Plane affine geometry of Anosov flows

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Abstract. We study Anosov flows on closed 3-manifolds. We define the notion of Anosov flows with the topological contact property (abreviation TCP Anosov flows): typical examples of TCP Anosov flows are contact Anosov flows, i.e. flows preserving a contact 1-form. We show that TCP Anosov flows are $\mathbf{R}$*-covered. The main tool is the study of the leaf spaces of lifted strong stable foliations: we exhibit on these leaf spaces a structure of (generalized) affine plane, in the sense of incidence geometry.*

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1 Introduction

Let M be a closed manifold equipped with an Anosov flow Φ^t, i.e. a flow generated by a nonsingular C^1 vector field X such that the tangent bundle TM of M splits continuously as a Whitney sum $TM = \mathbf{R}X \oplus E^{ss} \oplus E^{uu}$ of vector subbundles where:

- $\mathbf{R}X$ is the tangent line bundle defined by the flow direction X,

- E^{ss} (respectively E^{uu}) is preserved and exponentially contracted (respectively expanded) by the differential of the flow.

E^{ss} and E^{uu} are called the stable and unstable strong directions. In general, they are only Hölder continuous (see [21]). They are uniquely integrable and define two foliations, the so-called strong stable and unstable foliations, and denoted by $\mathcal{F}^{ss}$ and $\mathcal{F}^{uu}$. When we add the flow direction $\mathbf{R}X$ to the strong directions, we obtain two plane fields which are uniquely integrable: they define two foliations $\mathcal{F}^{s}$ and $\mathcal{F}^{u}$, transverse one to the other, and both tangent to the flow. They are called the weak stable and unstable foliations. A very interesting property of Anosov flows is the ***structural stability:*** if Y is a vector field sufficiently near to X in the C^1-topology, then it generates an Anosov flow Ψ^t which is *topologically equivalent* to Φ^t, i.e. there exists a homeomorphism from the ambient manifold to itself, mapping the orbits of Φ^t on orbits of Ψ^t (but the parametrizations are not necessarily preserved).

Typical examples of Anosov flow are the suspensions of Anosov diffeomorphisms: a diffeomorphism $f : M \to M$ is Anosov if the tangent bundle of M splits continuously as the sum of two subbundles, one which is exponentially contracted and the other exponentially expanded. The suspension of f is the flow induced by the vector field $\frac{\partial}{\partial t}$ on the quotient of $M \times \mathbf{R}$ by the relation identifying (x, t) with $(f(x), t-1)$. The characteristic property of Anosov flows topologically equivalent to suspensions is the existence of a cross-section. The only known examples of Anosov diffeomorphisms are hyperbolic automorphisms of infranilmanifolds ([11]), and it is conjectured that this is the only possibility. According to [25] and [11], if an Anosov diffeomorphism is such that its stable or unstable direction is of dimension one, then it is topologically conjugate to a linear hyperbolic automorphism of a torus. Moreover, A. Verjovsky ([34]) conjectured that every Anosov flow for which one of the strong direction is of dimension one, and on a closed manifold of dimension strictly higher than three, is topologically equivalent to the suspension of an Anosov diffeomorphism. S. Simić proved this conjecture with the additional hypothesis that the one-dimensional direction is α-Hölder for any $\alpha < 1$ ([31]).

Another important family of Anosov flows are the geodesic flows on unitary tangent bundles of closed manifolds equipped with a riemannian metric of negative curvature (cf. [1]). These flows are not suspensions, but there is no contradiction with the discussion above since such a flow has a strong direction of dimension one if and only if the riemannian manifold is of dimension two, i.e. only when the ambient manifold is of dimension three. From this observation we see that the the 3-dimensional case is quite particular. Actually, there is a great variety of Anosov flows in dimension three ([12, 20, 5, 2]). D. Fried gave a description of Anosov flows on 3-manifolds which are transitive, i.e. which admits a dense orbit, as a kind of twisted suspension of certain pseudo-Anosov maps on surfaces ([13]). This description, quite general, has the same advantage and the same limit than the description of 3-manifolds by surgeries along links of $\mathbf{S}^3$. It does not solve the problem of deciding, for example, what are the closed 3-manifolds admitting Anosov flows. Some facts are known in this direction: the universal covering of such a manifold is homeomorphic to $\mathbf{R}^3$ ([26]), its fundamental group has exponential growth ([27]). We gave partial results to this problem in the context of graphmanifolds ([4]). A particular feature in dimension 3 is the following: if X is sufficiently smooth (i.e. of class C^2), then $\mathcal{F}^s$ and $\mathcal{F}^u$ are more regular than expected: they are of class C^1; more precisely, their tangent bundles $\mathbf{R}X \oplus E^{ss}$ and $\mathbf{R}X \oplus E^{uu}$ are of class C^1 ([23]).

In dimension 3, we distinguish the special class of **R**-*covered* Anosov flows. An Anosov flow is **R**-covered if the lifting of one of the weak stable foliations in the universal covering of M is a foliation by planes homeomorphic to the product foliation of $\mathbf{R}^3$ by horizontal planes $\mathbf{R}^2 \times \{*\}$ (if one weak foliation lifts as a product foliation, the same is true for the other weak foliation [3, 8]). Not all Anosov flows are **R**-covered (see [2]), but this property appears naturally when studying Anosov flows. For example, the equivalent property for Anosov diffeomorphisms is a key step in the proof of Frank's theorem, asserting that Anosov diffeomorphisms of codimension one are linear. It is also a crucial step in [28] and [14]. The classical examples (i.e. geodesic flows and suspensions) are **R**-covered, and S. Fenley proved that the class of **R**-covered Anosov flows is stable under Dehn-Goodman surgeries ([18]) satisfying a positivity condition, providing by the way a huge family of examples.

In [3], we investigated some properties of **R**-covered Anosov flows not topologically equivalent to suspensions (see also [8] for similar and independent results). The main point is that these flows are characterized up to topological equivalence by some C^1-action of the fundamental group of the ambient manifold on the real line, the action commuting with a continuous homeomorphism without fixed point. This description of Anosov flows by an action on the line appears in [33]. We will recall in this paper the construction of this action. We will add the non-previously known fact (at least, by us) that the commuting homeomorphism is Hölder continuous.

In this paper, we study Anosov flows on closed 3-manifolds satisfying what we call the *topological contact property* (abbrevation TCP Anosov flows). This property means that there is no small loop, union of two small pieces of strong stable leaves and of two pieces of strong unstable leaves (for a precise definition, see Definition 3.5). This hypothesis is restrictive. For example, as a corollary of our results, the Bonatti-Langevin examples ([5]) are not TCP (because they are not **R**-covered, and because of Theorem A below), and thus, according to [2], it follows that there are 3-manifolds admitting Anosov flows, but no TCP Anosov flow. But the TCP hypothesis is valid in many interesting situations;

for example, in the case of geodesic flows, and more generally in the case of contact Anosov flows, i.e. Anosov flows such that the sum $E^{ss} \oplus E^{uu}$ is a contact plane of class C^1 (cf. Lemma 3.7). P. Foulon proved that the Anosov flows on graphmanifolds constructed by M. Handel and W. Thurston in [20] are contact. It is very presumable that the work of P. Foulon ([10], unfortunately a written version is not yet available) can be extended to a more general context. More precisely, the Dehn-Goodman surgery should be extended, in certain situations, to the context of contact Anosov flows. However, this optimistic point of view has to be restrained by the observation that Dehn-Goodman surgeries, applied to geodesic flows, can lead to suspensions of Anosov diffeomorphisms, and suspensions do not have the topological contact property (Proposition 5.1). It would be very interesting to decide whether the topological contact property is an open property amongst Anosov flows or not.

A nice feature of TCP Anosov flows is the following (see corollary 3.15):

Theorem A *Any Anosov flow admitting the topological contact property is* $\mathbf{R}$*-covered.*

This theorem, in the case of contact Anosov flows, has been announced by V.V. Solodov in [32]. The proof of Solodov has never been published. The only examples that we know of $\mathbf{R}$-covered Anosov flows which are not TCP are the flows topologically equivalent to suspensions (see Proposition 5.1).

As suggested by the title, there is a link between TCP Anosov flows and affine geometry. A paradigm of this phenomena is the case of geodesic flows of surfaces with constant negative curvature: the strong stable foliations associated to theses flows are transversely affine. Let's be more precise: the geodesic flow in the constant negative curvature case can be described as the action by right translations of positive diagonal matrices on left compact quotients M of $SL(2, \mathbf{R})$ by a discrete subgroup $\bar{\Gamma}$. The strong stables leaves are the orbits of the action by right translations of the group of upper unipotent matrices; the orbit space of this action is homeomorphic to the plane $\mathbf{R}^2$ minus the origin, and the natural action of $\bar{\Gamma}$ is conjugate to the usual linear action of $\bar{\Gamma}$ on $\mathbf{R}^2 \setminus \{0\}$.

In order to see how this result can be generalized, we have to remind the notion of (axiomatic) *affine plane*. On the subject, we used the recent quite complete reference books [7, 30]. An affine plane in the general meaning is a structure consisting of a set of points, with a collection of distinguished subsets called lines, satisfying the following three axioms:

- any two points lie on a unique line;

- if l is a line and p a point, then there is a unique line containing p and parallel to (i.e. either equal to or disjoint from) l;

- there exist three noncollinear points.

The construction of affine planes from any skew field K is well-known, especially in the case $K = \mathbf{R}$. We needn't to recall this construction here. Of course, there are many examples of affine planes which are not associated to a division ring. We have the following beautiful and fundamental theorem:

Theorem *An affine plane is associated to a skew field if and only if it satisfies Desargues' Theorem. It is associated to a commutative field if and only if satisfies Pappus' Theorem.*

Actually, we will consider only topological affine planes; i.e. we equip the set of points and the set of lines with a topology such that:

- the application mapping two different points on the line containing both is continuous,

- the application mapping two non parallel lines to their intersection point is continuous,

- the set of pairs of parallel lines is closed.

More precisely, we will actually assume that the topological affine plane is homeomorphic to $\mathbf{R}^2$, and that every line is closed in the plane and homeomorphic to $\mathbf{R}$: these planes are the so-called *affine* $\mathbf{R}^2$*-planes* (see chapter 3 of [30]).

The notion of affine $\mathbf{R}^2$-plane is not yet exactly the good notion associated to Anosov flows. The best picture to have in mind is as follows: remove a point a from an affine $\mathbf{R}^2$-plane A, and take the universal covering of the remaining part. We obtain then a topological space A^* with a collection of subsets which are the liftings of the lines in A, that we call *generalized lines.* There are two kinds of generalized lines: some of them projects to lines in A which do not contain the point a: we call them *complete lines.* The others projects in A as half-lines, connected components of a line in A minus a: we call them *rays.* We say that two complete lines are *parallel* if and only if they intersect the same rays. Finally, for any complete line l, we call *fundamental region associated to* l the union of the rays which intersect l. All the objects we defined satisfies the following axioms:

1. A^* is homeomorphic to $\mathbf{R}^2$;

2. The rays are the leaves of a product foliation of A^*;
3. any two points belonging to the same fundamental region lie on a unique generalized line;
4. if p is point in A^*, and l a complete line intersecting the ray through p, there is a unique line containing p and parallel to l;
5. if two complete lines intersects the same ray, either their intersection is non-empty, either they are parallel;
6. there is a topology on the set G of generalized lines such that the incidence maps $A^* \times A^* \to G$ and $G \times G \to A^*$ are defined on open subsets, and are continuous.

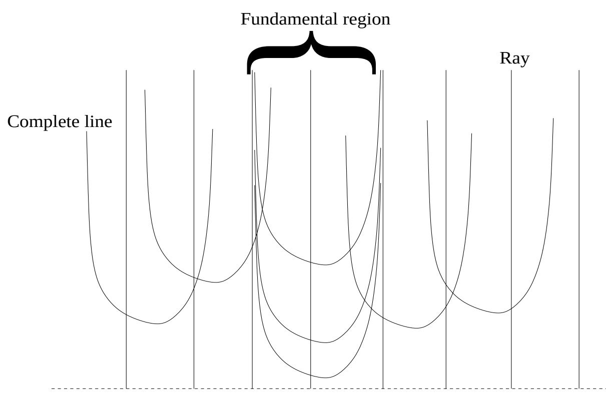


Figure 1: Picture of a lifted affine $\mathbf{R}^2$-plane.

A topological space equipped with a collection of rays and complete lines satisfying the 6 axioms above - with the conventions that a fundamental region is the union of rays meeting a given complete line, and that two complete lines are called parallel if they define the same fundamental region - is called a *lifted affine* $\mathbf{R}^2$*-plane.* A homeomorphism between two lifted affine $\mathbf{R}^2$-planes mapping rays on rays and complete lines on complete lines is called a collineation (Theorem 32.9 of [30], in the case of affine 2-planes, suggests that the continuity is a corollary of the preservation of the generalized lines. I didn't check the details).

Remark 1.1 Let A be an affine $\mathbf{R}^2$-plane, and select a line l in A. Call rays the lines parallel to l, and complete lines the lines which are not parallel to l. Then, all these objects satisfy the six axioms of lifted affine $\mathbf{R}^2$-planes, with the extra condition that there is only one fundamental region: the whole space itself. Inversely, any lifted affine $\mathbf{R}^2$-plane with a unique fundamental region is actually an affine $\mathbf{R}^2$-plane with a distinguished direction of parallel lines. In this case, an affine collineation of the affine $\mathbf{R}^2$-plane is a collineation of the associated lifted affine $\mathbf{R}^2$-plane if and only if it preserves the rays, i.e. it maps l on a line parallel to l.

Consider now a TCP Anosov flow Φ^t on a closed 3-manifold M. Lift the strong foliations to foliations $\widetilde{\mathcal{F}}^{ss}$ and $\widetilde{\mathcal{F}}^{uu}$ in the universal covering $\widetilde{M}$ of M. Let P^s be the leaf space of $\widetilde{\mathcal{F}}^{ss}$, i.e. the quotient of $\widetilde{M}$ by the equivalence relation identifying two points if they are on the same leaf of $\widetilde{\mathcal{F}}^{ss}$. It is not clear from this definition that the topology on this quotient set is Hausdorff, but it is true; actually we will show that P^s is homeomorphic to $\mathbf{R}^2$ (lemma 3.13). We denote by $p^s : \widetilde{M} \to P^s$ the quotient map. The fundamental group Γ of M acts naturally on P^s. The lifting of Φ^t to $\widetilde{M}$ preserves for any time t the foliation $\widetilde{\mathcal{F}}^{ss}$. Therefore, it induces a flow φ^t on P^s, which commutes with the Γ-action. Call rays the orbits of φ^t, and complete lines the projection by p^s of leaves of $\widetilde{\mathcal{F}}^{uu}$. Then (see Theorem 3.17):

Theorem B *For any Anosov flow with the topological contact property, the leaf space P^s, with the collection of rays and complete lines defined as above is a lifted affine $\mathbf{R}^2$-plane, whose collineation group contains Γ and the flow φ^t.*

Theorem A is precisely Axiom 2, its proof consists essentially in checking Axiom 1 and the uniqueness ingredient of Axiom 4; thus, one interest of Theorem B is that it provides an efficient way to remember the proof of Theorem A. In a forthcoming paper, we will see that the lifted affine $\mathbf{R}^2$-plane P^s is the universal covering of some punctured affine $\mathbf{R}^2$-plane if and only if the Anosov flow is a finite covering of an Anosov flow on the unitary tangent bundle of a surface. Therefore, contact Anosov flows (for example, geodesic flows) on unitary tangent bundles over surfaces provide examples of $\mathbf{R}^2$-planes. In general, these affine $\mathbf{R}^2$-planes are not isomorphic to the usual affine plane $\mathbf{R}^2$; we will see that it is the case if and only if the Anosov flow is topologically conjugate to the geodesic flow of a Riemannian surface with constant negative curvature, or to one of the "special reparametrizations" of it introduced in [16] (Proposition 6.2).

Finally, we should mention other important facts that are discussed in section 4:

- any TCP Anosov flow is topologically conjugate to the radial flow induced on the quotient of the flag variety associated to P^s by the action of the fundamental group Γ,

- up to constant factors, there is a unique Γ-invariant Borel measure ν on the leaf space P^s: the Margulis measure.

2 R-covered Anosov flows.

We recall here the properties of $\mathbf{R}$-covered Anosov flows stated in [3]. Let (M, Φ^t) a $\mathbf{R}$-covered Anosov flow. We assume that Φ^t is not topologically equivalent to a suspension. Let $\widetilde{\Phi}^t$ be the lifted flow in the universal covering $\widetilde{M}$ of M. The orbit space Q^Φ of this lifted flow is diffeomorphic to $\mathbf{R}^2$ (Theorem 3.1 of [3], [8]). Let $\widetilde{\mathcal{F}}^s$ and $\widetilde{\mathcal{F}}^u$ be the lifting of the weak foliations in $\widetilde{M}$. Let $\mathcal{L}^s$ and $\mathcal{L}^u$ be the leaf spaces of these foliations. The $\mathbf{R}$-covered hypothesis means that $\mathcal{L}^s$ or $\mathcal{L}^u$ is a manifold homeomorphic to $\mathbf{R}$; actually if it is true for one of them, it is true for the other (Theorem 4.1 of [3]).

The action of the fundamental group Γ of M on $\widetilde{M}$ induces natural actions on the various quotient spaces defined above.

There is a natural Γ-equivariant map $i : Q^\Phi \to \mathcal{L}^s \times \mathcal{L}^u$: for any orbit $\widetilde{\theta}$, the image $i(\widetilde{\theta})$ is the pair of stable and unstable leaves containing $\widetilde{\theta}$. The map i is a homeomorphism onto its image, and, since we assumed that Φ^t is not topologically equivalent to a suspension, it is not surjective (Theorem 2.7 of [3]). Actually, the image of i is the open set in $\mathcal{L}^s \times \mathcal{L}^u$ bounded by the graphs of two Γ-equivariant homeomorphisms α and β from $\mathcal{L}^s$ onto $\mathcal{L}^u$. Let τ^s and τ^u be the compositions $\alpha^{-1} \circ \beta$ and $\beta \circ \alpha^{-1}$: they are increasing homeomorphisms of $\mathcal{L}^s$ and $\mathcal{L}^u$. According to Theorem 4.6 of [3], there is a homeomorphism I of M onto itself, isotopic to the identity, realizing a non-trivial topological equivalence of Φ^t with its inverse (i.e. mapping any oriented orbit onto another orbit with the reversed orientation), and which lifts in $\widetilde{M}$ as a homeomorphism $\widetilde{I}$ which induces on the quotient $Q^\Phi \subset \mathcal{L}^s \times \mathcal{L}^u$ the map $(l^s, l^u) \mapsto (\alpha^{-1}(l^u), \beta(l^s))$.

Proposition 2.1 *The maps τ^s, α and β are Hölder continuous.*

Remark 2.2 Since $\mathbf{R}$-covered Anosov flows are topologically transitive (Theorem 2.5 of [3]), and since we assume that the flow is not topologically equivalent to a suspension, every strong leaf is dense (Theorem 1.8 of [29]). It is easy to show from this property that the flow is topologically mixing (see e.g. chapter 18.3 of [22]).

Proof Consider the topological equivalence I constructed above. According to Theorem 19.1.5 of [22], there is another Hölder continuous orbit equivalence I' arbitrarily C^0-close to I. Let $\widetilde{I}'$ be the lifting of I' C^0-close to $\widetilde{I}$. Let $\widetilde{\theta}$ be an orbit of $\widetilde{\Phi}^t$ preserved by an element γ of Γ. Then, $\widetilde{I}(\widetilde{\theta})$ and $\widetilde{I}'(\widetilde{\theta})$ are fixed points of γ in Q^Φ which are very near. Hence, they are equal since γ-fixed points are discrete (actually, we use also the fact that the γ-fixed points are precisely the $\widetilde{I}$-iterates of $\widetilde{\theta}$). In other words, the induced actions of $\widetilde{I}$ and $\widetilde{I}'$ on Q^{Φ} coincide on the projection of lifted periodic orbits. Since the flow is topologically transitive, periodic orbits are dense. It follows that the actions of $\widetilde{I}$ and $\widetilde{I}'$ on Q^{Φ} are equal. Since I' is Hölder continuous, the proposition follows. ∎

3 TCP Anosov flows.

We don't assume anymore in this section that the Anosov flow Φ^t is $\mathbf{R}$-covered. Even in this case, the orbit space Q^{Φ} is Hausdorff, homeomorphic to $\mathbf{R}^2$. We assume that the strong foliations $\mathcal{F}^{ss}$ and $\mathcal{F}^{uu}$ are oriented: this is not a restriction up to finite coverings. We first recall a crucial result by S. Fenley ([9]) about non $\mathbf{R}$-covered Anosov flow. Since the lifted weak foliations are tangent to $\widetilde{\Phi}^t$, they induce two foliations $\mathcal{G}^s$ and $\mathcal{G}^u$ on Q^{Φ}. They are foliations by closed lines; every leaf of one of these foliations intersects every leaf of the other foliation in at most one point. Moreover, they are Γ-invariant. Obviously, the flow is $\mathbf{R}$-covered if and only if the foliations $\mathcal{G}^s$ and $\mathcal{G}^u$ are individually conjugate to the product foliation of $\mathbf{R}^2$ by horizontal lines. We call open semileaf a connected component (in the leaf) of a leaf of $\mathcal{G}^s$ or $\mathcal{G}^u$ minus one point. A closed semileaf is the closure of an open semileaf. Obviously, we can speak of stable or unstable (semi)leaves.

Definition 3.1 *A lozenge is an open subset subset of Q^{Φ} bounded by four closed semileaves u_1, u_2, s_1, s_2 such that:*

- s_1 and s_2 are stable semileaves; u_1 and u_2 are unstable semileaves;

- s_i and u_i have a common point x_i $(i = 1, 2)$;

- a leaf of $\mathcal{G}^u$ (resp. of $\mathcal{G}^s$) meets s_1 (resp. u_1) if and only if it meets s_2 (resp. u_2).

The points x_1 and x_2 are called the vertices of the lozenge; the semileaves u_1, u_2, s_1 and s_2 are called the sides of the losenge. Let γ be an element of the fundamental group Γ; the lozenge is γ-invariant if its vertices are preserved by γ.

Observe that the sides of a γ-invariant lozenge are preserved by γ.

Definition 3.2 *Two lozenges L_1 and L_2 are (un)stably adjacent if they are disjoint but such that their closures contain both the same (un)stable semileaf.*

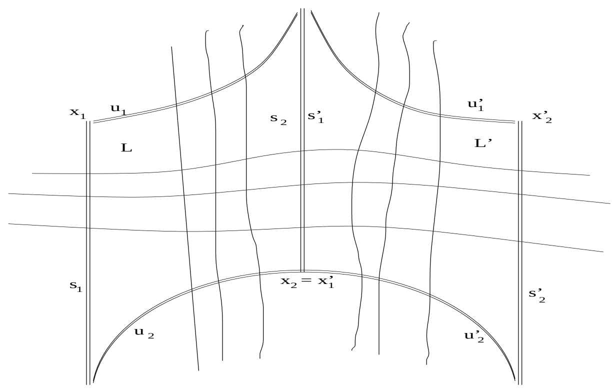


Figure 2: Adjacent lozenges.

Theorem 3.3 **(S. Fenley [9])** *Let Φ^t be an Anosov flow on a closed 3-manifold M. Assume that Φ^t is not $\mathbf{R}$-covered, i.e. that the leaf spaces $\mathcal{L}^s$ and $\mathcal{L}^u$ are not Hausdorff. Let u and u' two leaves of $\mathcal{G}^u$ which are not separated by the quotient topology of $\mathcal{L}^s$. Then, there is an element γ of Γ, and a finite sequence of distinct γ-invariant lozenges $L_1, \ldots, L_k$ such that:*

- *each L_i is stably adjacent to the following L_{i+1},*

- *u contains a side of L_1,*

- *u' contains a side of L_k.*

Choose an auxiliary metric on M. The length metric on leaves defines continuous parametrizations of the oriented foliations $\mathcal{F}^{ss}$ and $\mathcal{F}^{uu}$; i.e. the leaves of these foliations are the orbits of flows that we denote respectively by h^t_s and h^t_u. For any $\epsilon > 0$ and for any element x of M, we denote by $\mathcal{F}^{ss}_\epsilon(x)$ the ϵ-ball centered at x in the leaf of $\mathcal{F}^{ss}$ through x equipped with the induced length metric. We define similarly $\mathcal{F}^{uu}_\epsilon(x)$, $\mathcal{F}^s_\epsilon(x)$ and $\mathcal{F}^u_\epsilon(x)$.

Definition 3.4 *The Anosov flow has a ϵ-rectangle if there is a point x in M and four real numbers t_1, t_2, t_3 and t_4 different from zero and of absolute value less than ϵ such that $h^{t_1}_s \circ h^{t_2}_u(x) = h^{t_3}_u \circ h^{t_4}_s(x)$.*

Definition 3.5 *The Anosov flow has the topological contact property if there is a real positive number ϵ_0 such that the flow has no ϵ_0-rectangle.*

There is a local product structure for the pair $(\mathcal{F}^u, \mathcal{F}^{ss})$, i.e. there is a real positive number ϵ such that, for any point x in M and any point y in the 2ϵ-neighborhood of x, the local strong stable leaf $\mathcal{F}^{ss}_{4\epsilon}(y)$ meets the local leaf $\mathcal{F}^u_{4\epsilon}(x)$ at a unique point $\pi_x(y)$.

The following lemma is obvious:

Lemma 3.6 *The Anosov flow has the topological contact property if and only if there is a real positive number ϵ_0 less than ϵ such that, for any point x in M, and for any point $y \neq x$ in the local leaf $\mathcal{F}^{ss}_{\epsilon_0}(x)$, the projection $\pi_x(\mathcal{F}^{uu}_{\epsilon_0}(y))$ meets $\mathcal{F}^{uu}_{\epsilon_0}(x)$ only at the point x, and nowhere else.* ∎

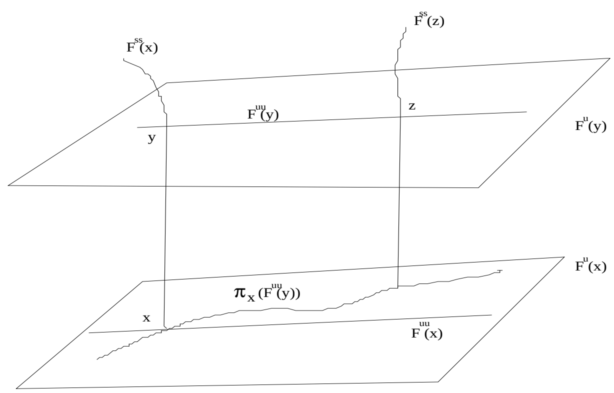


Figure 3: The topological contact property.

A contact Anosov flow is an Anosov flow such that the plane bundle $P = E^{ss} \oplus E^{uu}$ is a contact plane of class C^1. Then, E^{ss} is the intersection between the C^1-bundles $E^{ss} \oplus E^{uu}$ and $E^{ss} \oplus \mathbf{R}X$: it is of class C^1. It is worth to know the following theorem of S. Hurder and A. Katok (Theorem 2.3 of [19]): for any smooth contact Anosov flow, the contact plane is actually smooth. Moreover, a smooth Anosov flow which is not topologically equivalent to a suspension is contact as soon as its strong bundles are both of class C^1.

Lemma 3.7 *Contact Anosov flows have the topological contact property.*

Proof Let (M, Φ^t) be a contact Anosov flow. Since E^{ss} is of class C^1, the maps π_x are all of class C^1. Select a point x in M, and consider a point y in the local leaf $\mathcal{F}^{ss}_{\epsilon_0}(x) \quad (\epsilon_0 < \epsilon)$ and different from x. The local strong unstable leaves contained in $\mathcal{F}^u(y)$ are curves tangent to E^{uu}, which is the intersection between the tangent bundle of $\mathcal{F}^u(y)$ and P. Hence, $\pi_x(\mathcal{F}^{uu}_\epsilon(y))$ is tangent at every point $\pi_x(z)$ to the intersection of the tangent bundle of $\mathcal{F}^u(y)$ and $d\pi_x(P_z)$. But that P is a contact plane means precisely that, at least for sufficiently small ϵ_0, $d_z\pi_x(P_z)$ is different from $P_{\pi_x(z)}$. Therefore, $\pi_x(\mathcal{F}^{uu}_\epsilon(y))$ is everywhere tangent to a local vector field which is nowhere tangent to the strong stable leaves. The lemma follows. ∎

From now, we assume that the Anosov flow is TCP, i.e. admits the topological contact property. Let s, t be two real positive numbers less than ϵ_0; for any element x of M define $\tau_x(s,t)$ as the unique time τ less than 4ϵ such that $\Phi^{-\tau}\pi_x(h^s_u \circ h^t_s)$ belongs to $\mathcal{F}^{uu}(x)$. By definition of TCP Anosov flows, when s and t are not zero, $\tau_x(s,t)$ is not zero. Hence, all the $\tau_x(s,t)$ have the same sign. Reversing the orientation of one of the strong foliations, we can assume that this sign is positive.

Definition 3.8 *Let F_1 and F_2 be different leaves of $\widetilde{\mathcal{F}}^u$. Let $\Omega^s(F_1, F_2)$ be the intersection between F_1 and the $\widetilde{\mathcal{F}}^{ss}$-saturation of F_2. We denote by $h^s(F_1, F_2)$: $\Omega^s(F_1, F_2) \mapsto \Omega^s(F_2, F_1)$ the holonomy map along strong stable leaves.*

Lemma 3.9 *For any pair (F_1, F_2), $\Omega^s(F_1, F_2)$ is an open connected subset of F_1 which is $\widetilde{\Phi}^t$-invariant.*

Proof The openness and $\widetilde{\Phi}^t$-invariance are obvious. Let G_1 and G_2 be the projections of F_1 and F_2 in Q^Φ: they are leaves of $\mathcal{G}^u$. Since Q^Φ is diffeomorphic to $\mathbf{R}^2$, the set of points of G_1 whose $\mathcal{G}^s$-leaves meet G_2 is connected. The connectedness of $\Omega^s(F_1, F_2)$ follows immediatly. ∎

When $\Omega^s(F_1, F_2)$ is not empty, the sign of the time needed to reach F_2 along the flow $\widetilde{h}_s$ and starting from F_1 is constant on $\Omega^s(F_1, F_2)$. If this sign is positive we write $F_1 \prec F_2$, and we write $F_1 \succ F_2$ if it is negative.

Definition 3.10 *A graph on a leaf of $\widetilde{\mathcal{F}}^u$ is a continuous path in the leaf which meets every orbit of $\widetilde{\Phi}^t$ in at most one point.*

Let F be a leaf of $\widetilde{\mathcal{F}}^u$ and $\widetilde{x}$ a point of F. The flows $\widetilde{\Phi}^t$ and $\widetilde{h}^s_u$ provide a parametrization of F by $\mathbf{R}^2$, the point of coordinates (t, s) being $\widetilde{\Phi}^s \circ \widetilde{h}^t_u(\widetilde{x})$. We call *special parametrizations* this type of parametrization. When we identify F with $\mathbf{R}^2$ in this way, it is obvious that a continuous path c in F is a graph if and only if it is the graph in the usual meaning of some function f from an interval of $\mathbf{R}$ into $\mathbf{R}$. If f is (strictly) increasing (respectively decreasing), we say that c is increasing (resp. decreasing). Observe that the notion of increasing and decreasing graphs does not depend on the parametrization of the flows, it just depends on the orientations of $\widetilde{\Phi}$ and $\widetilde{\mathcal{F}}^{uu}$.

Remark 3.11 A decreasing or increasing graph in F is a path which intersects any $\widetilde{\Phi}^t$-orbit in at most one point, and which intersects any $\widetilde{\mathcal{F}}^{uu}$-leaf in at most one point.

Lemma 3.12 *Let F_1 and F_2 be two leaves of $\widetilde{\mathcal{F}}^u$. We select the orientations of the strong foliations such that $F_1 \prec F_2$ and such that $\tau_x(s,t)$ is positive for any pair (s,t) in $]0,\epsilon_0[\times]0,\epsilon_0[$. Then, for any strong stable leaf f^{uu} in F_1, the image of $f^{uu} \cap \Omega^s(F_1,F_2)$ by $h^s(F_1,F_2)$ is an increasing graph.*

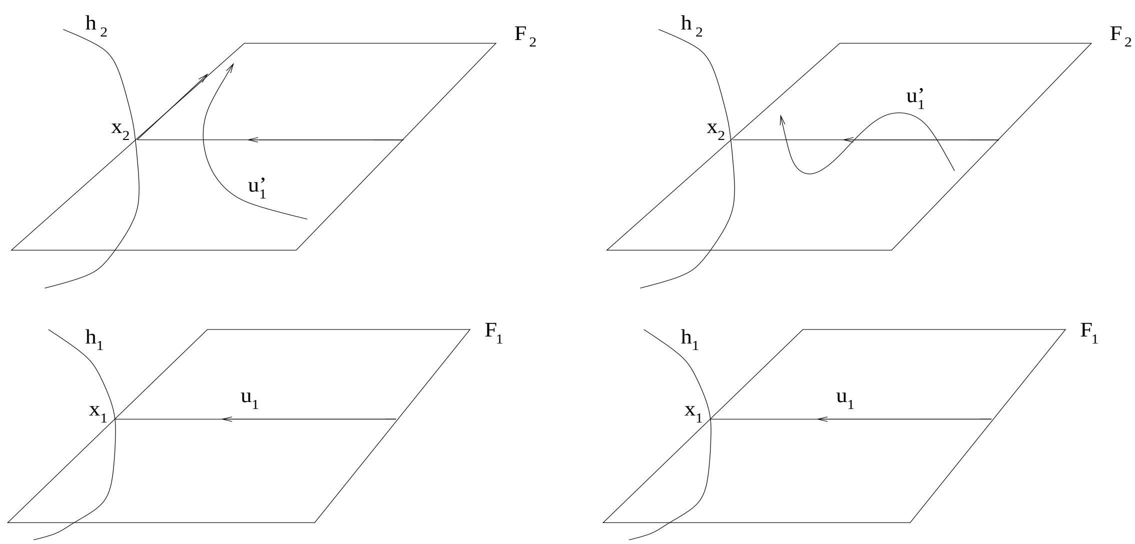
Proof Denote by c the image of $f^{uu} \cap \Omega^s(F_1,F_2)$ by $h^s(F_1,F_2)$. Since f^{uu} meets every leaf of $\widetilde{\mathcal{F}}^s$ in at most one point, c is a graph. To be an increasing graph is a local property, i.e. it is enough to see that for any point x in $f^{uu} \cap \Omega^s(F_1,F_2)$, the graph c is increasing near $h^s(F_1,F_2)(x)$. Replacing x by some $\widetilde{\Phi}^T(x)$, we can assume that, near x, the time needed to reach F_2 (and thus c) along $\widetilde{\mathcal{F}}^{ss}$ is less than ϵ_0. Then, the positivity of the $\tau_x(s,t)$ is precisely what we need in order to show that c is increasing near $h^s(F_1,F_2)(x)$. ∎

Let P^s be the leaf space of $\widetilde{\mathcal{F}}^{ss}$. Let $p^s : \widetilde{M} \to P^s$ denote the projection map.

Proposition 3.13 *P^s equipped with the quotient topology is homeomorphic to $\mathbf{R}^2$.*

Proof The main difficulty is to prove the Hausdorff separation property for P^s. We argue by contradiction, assuming the existence of two different leaves h_1 and h_2 of $\widetilde{\mathcal{F}}^{ss}$ such that any $\widetilde{\mathcal{F}}^{ss}$-saturated neighborhood of h_1 meets any $\widetilde{\mathcal{F}}^{ss}$-saturated neighborhood of h_2. It implies that the $\widetilde{\mathcal{F}}^s$-leaves containing h_1 and h_2 are not separated from one another. According to Theorem 3.3, there is an element γ of Γ preserving these two leaves. Hence, there are two real numbers T_1 and T_2 such that $\widetilde{\Phi}^{T_i}\gamma$ preserves h_i $(i=1,2)$. Moreover, h_i contains a (unique) fixed point x_i of $\widetilde{\Phi}^{T_i}\gamma$ $(i=1,2)$. Let F_i be the leaf of $\widetilde{\mathcal{F}}^u$ containing

x_i. Since h_1 and h_2 are not separated, x_1 belongs to the closure of $\Omega^s(F_1, F_2)$; actually, since h_1 and h_2 are different, x_1 belongs to the boundary of $\Omega^s(F_1, F_2)$. Let u_1 be the intersection between $\Omega^s(F_1, F_2)$ and the leaf of $\widetilde{\mathcal{F}}^{uu}$ containing x_1, and let u_1' be the image of u_1 by $h^s(F_1, F_2)$. Since F_1, F_2 are $\widetilde{\Phi}^{T_1}\gamma$-invariant, the same is true for u_1 and u_1'. Consider the special parametrization of F_2 by $\mathbf{R}^2$ for which x_2 is the origin. Then, the action on F_2 of $\widetilde{\Phi}^{T_1}\gamma$ is of the form $(t, s) \mapsto (\kappa(t), s+T_1-T_2)$, where κ is a contraction or dilatation of $\mathbf{R}$ admitting 0 as unique fixed point (it is conjugate to the action of $\widetilde{\Phi}^{T_2}\gamma$ on $\widetilde{\mathcal{F}}^{uu}(x_2)$). If $T_1 = T_2$, then every leaf of $\widetilde{\mathcal{F}}^{uu}$ in F_2 are $\widetilde{\Phi}^{T_1}\gamma$-invariant. But, according to Lemma 3.12, every such leaf meets u_1' in at most one point. This is impossible since u_1' is $\widetilde{\Phi}^{T_1}\gamma$-invariant (see figure 4). Therefore, $T_2 \neq T_1$. It follows that u_1' is asymptotic to the $\widetilde{\Phi}^t$-orbit of x_2 (see figure 4).



Case $T_2 < T_1$: u_1' is asymptotic to the orbit through x_2 Case $T_2 = T_1$: u_1' cannot be an increasing or decreasing graph

Figure 4: Non-Hausdorff pair of strong stable leaves.

Since h_1 and h_2 are not separated, there are elements y_n of $\Omega^s(F_1, F_2)$ converging to x_1 and such that $y_n' = h^s(F_1, F_2)(y_n)$ converge to x_2. Let t_n be the unique real number such that $\widetilde{\Phi}^{t_n}(y_n)$ belongs to u_1: the t_n tend to 0. On the other hand, $\widetilde{\Phi}^{t_n}(y_n')$ belongs to u_1'. We obtain a contradiction since u_1' is asymptotic to the $\widetilde{\Phi}^t$-orbit of x_2 and the y_n' tend to x_2.

This contradiction shows that P^s is Hausdorff. Now, every leaf of $\widetilde{\mathcal{F}}^u$ is homeomorphic to $\mathbf{R}^2$ and intersects every leaf of $\widetilde{\mathcal{F}}^{ss}$ in at most one point.

Therefore, the restrictions of p^s to leaves of $\widetilde{\mathcal{F}}^u$ are charts of some manifold structure on P^s. Moreover, if we select any parametrization f^t of $\widetilde{\mathcal{F}}^{ss}$, for any leaf F of $\widetilde{\mathcal{F}}^u$, the map $(x,t) \mapsto f^t(x)$ is a homeomorphism from $F \times \mathbf{R}$ It follows p^s is a locally trivial fibration by lines. Since $\widetilde{M}$ is homeomorphic to $\mathbf{R}^3$, P^s is homeomorphic to $\mathbf{R}^2$. ∎

The group Γ acts naturally on P^s. Observe that this action is free since no leaf of $\mathcal{F}^{ss}$ is a circle. The flow $\widetilde{\Phi}$ defines a flow φ^t on P^s. We call *rays* the orbits of φ^t, and *complete lines* the projections by p^s of the leaves of $\widetilde{\mathcal{F}}^{uu}$. A *generalized line* is a ray or a complete line. The following lemma is an immediate corollary of Lemma 3.12 (see Remark 3.11):

Lemma 3.14 *The intersection between two generalized lines contains at most one point.* ∎

We can now deduce Theorem A:

Corollary 3.15 *TCP Anosov flows are* $\mathbf{R}$*-covered.*

Proof Assume that the TCP Anosov flow Φ^t is not $\mathbf{R}$-covered. Then, according to Theorem 3.3, Q^Φ contains two unstably adjacent lozenges L and L'. Let $u_2 = u'_1$ be the common side of L and L'; let s_1 and s'_2 be sides of L and L' asymptotic to u_2, let s_2, s'_1 be the other stable sides of L, L', and let be u'_2 be the last unstable side of L'. The sides s_2, s'_1 are both contained in the same leaf s of $\mathcal{G}^s$; more precisely, there is an element $\widetilde{\theta}$ of s such that $s \setminus \{\widetilde{\theta}\}$ is the union of s_2 and s'_1. Denote by S, S_1, S'_2, U_2 and U'_2 the preimages in $\widetilde{M}$ of s, s_1, s'_2, u_2 and u'_2: the third ones are leaves of $\widetilde{\mathcal{F}}^s$, and the last ones are leaves of $\widetilde{\mathcal{F}}^u$. Project all these objects in P^s: we obtain three rays φ, φ_1 and φ'_2, and two complete lines λ_2 and λ'_2 (see figure 5).

Moreover, near to λ_2, there is a complete line λ which meets φ_1 and φ. Since the rays φ_1 and φ'_2 are not separated, there is a ray near φ_1 which intersects λ and λ'_2. Hence, it follows from topological properties of lines in the plane that there is a time T for which $\varphi^T(\lambda)$ intersects φ_1, λ'_2 and φ. But since φ_1 and φ belongs to the same connected component of $P^s \setminus \lambda'_2$, this is possible only if $\varphi^T(\lambda)$ intersects λ'_2 in at least two points. This is in contradiction with Lemma 3.14. ∎

Our goal is to prove Theorem B, i.e. to check the six axioms of lifted affine $\mathbf{R}^2$-planes for P^s. Observe that a fundamental region, which by definition is

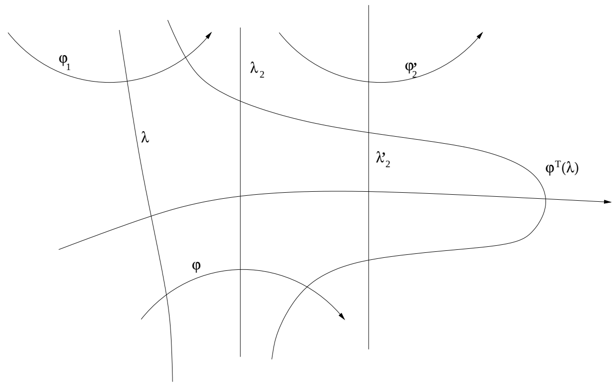


Figure 5: Non-Hausdorff pair of orbits.

the union of rays which meets a given complete line, is in the context of TCP Anosov flows the projection of a leaf of $\widetilde{\mathcal{F}}^u$. Therefore, it is open. Moreover, two complete lines are parallel if and only if one is the image of the other by some φ^t.

Lemma 3.16 *Two points in P^s belonging to the same fundamental region are contained in one and only one generalized line.*

Proof Let s_0 be any element of P^s, and V be any fundamental region containing s_0, i.e. the φ^t-saturation of some complete line l_0 containing s_0. Let r_0 be the ray through s_0. The line l_0 is the projection of some $\widetilde{\mathcal{F}}^{uu}$-leaf containing an element x of the $\widetilde{\mathcal{F}}^{ss}$-leaf s_0, and, as we observed previously, V is the projection of the $\widetilde{\mathcal{F}}^u$-leaf containing x. There is a special parametrization of V by $\mathbf{R}^2$ such that l_0 is the horizontal line $\mathbf{R} \times \{0\}$, r_0 the vertical line $\{0\} \times \mathbf{R}$, and s_0 the point $(0,0)$.

Define the maps $f_x^+ : \mathbf{R} \times \mathbf{R}_+^* \to P^s$ and $f_x^- : \mathbf{R} \times \mathbf{R}_+^* \to P^s$ by $f_x^+(s,t) = p^s \circ \widetilde{h}_u^s \circ \widetilde{h}_s^t(x)$ and $f_x^-(s,t) = p^s \circ \widetilde{h}_u^s \circ \widetilde{h}_s^{-t}(x)$ (where $\widetilde{h}_u^s$ and $\widetilde{h}_s^t(x)$ are parametrizations of $\widetilde{\mathcal{F}}^{ss}$ and $\widetilde{\mathcal{F}}^{uu}$). According to Lemma 3.14, these two maps are injective. Since they are continuous, and since P^s is homeomorphic to $\mathbf{R}^2$, they are open. Denote by U^+ and U^- their images: their union is the set of points of P^s belonging to a complete line containing s_0 (except s_0 itself). For any real number t, denote by $c_t^\pm$ the image of the map $s \mapsto f_x^\pm(s,t)$.

Let V^+ and V^- be the connected components of V minus r_0 containing respectively $U^+ \cap V$ and $U^- \cap V$. In order to prove the lemma, we just have to prove that U^+ contains V^+ and that U^- contains V^-.

Since U^+ is the union of the c_t^+, $U^+ \cap V$ is actually the union of increasing and decreasing graphs.

We distinguish now two cases:

Case 1*: The Anosov flow is not topologically equivalent to a suspension:*

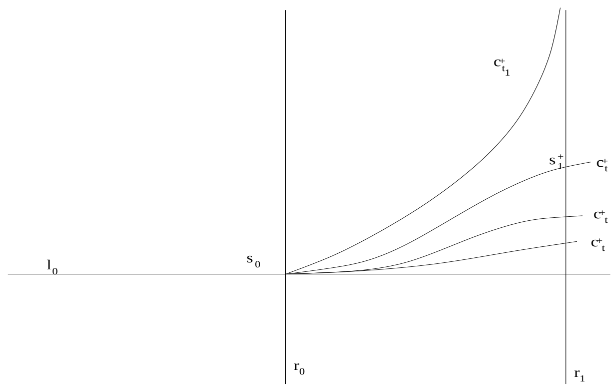


Figure 6: Case 1.

We can apply the results of the section 2. The set of rays is naturally identified with the leaf space $\mathcal{L}^s$. We can interpret the translation $\tau^s : \mathcal{L}^s \to \mathcal{L}^s$ as a permutation of rays. Nearly by definition, the fundamental regions are the open subsets bounded by r and $\tau^s(r)$, where r is a ray. Denote by ρ the function from U^+ into $\mathcal{L}^s$ which maps a point s to the supremum in $\mathcal{L}^s \approx \mathbf{R}$ of the set of rays which intersect the complete line containing s_0 and s, i.e. the unique graph c_t^+ containing s. This function is very similar to the functions α and β defined in the section 2, and since α and β are homeomorphisms, it is easy to deduce that ρ is continuous, that the image of ρ is the interval $]r_0, \tau^s(r_0)[$, and that two points in U^+ have the same image by ρ if and only if they belong to the same c_t^+.

Assume that V^+ is not contained in U^+, i.e. that there is a ray r_1 in V^+ which is not contained in U^+. The intersection between r_1 and U^+ is a segment $]s_1^-, s_1^+[$. We consider here only the case $s_1^+ < +\infty$; the other case $s_1^- > -\infty$ can be treated in a similar way. Let t_1 be the unique parameter for which $\rho(c_{t_1}^+) = r_1$. Observe that t_1 is positive. Then, for any positive t smaller than t_1, c_t^+ meets r_1. In other words, for t in $[0, t_1]$, the c_t^+ are graphs of continuous functions ψ_t defined on $[r_0, r_1[$, taking value in $\mathbf{R}_+^*$, and such that:

$$\lim_{r \to r_1} \psi_t(r) < b \quad (0 < t < t_1)$$

$$\lim_{r \to r_1} \psi_{t_1}(r) = +\infty$$

In the first equation, b is the second coordinate of s_1 in $V \approx \mathbf{R}^2$. The second equation follows from the fact that $c_{t_1}^+$ is an embedded path which does not intersect r_1.

Since the c_t^+ are increasing graphs, the functions ψ_t are increasing. Therefore, for t less than t_1, the function ψ_t takes value only in $[0, b]$. On the other hand, since the function f^+ is continuous, $\psi_t(r)$ is continuous in the parameter t. We obtain a contradiction since ψ_{t_1} admits arbitrarly large values. This contradiction shows that any ray contained in V^+ belongs to U^+. Hence, U^+ contains V^+. Similarly, U^- contains V^-. This achieves the proof of the lemma in this case.

Case 2: the Anosov flow is topologically equivalent to a suspension: We will prove later that actually this case cannot occur (Proposition 5.1). But the impossibility of this case is still far from obvious and will depend on the present study.

In this case, every leaf of $\widetilde{\mathcal{F}}^s$ meets every leaf of $\widetilde{\mathcal{F}}^u$. It follows that the whole P^s is the unique fundamental region. In other words, any complete line containing s_0 is the graph of a function $\psi_t : l_0 \to r_0$. When t is positive, ψ_t is increasing, and when t is negative, ψ_t is decreasing. By continuity of f^+ and f^-, the map $t \mapsto \psi_t(r)$ is continuous for every r. Let g^+ (resp. g^-) be the limit of the functions ψ_t when t tends to $+\infty$ (resp. $-\infty$). The function $g^+ : l_0 \to r_0 \cup \{+\infty\}$ is increasing, and $g^- : l_0 \to r_0 \cup \{-\infty\}$ is decreasing. Then, U^+ (resp. U^-) is the open set in V^+ bounded by the graphs of g^+ and g^- over the positive (resp. negative) part of l_0.

Assume for a moment that the ray r_0 is preserved by some element γ of Γ. Then, there is a real number T such that r_0 is fixed pointwise by $\varphi^T \circ \gamma$. There is a special parametrization of P^s by $\mathbf{R}^2$ such that s_0 is of coordinates $(0,0)$, such that r_0 is the vertical line, and such that the horizontal line is the unique $\varphi^T \circ \gamma$-invariant complete line through s. The action of $\varphi^T \circ \gamma$ on P^s is of the form $(t, s) \mapsto (\kappa(t), s)$ where κ is a contraction or a dilatation. But $\varphi^T \circ \gamma$ must preserve U, and therefore the boundary of it. Since g^+ is increasing and g^- is decreasing, the only possibility is $g^+ = +\infty$ and $g^- = -\infty$.

In other words, when the ray r_0 through s_0 is invariant by some element of Γ, every point of P^s belongs to a generalized line containing s_0.

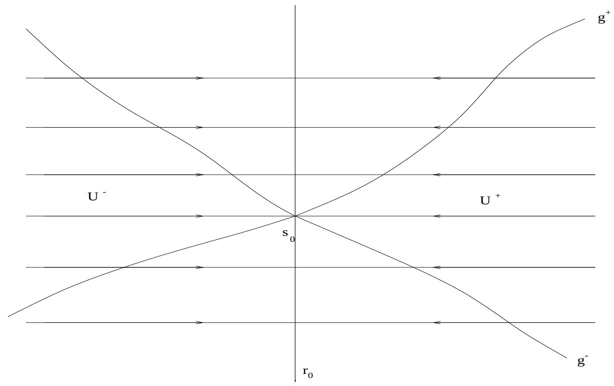


Figure 7: Action of $\varphi^T \circ \gamma$ when r is γ-invariant.

We go back to the general case: what we did above shows that $U^+ \cup U^- \cup r_0$ contains all the rays preserved by some element of Γ. The lemma follows since the union of these rays is dense in P^s (this last assertion follows from the density of periodic orbits of topologically transitive Anosov flows). ∎

We proved all the statements necessary for the proof of Theorem B:

Theorem 3.17 *The space P^s, equipped with the generalized lines defined as above, is a lifted affine $\mathbf{R}^2$-plane.*

Proof We check the six axioms one by one:

(1) Axiom 1 is Proposition 3.13.

(2) Axiom 2 follows from Theorem 3.15 since the rays are the projections in P^s of the leaves of $\widetilde{\mathcal{F}}^s$.

(3) The uniqueness part of Axiom 3 is Lemma 3.14. The existence part is lemma 3.16.

(4) Axiom 4 is obvious since the parallels of a complete line are its φ^t-iterates.

(5) The TCP property is symmetric: we always privileged the stable foliation, but all what we did apply if we exchange the role of $\widetilde{\mathcal{F}}^{ss}$ and $\widetilde{\mathcal{F}}^{uu}$. Let P^u be the leaf space of $\widetilde{\mathcal{F}}^{uu}$: according to our previous work, Axiom 3 is true in P^u. But P^u can be considered as the dual of P^s, i.e. the set of complete lines of P^s. Axiom 5 is nothing but the dual version in P^s of Axiom 3 in P^u.

(6) The topology that we have to consider on the set of generalized lines is obvious; and Axiom 6 is obvious also for this topology.

∎

4 Margulis measure and flag variety

For any topologically transitive Anosov flow, Margulis ([24]) constructed a foliated measure supported on weak unstable leaves, depending continuously on the leaf, multiplied by the Anosov flow, and preserved by the holonomy along strong stable leaves. Hence, it induces a Borel measure ν on P^s such that:

- ν is nonatomic, and any open subset is of positive ν-measure,

- ν is preserved by the fundamental group Γ,

- there is a constant $\lambda > 1$ such that, for any t, the measure is multiplied by λ^t under the action of φ^t.

Moreover, the strong stable foliation is uniquely ergodic [1]. It means that ν is the unique Γ-invariant Borel measure on P^s up to constant factor.

Definition 4.1 *A Γ-invariant measure ν is called a Margulis measure*

We now define another important notion:

Definition 4.2 *Let P be a lifted affine $\mathbf{R}^2$-plane. The flag variety associated to P is the set of pairs (p, l) where p is a point of p and l a complete line containing p.*

The flag variety associated to any lifted affine $\mathbf{R}^2$-plane is always homeomorphic to $\mathbf{R}^3$. Any collineation induces naturally a transformation on the flag variety. Consider a TCP Anosov flow Φ^t. There is a natural map from the universal covering $\widetilde{M}$ onto the flag variety F associated to P^s: the image of a point x of $\widetilde{M}$ is the pair $(\widetilde{\mathcal{F}}^{ss}(s), l)$, where l is the projection in P^s of $\widetilde{\mathcal{F}}^{uu}(x)$.

[1] Unique ergodicity of strong foliations is proved in [6] when the Anosov flow is topologically mixing, and we will see that TCP Anosov flows are topologically mixing (Remark 5.4).

Obviously, this map is a Γ-equivariant homeomorphism. Moreover, it maps the flow $\widetilde{\Phi}^t$ on the flow Ψ^t induced by the collineations φ^t of P^s.

It follows that *the TCP Anosov flow Φ^t is topologically conjugate to the flow induced by Ψ^t on the quotient of the flag variety F by the collineation group Γ.*

5 Suspensions and the topological contact property.

Proposition 5.1 *A TCP Anosov flow is not topologically equivalent to a suspension.*

Remark 5.2 This lemma is well-known in the particular case of contact Anosov flow, but the proof is more subtle in the general TCP setting. Our proof is quite sophisticated and relies on all the previous results.

Remark 5.3 According to a Theorem of V.V. Solodov, an Anosov flow on a 3-manifold is topologically equivalent to a suspension if and only if it has the splitting property, i.e. if every leaf of $\widetilde{\mathcal{F}}^s$ meets every leaf of $\widetilde{\mathcal{F}}^u$ ([32] or Theorem 2.7 of [3]). In the context of TCP Anosov flows, and according to lemma 3.16, the splitting property would mean that any two points in P^s belongs to some generalized line. Hence, an equivalent formulation of Proposition 5.1 is: *the lifted affine $\mathbf{R}^2$-plane associated to a TCP Anosov flow is never an affine $\mathbf{R}^2$-plane* (see Remark 1.1).

Remark 5.4 According to Remark 2.2, it follows that TCP Anosov flows are topologically mixing.

We will need the following definition:

Definition 5.5 *Let I be a compact segment in a complete line of P^s. The radial triangle $T(I)$ of base I is the union of all the $\varphi^t(I)$ for negative t.*

Observe that different basis define different radial triangles. Let ν be a Margulis measure (see Definition 4.1).

Lemma 5.6 *Every radial triangle is of finite ν-measure.*

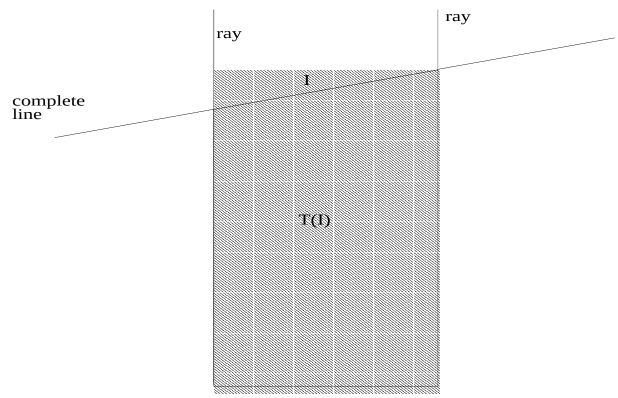


Figure 8: A radial triangle.

Proof Let T be a radial triangle of base I. For any natural integer i, let T_i be the union of the $\varphi^t(I)$ for t in $[-i-1, -i]$. Every T_i is compact. Hence, its ν-measure m_i is finite. The ν-measure of T is less than the sum of the m_i. Since φ^t multiplies ν by λ^t, the measure m_i is less than $\lambda^{-i}m_0$. The lemma follows since $\lambda > 1$. ∎

Proof of 5.1 Assume *a contrario* that the TCP Anosov flow is topologically equivalent to the suspension of some Anosov diffeomorphism of the torus T. According to Remark 5.3, the whole P^s is a fundamental region. According to Lemma 3.16, any two points in P^s belongs to a generalized line. Since the flow is a suspension, the ambient manifold is a torus bundle over the circle. There is a cyclic covering $\widehat{M}$ of M which is homeomorphic to $T \times \mathbf{R}$, and the lifting of Φ^t in $\widehat{M}$ is a flow $\widehat{\Phi}^t$ such that any orbit of $\widehat{\Phi}^t$ is a closed embedding of $\mathbf{R}$ which intersects every $T \times \{*\}$ in one and only one point. In particular, the orbit space of $\widehat{\Phi}^t$ is homeomorphic to T. The Galois group of the covering $\widehat{M} \to M$ is a normal subgroup H of Γ isomorphic to $\mathbf{Z}^2$. Since no leaf of $\mathcal{F}^{ss}$ is a circle, the action of Γ on P^s is free. Actually, since $\widehat{\Phi}^t$ has no periodic orbit, no periodic orbit of Φ^t is freely homotopic to a loop in a fiber. It follows that H acts freely on the set of rays of P^s. Select an element a of H. Consider an element $m = (s, l)$ of the flag variety F associated to P^s. Let r be the ray through s: the image $a(r)$ intersect the complete line l at a unique point s'. Then, $[s, s']$ is a compact segment of l. Denote by $T(m)$ the radial triangle of basis $[s, s']$ (see definition 5.5), and by $\mu(m)$ the ν-measure of $T(m)$. Then, $\mu : F \to \mathbf{R}$ is a continuous function. Since any element of H commutes with a, μ is H-invariant. Moreover, $\mu(\Psi^t(m)) = \lambda^t\mu(m)$, where Ψ^t is the flow induced by φ^t on F. Therefore, μ is not constant, and the set-locus $\mathcal{E}$ where μ equals one is a H-invariant closed subset of F which meets every orbit of Ψ^t at one and only one point. We have seen in the section 4 that Φ^t is topologically conjugate to the flow induced by Ψ^t on the quotient of F by Γ. It follows that the quotient of $\mathcal{E}$ by H is homeomorphic to the orbit space of $\widehat{\Phi}^t$, i.e. to T. Finally, μ is injective along every leaf of $\widetilde{\mathcal{F}}^{ss}$ since if $m = (s, l)$ and $m' = (s, l')$ are two different points of the same strong stable leaf, one of the radial triangles $T(m)$ and $T(m')$ contains the other. It follows that $\mathcal{E}$ can be interpreted also as the leaf space of $\widetilde{\mathcal{F}}^{ss}$, i.e. P^s.

What we finally obtained is the following: *the action of H on P^s is free and properly discontinuous; the quotient Q of this action is homeomorphic to the torus T.* The Borel measure ν induces a mesure $\bar{\nu}$ on the compact Q, and φ^t induces a flow $\bar{\varphi}^t$ on Q. The measure $\bar{\nu}$ has a total mass which is finite and not null. But this measure is multiplied under the action of $\bar{\varphi}^t$ by the factor λ^t: contradiction. ∎

6 Desarguian Anosov flows

Definition 6.1 *A TCP Anosov flow is called Desarguian if the leaf space P^s is affinely isomorphic to the universal covering of the usual punctured affine plane $\mathbf{R}^2 \setminus \{0\}$.*

As we have indicated in the introduction, an alternative definition of (generalized) geodesic flow of Riemanian surfaces with constant negative curvature is the following: consider a discrete uniform subgroup Γ of $\widetilde{SL}(2,\mathbf{R})$, the universal covering of $SL(2,\mathbf{R})$. Let $\widetilde{\mathcal{R}}$ be the universal covering of $\mathbf{R}^2 \setminus \{0\}$, viewed as a lifted affine $\mathbf{R}^2$-plane. Observe that the affine action of $SL(2,\mathbf{R})$ on $\mathbf{R}^2 \setminus \{0\}$ lifts as an action of $\widetilde{SL}(2,\mathbf{R})$ on $\widetilde{\mathcal{R}}$. The collineation group of $\widetilde{\mathcal{R}}$ contains another 1-parameter subgroup: the flow φ_0^t, lifting of the radial flow of $\mathbf{R}^2 \setminus \{0\}$.

Let F be the flag variety associated to $\widetilde{\mathcal{R}}$ (see Definition 4.2). It is equipped with an action of $\widetilde{SL}(2,\mathbf{R})$, which commutes with the flow Ψ_0^t induced by φ_0^t. Let M_Γ be the quotient of F by Γ: it is a 3-manifold (actually, a Seifert manifold). Let Φ_Γ^t be the flow induced by Ψ_0^t on M_Γ. Then, Φ_Γ^t is an Anosov flow which is smoothly conjugate (up to finite covers) to the geodesic flow of a Riemannian surface with constant negative curvature. The strong stable leaves of Φ_Γ^t lifted in the universal covering are nothing but the fibers of the fibration of F over $\widetilde{\mathcal{R}}$. It follows that Φ_Γ^t is Desarguian.

Now, we can modify the action of Γ on F: let $\rho : \Gamma \to \mathbf{R}$ be a morphism. We now define a new action of Γ on F: the element γ of Γ maps the element x of

F on $\Psi_0^{\rho(\gamma)} \circ \gamma(x)$. Then, this new action of Γ is still an action by collineations, and it commutes with Ψ_0^t. It happens that for some morphisms ρ, the new associated Γ-action on F remains free and properly discontinuous. Then, the quotient is a 3-manifold diffeomorphic to M_Γ, and Ψ_0^t induces on this quotient an Anosov flow $\Phi_{\Gamma,\rho}^t$. The Anosov flows constructed in this way are the exotic Anosov flows defined in [16]: indeed, what we did here is just to reformulate the definition given in [16]. The fact that $\Phi_{\Gamma,\rho}^t$ is an Anosov flow is not completely immediate, we refer to [16] for the details. The liftings in F of the leaves of the strong foliations of the exotic Anosov flows are still the fibers of the fibration $F \to \widetilde{\mathcal{R}}$. Therefore, the exotic Anosov flows are Desarguian. Inversely:

Proposition 6.2 *A TCP Anosov flow is Desarguian if and only if it is topologically conjugate to an exotic Anosov flow.*

Proof Let Φ^t be a Desarguian TCP Anosov flow. Here, Γ will denote the fundamental group of the ambient manifold of Φ^t. Let F be the flag variety of $\widetilde{\mathcal{R}}$. Then, Φ^t is topologically conjugate to the flow Φ_0^t, which is the flow induced by the radial flow Ψ^t on the quotient of F by Γ. It follows that Φ_0^t is an Anosov flow. Observe that Φ_0^t is smooth. Moreover, the strong stable leaves in the universal covering $\widetilde{M} \approx F$ are the fibers of the projection of F over P^s. Hence, the strong stable foliation is smooth. A similar argument shows that the strong unstable foliation is smooth also. The proposition then follows from the main theorem of [16]. ∎

Remark 6.3 The proof given here is short but very inelegant. The main difficulty in [16] consists in showing that if an Anosov flow has smooth strong foliations, then the leaf space of $\widetilde{\mathcal{F}}^{ss}$ is locally modelled on $\mathbf{R}^2$ so that the leaves of $\widetilde{\mathcal{F}}^{uu}$ project in this leaf space as straight lines (Proposition 3.7 of [16]). In our case, we know that immediately. The real difficulties here for the proof of Proposition 6.2 are:

- first, we have to show that the collineation group of $\widetilde{\mathcal{F}}$ is $\widetilde{SL}(2,\mathbf{R}) \times \mathbf{R}$, where the factor $\mathbf{R}$ is the 1-parameter subgroup φ_0^t: this is quite easy, but we won't discuss it here.

- then, we have to show that any subgroup of $\widetilde{SL}(2,\mathbf{R}) \times \mathbf{R}$ which acts freely and properly discontinuously on F is of the form $(\gamma, \rho(\gamma))$, where γ describes a discrete uniform subgroup Γ of $\widetilde{SL}(2,\mathbf{R})$, and where $\rho : \Gamma \to \mathbf{R}$ is a morphism. This is precisely the matter of Theorem 6.4 in [16].

Remark 6.4 The main result of [16] can now be rewritten as follows: *the only Anosov flows in dimension* 3 *admitting smooth splittings are the Desarguian Anosov flows.*

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