $\mathbf{R}^2_{xy}$ denotes the xy-plane, by

$$(a, b, 0) \mapsto \begin{pmatrix} e^a & be^a & 0 \\ 0 & e^a & 0 \\ 0 & 0 & e^c \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = ((b+1)e^a, e^a, 0).$$

Under $\mathcal{F}$, each $\tilde{H}$-orbit corresponds to a translate of $P(L)$. We may add $\mathbf{R}^2_{xz}$ and $\mathbf{R}^2_{xy}$ to $\mathbf{R}^3$ by considering $(x + t, e^{-t} - 1, z)$, $t < 0$, to converge to $(x, 0, z)$ as

$t \to -\infty$ and considering (x, y, t), $t < 0$, to converge to $(x, y, 0)$ as $t \to -\infty$. Then $\mathbf{R}^3$ is an open subspace of the completed space C. By adding $\mathcal{F}_{xz}$ and $\mathcal{F}_{xy}$ to $\mathcal{F}$, we obtain a homeomorphism $C \to U$ with appropriate topology on C.

In U^o, a $\tilde{H}$-orbit separates U_{xz} and U_{xy} since they correspond to two boundary components of $T^2 \times I$. In C, $P(L)$ must separate $\mathbf{R}^{xz}$ and $\mathbf{R}^{xy}$. Considering g to be a map $\mathbf{R}^2 \to \mathbf{R}^2$, as $g(x, y) = (x, y')$ for each (x, y) and some y', it follows that g is represented by a matrix

$$\begin{pmatrix} 1 & l \\ 0 & m \end{pmatrix}$$

where the plane $P(L)$ is given by $z = lx + my$.

Consider $\mathbf{R}^3$ as an open hemisphere, i.e., as an affine patch, in the projective sphere $\mathbf{S}^3$ with boundary $\mathbf{S}^2$ (see [**17**] where the correspondence is realized by moving the subspace $\mathbf{R}^3$ in $\mathbf{R}^4$ in the orthogonal direction by a unit and stereographically project from the origin onto a hemisphere $\mathbf{S}^3$). Since the point that $(x + t, e^{-t} - 1, z)$ converges on $\mathbf{S}^2$ as $t \to -\infty$ equals the ray through $(0, 1, 0, 0)$ and (x, y, t) converges to the ray through $(0, 0, -1, 0)$, for separation to hold, we must have that the function $z - lx - my$ takes different signs at these two points. This means that $m < 0$. Therefore the determinant of the matrix equals m which is negative.

We now prove the converse. Using the notation above, we see that $m < 0$. Let l_1 be the arc given by $\{(t, e^{-t} - 1, 0) | t \leq 0\}$ and l_2 one by $\{(0, 0, s) | s \leq 0\}$. Then $l_1 \cup l_2$ is an arc with well-defined endpoints at $\mathbf{R}^2_{xz}$ and $\mathbf{R}^2_{xy}$ in C. The arc α meets each of the translates of $P(L)$ at a unique point eventually far away. We can easily choose an arc α' which eventually agrees with α far away and meets each translates of $P(L)$ at a unique point. The image of α' by $\mathcal{F}$ is an arc in U with endpoints in U_{xz} and U_{xy} which meets each $\tilde{H}$-orbit at exactly one point x. As L acts on each $\tilde{H}$-orbits to produced a torus, it follows that this condition is enough to give us a compact fundamental domain in U of the L-action. Hence, U/L is a compact manifold homeomorphic to $T^2 \times I$. □

REMARK C.1. There are analogous statements in case D also. But we omit them here.

Let L' be the lattice in $\mathbf{R}^3$ corresponding to our group G by above correspondence. Let ϑ be an element of L' so that $x - z$ and y are positive on it. We can always choose such ϑ since $m < 0$ for our group L' by the above lemma C.13. We may further assume that ϑ is not a power of an element of L'. Let ϑ' be the corresponding element of G. The condition implies that for the projective automorphism ϑ'' corresponding to ϑ' acting on $\mathbf{S}^2$, ϑ'' acts properly and freely on the subset U' of $\mathbf{S}^2$ corresponding to U under the radial projection (see Section 1.4 of [**14**]). Hence, $U'/ < \vartheta'' >$ is a compact real projective annulus with totally geodesic boundary (of type *IIb* in [**14**]). Hence, $U/ < \vartheta' >$ is a $\mathbf{R}$-bundle over this quotient space $U'/ < \vartheta >$. We see easily that $U/ < \vartheta' >$ has a total cross-section A with real projective structure isomorphic to $U'/ < \vartheta'' >$. As the holonomy group is Abelian of rank 2, there exists an element φ of G inducing an automorphism φ' on $U/ < \vartheta' >$ so that $\varphi'^{,i}(A)$ is disjoint from A for $i \neq 0$. Since $U'/ < \vartheta, \varphi >$ is a finite cover of N_1 with a total cross section coming from A, N_1 has a total cross-section.

We end with an example.

EXAMPLE C.1. Let ϑ be equal to

$$\begin{pmatrix} e^2 & e^2 & 0 \\ 0 & e^2 & 0 \\ 0 & 0 & e^{-4} \end{pmatrix}$$

and φ equal to

$$\begin{pmatrix} e^3 & -e^3 & 0 \\ 0 & e^3 & 0 \\ 0 & 0 & e^3 \end{pmatrix}.$$

Then the group $\langle \vartheta, \varphi \rangle$ acts on U freely and properly discontinuously. The above coefficients are given by $l = -1/5$ and $m = -18/5$. This has a total cross-section.

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