On certain radiant affine 3-manifolds

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Let G be a group acting on a analytic manifold X. A (G, X)-manifold is a manifold admitting an atlas with charts with value in X and whose coordinate change mappings are restrictions of element of G. It is well-known that equipping a manifold M with a (G, X)-structure is equivalent to giving a pair $(\mathcal{D}, \rho)$, where $\mathcal{D}$ is a submersion from the universal covering $\widetilde{M}$ of M into X, and where ρ is a morphism from the fundamental group Γ of M into G, such that:

$$\forall \gamma \in \Gamma \quad \mathcal{D} \circ \gamma = \rho(\gamma) \circ \mathcal{D}$$

Here, the action of Γ on $\widetilde{M}$ is of course the action by covering automorphisms. The map $\mathcal{D} : \widetilde{M} \to X$ is the *developping map* of the structure, and $\rho : \Gamma \to G$ is the *holonomy map.*

A radiant affine n-manifold is a (G, X)-manifold, where X is the vector space R^n, and where G is the group $GL(n, R)$ of linear transformations (cf. [6]). In this paper, we only consider closed radiant affine manifolds. Such a manifold is naturally equipped with a transversely projective flow, the so-called *radiant flow,* defined as follows: if $(x_1, \ldots, x_n)$ are local coordinates, the vector field generating the radiant flow is:

$$X(x_1, \ldots, x_n) = \Sigma_{i=1}^{n} x_i \partial_{x_i}$$

Observe that this vector field does not depend from the coordinate system, and thus is well-defined. The radiant flow has no singularity (cf. [6]).

Let N be a closed real projective $n-1$-manifold, i.e. a $(PGL(n, R), RP^{n-1})$-manifold, where RP^{n-1} is the real projective space of dimension $n-1$, and $PGL(n, R)$ is the group of projective transformations. Let φ be a projective automorphism of N. We can associate to the pair (N, φ) a family of radiant affine closed n-manifold: the Benzécri-Smillie's examples ([4, 13, 6, 1]). We recall this construction at section 1. They can be characterized by the following property: *a closed radiant affine manifold is finitely covered by a Benzécri-Smillie's example if and only if its radiant flow admits a cross-section, i.e. there is a closed embedded submanifold transverse everywhere to the flow and which* *meets every orbit of the flow.* In these examples, the radiant flow is topologically equivalent to the topological suspension of $\varphi : N \to N$. Y. Carrière arose the following question: is every closed radiant affine 3-manifold projectively isomorphic to a finite covering of some Benzécri-Smillie's example? (the answer is known to be no in greater dimension ([11])). The answer is yes with the additional assumption that the radiant flow preserves some volume form ([6]). In preceding papers, we have shown that the same is true if the holonomy group is virtually solvable (and more generally, if some finite index subgroup preserves a plane of R^3), or if the manifold is homeomorphic to a seifert manifold [1, 3]).

In the present paper, we deal with the following particular case:

Theorem *There is no closed radiant affine* 3*-manifold whose developping map is a cyclic covering over* R^3 *minus a line.*

The whole paper is devoted to the proof of this theorem. The proof goes as follows: we assume the existence of a radiant affine 3-manifold whose developping map is a cyclic covering over $R^3 \setminus \{x = y = 0\}$. In the first section, we prove that the holonomy group is solvable: indeed, if not, it contains a hyperbolic element $\rho(\gamma)$ with one eigenvalue greater than 1, another less than 1 (and positive), and the last exactly equal to 1. The contradiction nearly arises from the fact that such a linear transformation do not act properly discontinously on $R^3 \setminus \{x = y = 0\}$, whereas γ must act properly discontinously on $\widetilde{M}$.

Since the holonomy group is solvable, the affine manifold is a Benzécri-Smillie exemple ([1]). Therefore, to achieve the proof, we have just to observe that no projective surface has a developping map which is a cyclic covering over the sphere minus two points (it is more suited for our arguments to consider real projective surfaces as $(P^+GL(3,R), S^2)$-manifolds, where S^2 is the sphere of half-lines, and $P^+GL(3,R)$ the quotient of $GL(3,R)$ by the positive homotheties).

S. Choi announced a positive answer in the general case, using our partial results and additionnal techniques of convex-concave decomposition. This positive answer provides a very satisfactory understanding of radiant affine 3-manifolds, since real projective surfaces are fairly well-known ([7, 8, 9, 10]). The present work follows from a conversation with S. Choi, who asked me how to manage with the case treated here. I want to express here my consideration to S. Choi. It is a real pleasure for me to have the opportunity to collaborate with him. I thank A. Zeghib too for its precious collaboration.

1 Conventions and notations

In all the paper, M is a closed radiant affine 3-manifold (actually, the compactness is not needed for the first section). We denote by Φ^t its radiant flow. We denote by $p : \widetilde{M} \to M$ the universal covering (we don't worry about the choice of base point). Let Γ be the fundamental group of M: it acts naturally on $\widetilde{M}$.

Let $\mathcal{D} : \widetilde{M} \rightarrow R^3$ be the developping map, and $\rho : \Gamma \rightarrow GL(3, R)$ be the holonomy morphism (for the definitions, see for example [5]). They satisfy:

$$\forall \gamma \in \Gamma \quad \mathcal{D} \circ \gamma = \rho(\gamma) \circ \mathcal{D}$$

We assume that $\mathcal{D}$ is a cyclic covering over $R^3 \setminus \Delta$, where Δ is some lines through 0. Our aim is to obtain a contradiction.

Since we want to show that such a M does not exist, we are free to replace M by any finite covering of itself. For example, we can consider only the case where M is oriented, i.e. the case where every element of the holonomy group is of positive determinant.

Since $\mathcal{D}$ is well-defined up to composition by a linear transformation, we can assume that Δ is the line $\{x = y = 0\}$. Then, since Δ has to be $\rho(\Gamma)$-invariant, every element $\rho(\gamma)$ of the holonomy group is of the form:

$$\rho(\gamma) = \begin{pmatrix} \multicolumn{2}{c}{\bar{\rho}(\gamma)} & \begin{matrix} 0 \\ 0 \end{matrix} \\ * & * & \lambda(\gamma) \end{pmatrix}$$

where $\lambda(\gamma)$ is a non-zero real number, and $\bar{\rho}(\gamma)$ an element of $GL(2, R)$. Clearly, λ and $\bar{\rho}$ are morphisms.

We will need the following easy lemma:

Lemma 1.1 *Let G be a Lie group acting on two manifolds X and Y. Let $f : X \rightarrow Y$ be a function equivariant for the actions of G. Let x be an element of X such that $f(x)$ is fixed by no element of G. Then, the restriction of f to the G-orbit of x is injective.*

Proof For every element g of G we have $f(gx) = g.f(x)$. ∎

We give now the construction of Benzécri-Smillie examples. Let $\varphi : N \rightarrow N$ a projective diffeomorphism of a real-projective manifold N. Let $f_i : U_i \rightarrow V_i \subset S^{n-1}$ be a family of projective charts covering N. When U_i meets U_j, we have an element $\bar{g}_{ij}$ of $P^+GL(n, R)$ such that on $U_i \cap U_j$:

$$f_i = \bar{g}_{ij} \circ f_j$$

Let's choose representatives g_{ij} of the $\bar{g}_{ij}$ in $GL(n, R)$. We impose the condition $g_{ij}g_{jk}g_{ki} = id$ if $U_i \cap U_j \cap U_k$ is not empty. Such a choice is always possible: take for example the unique representative of $\bar{g}_{ij}$ with determinant ± 1. The set of the possible choices is parametrized by $H^1(N, R)$. For every i, let W_i be the open cone in R^n with vertex at 0, union of the half lines belonging to V_i. Let denote by W the quotient of the disjoint union of the W_i by the relation identifying each element x_j of W_j with the element $g_{ij}(x_j)$ of W_i (when $g_{ij}(x_j)$ belongs effectively to W_i, of course). This quotient is a noncompact radiant affine manifold, equipped with a complete radiant flow $\hat{\Phi}^t$. The quotient of W

by the relation 'being on the same orbit of $\hat{\Phi}^t$' is homeomorphic to N. The quotient map is a fibration by lines. Let N_0 be any section of this fibration. W is diffeomorphic to $N \times R$.

Remember the projective transformation φ of N. It lifts[1] to an affine diffeomorphism $\hat{\varphi}$ of W well-defined up to composition by $\hat{\Phi}^t$. If T is big enough, $\hat{\Phi}^T\hat{\varphi}(N_0)$ is a section of $\hat{\Phi}^t$ disjoint from N_0. Therefore, for every real positive t, $\hat{\Phi}^t\hat{\varphi}$ acts freely and properly discontinously on W. The quotient of this action is a closed radiant affine manifold homeomorphic to the topological suspension N_φ of $\varphi : N \to N$. This is a Benzécri-Smillie example.

Observe that the construction is not uniquely defined: we made some choices. These choices are parametrized by an open subset of $H^1(N_\varphi, R)$. These parameters are the morphisms $H_1(N_\varphi, R) \to R$ represented by $det \circ \rho$, where ρ is the morphism of monodromy and det the determinant map.

Actually, it is possible to perform a similar construction in the case where N is a projective orbifold whose singularities are isolated points.

By construction, the radiant flow of a Benzécri-Smillie example admits a closed cross-section homeomorphic to Σ. Note that this section, equipped with the projective structure induced by the transverse projective structure of the radiant flow is isomorphic to the initial projective surface Σ. The converse is true:

Proposition 1.2 *A radiant affine closed manifold admits a finite covering by a Benzécri-Smillie example if and only if its radiant flow admits a closed cross-section.* ∎

2 Solvability of the holonomy group

Proposition 2.1 *The holonomy group $\rho(\Gamma)$ is solvable.*

Proof Denote by Γ' the first commutator subgroup of Γ. Since λ and $\bar{\rho}$ are morphisms, for every element of Γ' we have:

- $\bar{\rho}(\gamma)$ belongs to $SL(2, R)$,
- $\lambda(\gamma) = 1$.

Observe that by definition $\rho(\Gamma)$ is solvable if and only if $\bar{\rho}(\Gamma')$ is solvable.

Let $\mathcal{F}^0$ be the foliation of $R^3 \setminus \Delta$ whose leaves are the half-planes containing Δ in their boundaries. The leaf space of this foliation, i.e. the quotient of $R^3 \setminus \Delta$ by the relation "being on the same leaf of $\mathcal{F}$", is naturally identified

[1]Actually, such a lifting does not always exist for any choice of W, but for many of W above the given Σ, we can perform such liftings. The condition is: let $\bar{\rho} : \pi_1(W) \to GL(n, R)$ be the monodromy morphism of W. Observe that $\pi_1(W)$ is isomorphic to $\pi_1(N)$. Let φ_* be the automorphism of $\pi_1(N)$ induced by φ. Then, φ lifts if and only if $det \circ \bar{\rho}$ is constant on the orbits of φ_*. For example, the choice of the g_{ij}'s of determinant ± 1 works. We don't want to go into further details.

with the double covering of the real projective line RP^1. Let $\mathcal{F}$ be the pull-back of $\mathcal{F}^0$ by $\mathcal{D}$. Since $\mathcal{D}$ is a cyclic covering, $\mathcal{F}$ is a foliation whose leaf space is naturally identified with the universal covering $\tilde{P}^1$ of RP^1. The action of Γ' on the leaf space induced by the action of Γ on $\widetilde{M}$ is a lifting of the projective action of $\bar{\rho}(\gamma') \in SL(2,R)$ over RP^1. According to lemma 2.2 below, if $\bar{\rho}(\Gamma')$ is not solvable, there is an element γ of Γ' preserving a leaf F of $\mathcal{F}$, and such that in an adequate coordinate system:

$$\rho(\gamma) = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for some real positive λ different from 1.

We fix this coordinate system. Let P be the inverse image by $\mathcal{D}$ of the punctured plane $\{z = 0\} \setminus \{(0,0,0)\}$. Since $\mathcal{D}$ is a cyclic covering, P is connected, and the restriction of $\mathcal{D}$ to P is a cyclic covering. Moreover, γ preserves P. Since γ preserves the leaf F too, every connected component of $\mathcal{D}^{-1}(\{z = y = 0\})$ or $\mathcal{D}^{-1}(\{z = x = 0\})$ is preserved by γ. Let C be a connected component of $\mathcal{D}^{-1}(\{z = 0, x \geq 0, y \geq 0\})$. It is preserved by γ, and the restriction of $\mathcal{D}$ to C is a homeomorphism over $\{z = 0, x \geq 0, y \geq 0\} \setminus \{(0,0,0)\}$. The action of $\rho(\gamma)$ on $\{z = 0, x \geq 0, y \geq 0\} \setminus \{(0,0,0)$ is given by $(x, y, 0) \mapsto (\lambda x, \lambda^{-1} y, 0)$. It is not properly discontinous, since any path joining $\{z = 0, x = 0, y \geq 0\}$ to $\{z = 0, x \geq 0, y = 0\}$ intersects all its iterates by $\rho(\gamma)$. This is a contradiction since the action of γ on C has to be properly discontinous. It follows that $\bar{\rho}(\Gamma)$, and therefore $\rho(\Gamma)$, is solvable. ∎

For the proof of lemma 2.2, we must first recall some facts about the actions of $PSL(2,R)$ and its universal covering $\tilde{S}L(2,R)$ on RP^1 and its universal covering, respectively. Let $q : \tilde{S}L(2,R) \to PSL(2,R)$ denote the covering map. Every element g of $PSL(2,R)$ is either:

- *elliptic:* g has no fixed point in RP^1. It is conjugate to a rotation,

- *parabolic:* g has one and only one fixed point. This fixed point is of saddle-node type, i.e. attractive on one side, and repulsive on the other side,

- *hyperbolic:* g has two fixed points: a repulsive one and a attractive one. It is conjugate to the element represented by:

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

An element of $\tilde{S}L(2,R)$ is said elliptic, parabolic or hyperbolic according to the nature of its projection $q(g)$. If this projection is trivial, g belongs to the center H of $\tilde{S}L(2,R)$. The group H is cyclic. Let h be a generator of H. If g is not trivial and admits fixed points on the universal covering $\tilde{P}^1$, it is parabolic or hyperbolic. In the first case, its fixed points are of saddle-node type; in the second case, they are attractive or repulsive.

Lemma 2.2 *Let Γ be a subgroup of $\tilde{SL}(2,R)$. We assume that $q(\Gamma)$ is not solvable. Then, it contains a hyperbolic element that fixes a point of $\tilde{P}^1$.*

Proof Note that Γ is not solvable since it is a cyclic extension of $q(\Gamma)$ which is not solvable. According to Hölder's theorem (see e.g. [12], IV.3.1), a group acting freely on the real line is abelian. Therefore, the action of Γ on $\tilde{P}^1$ is not free: some element γ_0 of Γ admits a fixed point x_0 in $\tilde{P}^1$. If γ_0 is hyperbolic, we are done. If not, γ_0 is parabolic. Then, the fixed points of γ_0 are the H-iterates of x_0. We denote by x_i the image of x_0 by h^i. Observe that since $\tilde{P}^1$ is homeomorphic to R, orienting $\tilde{P}^1$ is equivalent to equip it with an archimedian total order. We orient $\tilde{P}^1$ in such a way that x_1 is greater than x_0. Inversing γ_0 if necessary, we can assume that all the γ_0-orbits in $]x_0, x_1[$ go from x_0 to x_1.

The stabilizer in $PSL(2,R)$ of a point in P^1 is isomorphic to the group of affine transformations of the line. It is therefore solvable. It follows that there is an element γ of Γ such that γx_0 is not one of the x_i's. Let γ_1 be the conjugate $\gamma\gamma_0\gamma^{-1}$. It is parabolic and fixes γx_0. Therefore, it admits a fixed point x'_0 in $]x_0, x_1[$. Then, $\gamma_1^{-1}\gamma_0 x_0 = \gamma_1^{-1} x_0$ is less than x_0, and $\gamma_1^{-1}\gamma_0 x'_0$ is greater than x'_0, since x'_0 is a fixed point of γ_1^{-1} and that $\gamma_0 x'_0$ is greater than x'_0. Therefore, $[x_0, x'_0]$ is contained in its image by $\gamma_1^{-1}\gamma_0$. It follows that $\gamma_1^{-1}\gamma_0$ is a hyperbolic element admitting a repulsive fixed point in $]x_0, x'_0[$. ∎

3 Benzécri-Smillie's examples

We know from the proposition 2.1 that the holonomy group is solvable. It follows from the theorem A of [1] that M is affinely isomorphic to a Benzécri-Smillie's example. In particular, the radiant flow admits a cross-section. Let Σ be such a cross section, and $\tilde{\Sigma}$ a lifting of Σ in $\widetilde{M}$, i.e. a connected component of $p^{-1}(\Sigma)$. Let $\tilde{\Phi}^t$ be the lifting of Φ^t in $\widetilde{M}$. Since Σ is a cross-section, it is a fiber of some fibration of M over the circle. Hence, $\widetilde{M} \setminus \tilde{\Sigma}$ is not connected. Every orbit of $\tilde{\Phi}^t$ meets $\tilde{\Sigma}$. This orbit remains in the past in one connected component of $\widetilde{M} \setminus \tilde{\Sigma}$, and in the future, it remains in the other connected component. In other words, every orbit of $\tilde{\Phi}^t$ meets $\tilde{\Sigma}$ at one and only one point. The developping map sends injectively every orbit of $\tilde{\Phi}^t$ over a half-line in R^3 (lemma 1.1 applied to the R-actions). Therefore, it induces a local homeomorphism $\hat{\mathcal{D}}$ from $\tilde{\Sigma}$ on the sphere S^2 of half-lines. We denote by S_* the sphere S^2 punctured at $(0,0,1)$ and $(0,0,-1)$. Since $\mathcal{D}$ is a cyclic covering over $R^3 \setminus \Delta$, $\hat{\mathcal{D}}$ is a cyclic covering over S_*. Therefore, $\tilde{\Sigma}$ is the universal covering of Σ, and $\hat{\mathcal{D}}$ is the developping map of a real projective structure on Σ. The holonomy morphism $\hat{\rho}$ of this structure is the composition of the restriction of ρ to $\hat{\Gamma}$ with the projection of $GL(3,R)$ in $P^+GL(3,R)$, where $\hat{\Gamma}$ is the group of elements of Γ which preserve $\tilde{\Sigma}$.

In order to find a contradiction, i.e in order to achieve the proof of the theorem, it suffices to show:

Proposition 3.1 *There is no RP^2-structure on the closed surface Σ such that the developping map $\hat{\mathcal{D}}$ is a cyclic covering over S_*.*

Proof Up to a finite covering, we can assume that Σ is oriented, and that $\hat{\rho}(\Gamma)$ is contained in the identity component of $P^+GL(3,R)$. This identity component is naturally isomorphic to $SL(3,R)$, so we will consider $\hat{\rho}(\hat{\Gamma})$ as a subgroup of $SL(3,R)$.

Consider the foliation $\mathcal{G}_0$ whose leaves are the restriction to S_* of the great circles passing through $(0,0,1)$ (and therefore through $(0,0,-1)$). Then $\hat{\mathcal{D}}^*(\mathcal{G}_0)$ is a regular foliation on $\tilde{\Sigma}$ which is $\hat{\Gamma}$-invariant. It induces a regular foliation on the surface Σ. By the Poincaré-Hopf formula, the Euler characteristic of Σ is zero.

Hence, Σ is a torus. In particular, $\hat{\Gamma}$ is isomorphic to Z^2.

The proposition then follows from the classification of the projective structures on the torus (see for example [2]). In order to have a self-contained proof, and since we don't know a satisfactory reference, we produce the following arguments:

Let G be the identity component of the Zariski closure $\overline{\hat{\rho}(\hat{\Gamma})}$ in $SL(3,R)$. Up to a finite covering, we can assume that G contains $\hat{\rho}(\hat{\Gamma})$. The group G is an abelian algebraic group fixing the line Δ. The picture is very easy to understand, since connected abelian subgroups of $GL(2,R)$ are well-known. We obtain that, up to conjugacy, G and $\hat{\rho}(\hat{\Gamma})$ are contained in a group $A(G)$, where $A(G)$ is one of the following groups:

$$D = \left\{ \begin{pmatrix} u & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & w \end{pmatrix} \middle| \quad u>0, v>0, w>0, uvw=1 \right\}$$

$$P = \left\{ \begin{pmatrix} u & s & 0 \\ 0 & u & 0 \\ 0 & 0 & v \end{pmatrix} \middle| \quad u>0, v>0, u^2v=1 \right\}$$

$$S = \left\{ \begin{pmatrix} rcos\theta & rsin\theta & 0 \\ -rsin\theta & rcos\theta & 0 \\ 0 & 0 & u \end{pmatrix} \middle| \quad r>0, u>0, r^2u=1 \right\}$$

$$T = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ s & 1 & 0 \\ t & 0 & 1 \end{pmatrix} \right\}$$

$$C = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ s+\frac{t^2}{2} & t & 1 \end{pmatrix} \right\}$$

$$U = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ s & t & 1 \end{pmatrix} \right\}$$

In every case, $A(G)$ is isomorphic to R^2, except if $A(G) = S$. Let $\tilde{A}(G)$ be the universal cover of $A(G)$, and $\mathcal{A}(G)$ its Lie algebra. The action of $A(G)$ on S^2 induces a morphism of Lie algebra from $\mathcal{A}(G)$ to the algebra of vector fields on S^2. Pulling back by $\hat{\mathcal{D}}$ we obtain a commutative algebra of vector fields on $\tilde{\Sigma}$. Since $\hat{\rho}(\hat{\Gamma})$ is contained in $A(G)$ which is abelian, the action of $\hat{\Gamma}$ on $\tilde{\Sigma}$ preserves every vector field in this algebra. These vector fields project on Σ. By compactness, each of them generates a flow. We obtain thus an action of $\tilde{A}(G)$ on Σ, which can be lifted as an action on $\tilde{\Sigma}$, such that $\hat{\mathcal{D}}$ is equivariant.

Lemma 3.2 *The action of $A(G)$ on S_* has no common fixed points.*

Proof Assume that the action of $A(G)$ on S_* preserves a point x_0. Consider the foliation $\mathcal{G}$ whose leaves are the restriction to S_* of the great circles passing through x_0. We have the same picture as for $\mathcal{G}_0$: it induces a foliation on Σ. But, there is a difference with $\mathcal{G}_0$: the foliation $\mathcal{G}$ has singularities, and these singularities are all of index one. This is in contradiction with the Poincaré-Hopf formula. ∎

In the cases $A(G) = D$ and $A(G) = P$, the half-line $\{y = 0, z = 0, x > 0\}$ is $A(G)$-invariant. In the case $A(G) = T$, the half-line $\{x = 0, z = 0, y > 0\}$ is $A(G)$-invariant. According to the lemma 3.2, it follows that $A(G) = S$, $A(G) = U$ or $A(G) = C$. We will eliminate all these remaining cases one by one. It will give the final contradiction and prove the proposition.

- *The case $A(G) = S$:*

Then $\tau = \{z = 0\} \cap S_*$ is $\hat{\rho}(\hat{\Gamma})$-invariant. Since $\hat{\mathcal{D}}$ is a cyclic covering, the inverse image $\hat{\mathcal{D}}^{-1}(\tau)$ is connected. It is homeomorphic to the real line. But it has to be preserved by $\hat{\Gamma}$, and the action of $\hat{\Gamma}$ on it has to be free and properly discontinous. It is impossible since $\hat{\Gamma}$ is isomorphic to Z^2.

- *The case $A(G) = U$:*

All the orbits of $A(G)$ on S_* are of dimension one. They are lines joining the two ends $(0, 0, -1)$ and $(0, 0, 1)$ of S_*. Therefore, the orbits of $A(G) = \tilde{A}(G)$ on Σ are of dimension one. Let θ be an orbit of $A(G)$ in S_*. It is $\hat{\rho}(\hat{\Gamma})$-invariant. Therefore, $\hat{\mathcal{D}}^{-1}(\theta)$ is a closed subset of $\tilde{\Sigma}$ which is $\hat{\Gamma}$-invariant. More precisely, since $\hat{\mathcal{D}}$ is a cyclic covering, it is the image of an embedding of a countable union of copies of the real line. It follows that the projection of $\hat{\mathcal{D}}^{-1}(\theta)$ in Σ is a one-dimensional embedded submanifold. In other words, the orbits of $A(G)$ in Σ are closed: they are circles. All these circles, being disjoints, are isotopic one to the other. It means that there is an element γ_0 of $\hat{\Gamma}$ which fixes every orbit of $A(G)$ on $\tilde{\Sigma}$. Since the restriction of $\hat{\mathcal{D}}$ to each $A(G)$-orbit is injective, the action of $\hat{\rho}(\gamma_0)$ on each orbit of $A(G)$ in S_* is free. But every element of $A(G)$ admits a fixed point in S_*: if this element is:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ s & t & 1 \end{pmatrix}$$

the fixed point is the half-line containing $(-t, s, 1)$.

We obtain a contradiction with lemma 3.2 since $\hat{\rho}(\gamma_0)$ belongs to $A(G)$.

- *The case* $A(G) = C$*:*

The action of $A(G)$ on S_* has two orbits of dimension one, corresponding respectively to the half-planes $\{x = 0, y > 0\}$, $\{x = 0, y < 0\}$, and two orbits of dimension two corresponding to the half-spaces $\{x > 0\}$, $\{x < 0\}$. The bidimensional orbits are projectively isomorphic to complete affine planes.

Since every orbit of dimension one is preserved by $\hat{\rho}(\hat{\Gamma})$, their inverse image by $\hat{\mathcal{D}}$ are lifting of closed orbits of $\tilde{A}(G) = A(G)$ on the torus. Hence, they are circles disjoint one to the other. It follows as in the preceding case that there is an element γ of $\hat{\Gamma}$ preserving every orbit of dimension one of $A(G)$ on $\tilde{\Sigma}$. Let V be a connected component of $\tilde{\Sigma}$ minus the orbits of dimension one: it is a bidimensional orbit of $A(G)$, and its boundary is the union of two orbits of dimension one. V is γ-invariant. According to the lemma 1.1, the restriction of $\hat{\mathcal{D}}$ to the closure $\bar{V}$ of V is injective, the image being either $S_* \cap \{x \geq 0\}$, or $S_* \cap \{x \leq 0\}$. Thus, $\hat{\rho}(\gamma)$ is an element of $A(G)$ acting freely and properly discontinously on $S_* \cap \{x \geq 0\}$. Let γ^t a one parameter subgroup of $A(G)$ such that $\gamma^1 = \gamma$. It is of the form:

$$\gamma^t = \begin{pmatrix} 1 & 0 & 0 \\ tt_0 & 1 & 0 \\ ts_0 + t^2\frac{t_0^2}{2} & tt_0 & 1 \end{pmatrix}$$

where s_0 and t_0 are constants. Assume that the real t_0 is zero. Then the half-line containing $(0, 1, 0)$ is a fixed point of γ^t in S_*, and therefore of $\hat{\rho}(\gamma)$: contradiction. Therefore, $t_0 \neq 0$. It follows that the orbits of γ^t in the affine plane $S_* \cap \{x \geq 0\}$ are parabols. In other words, the action of γ^t on $S_* \cap \{x \geq 0\}$ have the following property: every γ^t-invariant neighborhood of $\{x = 0, y < 0\} \cap S_*$ meets every γ^t-invariant neighborhood of $\{x = 0, y < 0\} \cap S_*$. On the other hand, since γ acts properly discontinously, the quotient of its action on $S_* \cap \{x \geq 0\}$ is a closed annulus. On this quotient, γ^t induces a free action of the circle. But the orbits of an action of the circle on the annulus are isolated one to the others by invariant neighborhoods (in other words, the orbit space is Hausdorff). We have obtained a contradiction. The case $A(G) = C$ is impossible. This completes the proof of the theorem.

References

[1] T. Barbot, *Structures affines radiales sur les 3-variétés à monodromie résoluble,* preprint Universidade Federal Fluminense (1997).

[2] T. Barbot, *La classification des surfaces affines fermées d'après Benzécri et Nagano-Yagi, et classification des tores projectifs réels,* preprint Universidade Federal Fluminense (1997).

[3] T. Barbot, *Structures affines radiales sur les variétés de Seifert,* preprint (1997).

[4] J.P. Benzécri, *Variétés localement affines et projectives,* Bull. Soc. Math. France **88** (1960), 229-332.

[5] R.D. Canary, D.B.A. Epstein, P. Green, *Notes on notes of Thurston,* In Analitical and Geometric Aspects of Hyperbolic Space, ed. D.B.A. Epstein, London Math. Soc. Lect. Notes Series **111** (1986), 3-92.

[6] Y. Carrière, *Questions ouvertes sur les variétés affines,* Séminaire Gaston Darboux de Géométrie et Topologie Différentielle, 1991-1992 (Montpellier), 69-72, Univ. Montpellier II, Montpellier, 1993.

[7] S. Choi, *Convex decomposition of real projective surfaces. I: π-annuli and convexity.* J. Diff. Geom. **40** (1994), 165-208.

[8] S. Choi, *Convex decomposition of real projective surfaces. II: Admissible decompositions.* J. Diff. Geom. **40** (1994), 239-283.

[9] S. Choi, *Convex decomposition of real projective surfaces. III* J. Korean. Math. Soc. **33** (1996).

[10] S. Choi, W.M. Goldman, *The classification of real projective structures on compact surfaces,* Bull. Amer. Math. Soc. **34** (2) (1997), 161-171.

[11] D. Fried, *Affine 3-manifolds that fiber by circles,* preprint I.H.E.S (1992).

[12] G. Hector et U. Hirsch, *Geometry of foliations, part B,* Aspects of Math., (1987) 2^{nd} edition.

[13] J. Smillie , *Affinely flat manifolds,* Doctoral Dissertation, University of Chicago, 1977.