Hydrodynamics near the liquid-vapor critical point

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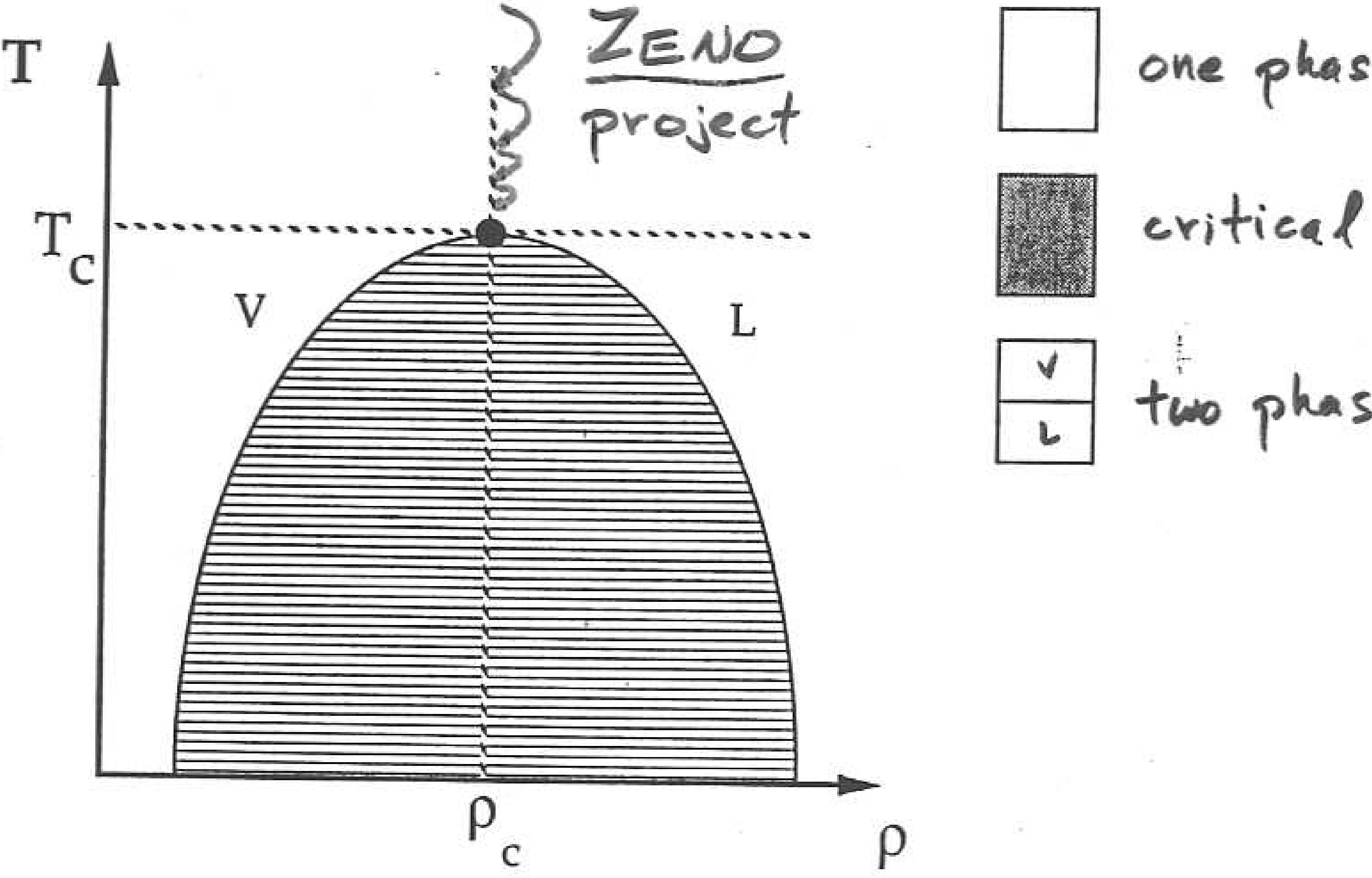
Outline

- A brief course on critical phenomena
- Hydrodynamic phenomena near T_c

⋆ Asymptotic modeling: incompressible flow for highly compressible fluids

⋆ Internal wave damping

Phase Diagram with Critical Point



Anomalous scaling in liquid-vapor phase transitions is universal

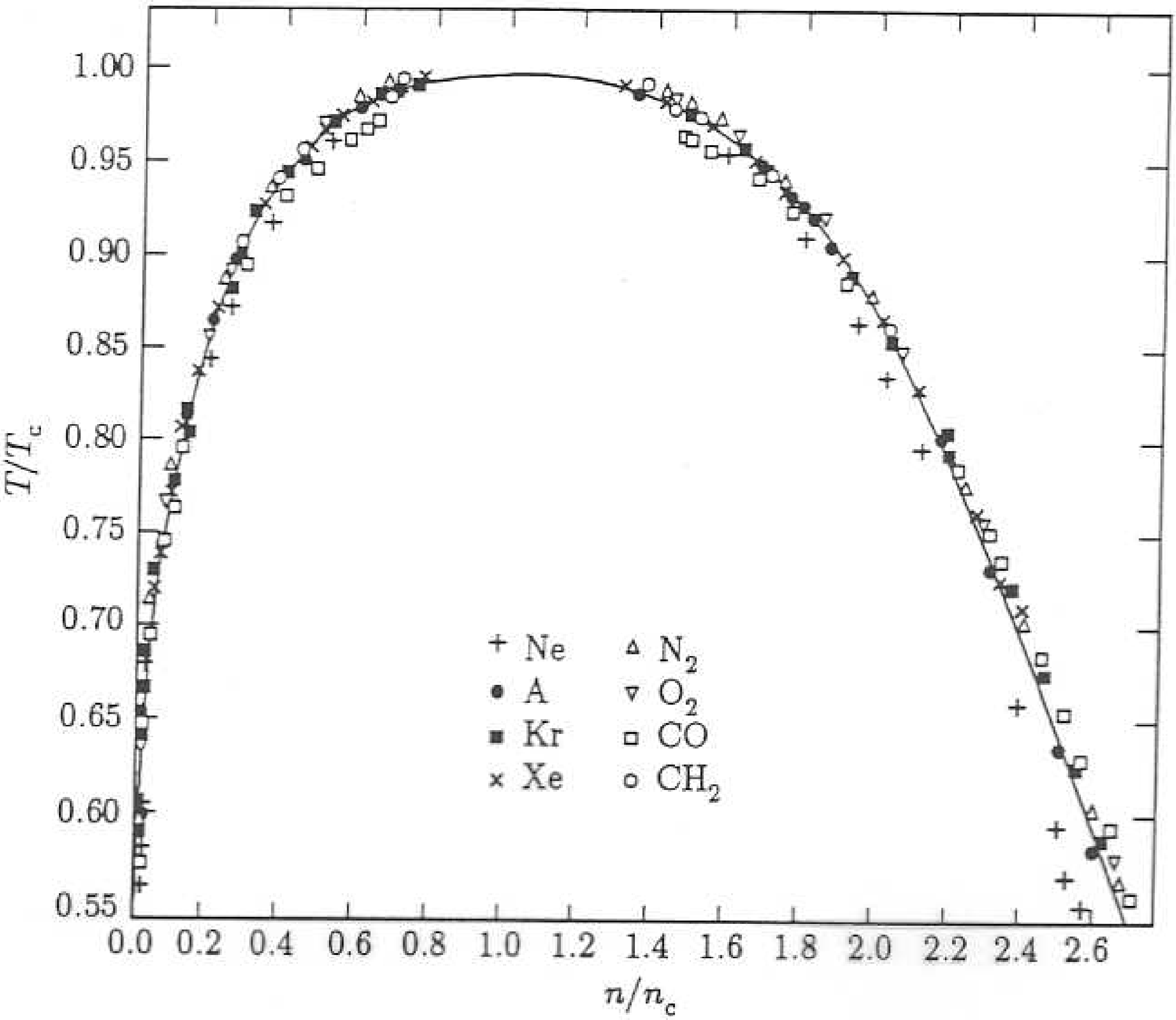


Fig. 4.4.4. Phase boundary in units of reduced temperature and density for eight different molecular fluids near their liquid-gas transitions. Note the universal behavior and the fact that the solid line is $\Delta\phi \propto (T_c - T)^\beta$ with $\beta = 1/3$ rather than the mean-field result $\beta = 1/2$. [E.A. Guggenheim, *J. Chem. Phys.* 13, 253 (1945).]

Critical phenomena on the critical isochore

As

$$\Delta T = \frac{T - T_c}{T_c} \to 0^+ \quad \text{with} \quad \rho = \rho_c$$

the heat capacities, compressibility, and thermal expansivity all *diverge*:

$$\begin{aligned} c_p,\ \alpha_P,\ K_T \sim \Delta T^{-\gamma}, &\quad \gamma = 1.24 \\ c_v \sim \Delta T^{-\alpha}, &\quad \alpha = 0.11 \end{aligned}$$

The diffusion time grows:

$$\tau_d = \frac{L^2 \rho c_p}{\kappa} \sim \Delta T^{-0.67}$$

Sound speed decreases weakly:

$$c_s = \left(\frac{c_p}{c_v \rho K_T} \right)^{1/2} \sim \Delta T^{0.06}$$

Critical point for Xenon:

$T_c = 289.72\,\mathrm{K}, \quad \rho_c = 1110\,\mathrm{kg/m^3}, \quad P_c = 5.84\,\mathrm{MPa}$

Experimental regime for ZENO:

$$L \sim 1 \text{ mm}$$
$$0.01 \text{ mK} \leq T - T_c \leq 100 \text{ mK}$$
$$(10^{-7} \leq \Delta T \leq 10^{-3})$$

At $T - T_c = 30$ mK $(\Delta T = 10^{-4})$:

- correlation length: $\xi \sim 0.1\ \mu\text{m} \sim 10^{-4} L$
- sound speed $c_s \sim 80$ m/s
- diffusion time: $\tau_d \sim 4$ hours

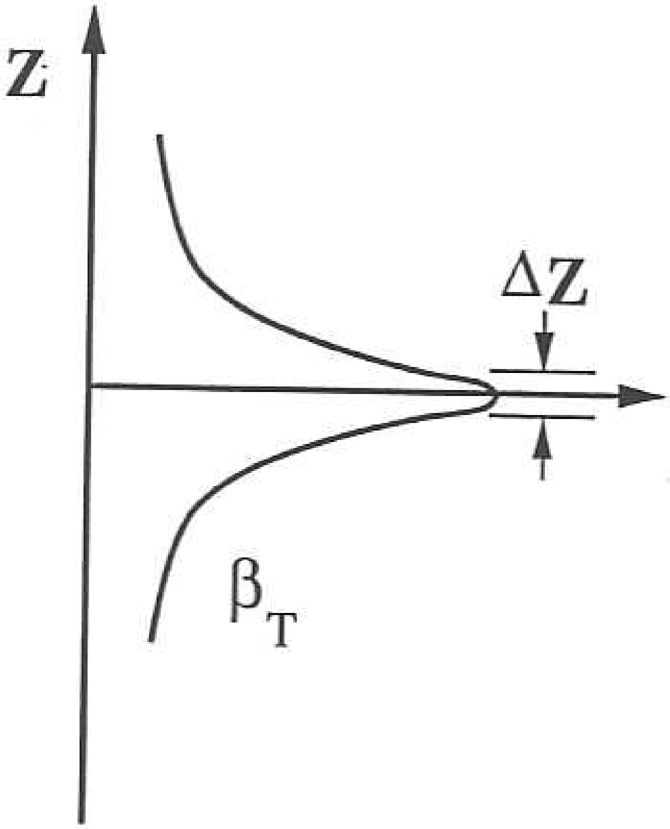
Hydrodynamic phenomena near T_c

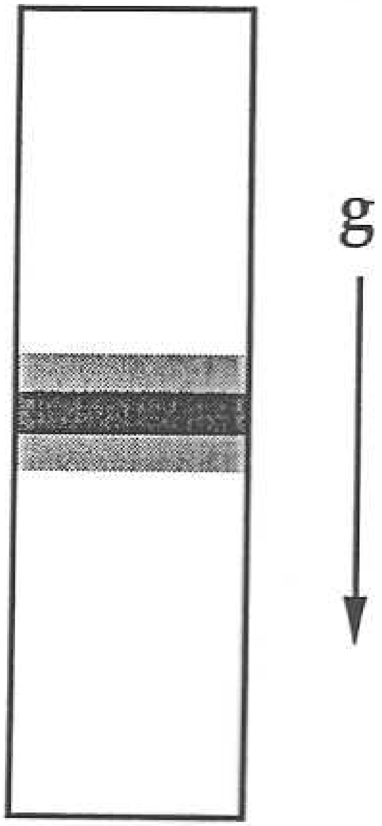
- strongly stratified equilibrium density at 1G
- *fast* thermal response through thermo-acoustic coupling (Onuki & Farrell 1990)
- *slow, diffusive* density relaxation
- non-Boussinesq convection

Experimental groups:

- Berg, Moldover (NIST), Gammon (U Md)
- D. Beysans (CEA), Y. Garrabos (CNRS) & B. Zappoli (CNES) in France
- H. Meyer (Duke)
- A. Wilkinson (NASA Lewis)

Compressibility with Height





Equilibrium density profiles from various model EOS

Ideal Gas: Exponential $\rho = \rho_c \, e^{-\alpha z}$

van der Waals expansion:

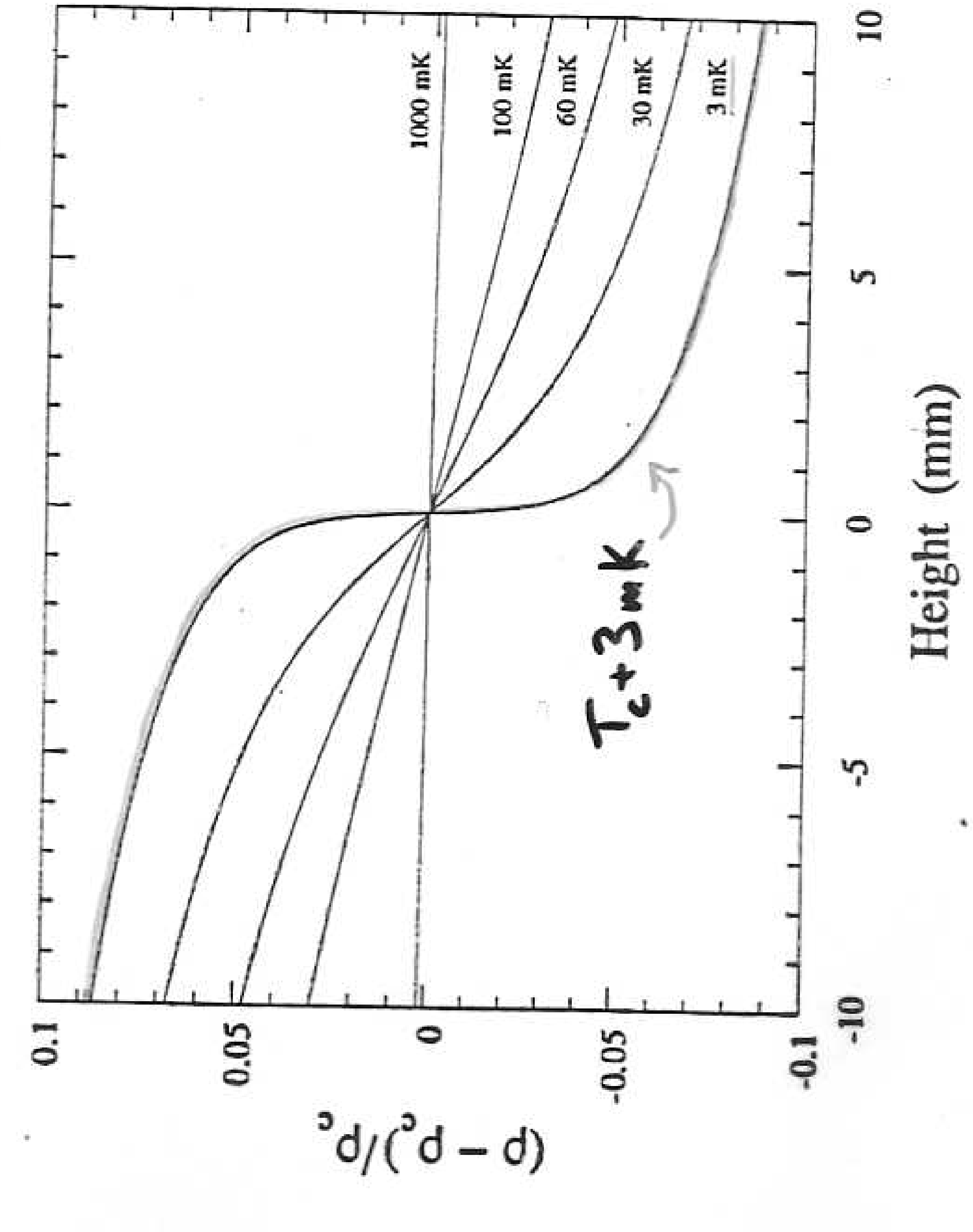
Anderson & McFadden (1997)

$$0 = \tilde{g}\tilde{z} + a^2 \Delta T^* \Delta\rho^* + (\Delta\rho^*)^3$$

Restricted Cubic: Ho and Listner (1970), Moldover, Sengers, Gammon, & Hocken (1979) Thermodynamic state parametrized by (r, θ):

$$\begin{aligned}
\Delta T^* &= \left(\frac{T - T_c}{T_c}\right) = r(1 - b^2\theta^2) \\
\Delta\rho^* &= \left(\frac{\rho - \rho_c}{\rho_c}\right) \ = kr^\beta \theta(1 + c\theta^2) \\
\Delta\mu^* &= \left(\frac{\rho_c}{P_c}\right)\Big(\mu(\rho, T) - \mu(\rho_c, T)\Big) \\
&= ar^{\beta\,\delta}(1 - \theta^2)\theta
\end{aligned}$$

Fig.(3.1): Gravity-induced density gradient at different temperature near Tc. Each curve is labeled by its temperature difference from Tc.



$T_c + 3\,mK$

⋆ Asymptotic modeling of low-speed flow

- H. Boukari, R. L. Pego & R. Gammon,
  Phys. Rev. E **52** (1995) 1614–1626.
- D. Denny & R. L. Pego,
  Quart. Appl. Math. **58** (2000) 103-125.

Scaling:

$$x/L = x^*, \quad t/\tau = t^*, \quad u/U = u^*$$
$$\frac{\rho}{\rho_c} = \rho^*, \quad \frac{T - T_c}{T_a} = T^*, \quad \frac{p - p_c}{p_a} = p^*$$

Dimensionless nos. (w/ $L = 10$ mm, $\tau = 10$ s):

$$\text{“Mach no.”} \quad \mathrm{M}^2 = \frac{U^2}{p_a/\rho_c} \approx 3 \times 10^{-7}$$
$$\text{Reynolds no.} \quad \mathrm{Re} = \frac{UL}{\nu_a} \approx 200$$
$$\text{Prandtl no.} \quad \mathrm{Pr} = \nu_a \frac{\rho_c c_{pa}}{\kappa_a} \approx 250$$
$$\text{Froude no.} \quad \mathrm{Fr} = \frac{L}{g\tau^2} \approx 10^{-5}$$

No heat conduction $(\mathrm{Pr} = \infty)$ leads to variable-density incompressible flow

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla \cdot u = 0$$

$$\rho\frac{Du}{Dt} + \frac{1}{\mathrm{M}^2}\nabla p = -\frac{\rho\hat{\mathbf{z}}}{\mathrm{Fr}} + \frac{1}{\mathrm{Re}}\left(\nabla \cdot \mu(\nabla u + \nabla u^T) + \nabla(\lambda \nabla \cdot u)\right)$$

$$\frac{Ds}{Dt} = \underline{0}$$

Formal expansion in M^2:

$$p = p_0(t) + \mathrm{M}^2 p_1, \quad \rho = \hat{\rho}(p, s)$$

Continuity equation with EOS yields at $O(1)$:

$$\frac{1}{\rho}\left(\left.\frac{\partial\rho}{\partial p}\right|_s \frac{dp_0}{dt} + \left.\frac{\partial\rho}{\partial s}\right|_p \frac{Ds}{Dt}\right) + \nabla \cdot u = 0$$

whence integration and BCs yield

$$\frac{dp_0}{dt} = 0, \quad \underline{\nabla \cdot u = 0}, \quad \underline{\frac{D\rho}{Dt} = 0}$$

Asymptotic equations with heat conduction and capillary stress

$$\rho = \tilde{\rho}(p_0, T), \quad \check{\rho} = \tilde{\rho}(p_0 + M^2 p_1, T)$$

$$\frac{dp_0}{dt} = H(t) = -\frac{\int A(p_0, T)\nabla \cdot (\kappa \nabla T)\, dx}{\int (\rho c_s^2)^{-1}\, dx}$$

$$\frac{DT}{Dt} = B(p_0, T)H(t) + \frac{1}{\mathrm{Re}\,\mathrm{Pr}}\frac{1}{\rho c_p}\nabla \cdot (\kappa \nabla T)$$

$$\nabla \cdot u = -\frac{1}{\rho}\left.\frac{\partial \rho}{\partial p}\right|_s H(t) - A(p_0, T)\nabla \cdot (\kappa \nabla T)$$

$$\rho \frac{Du}{Dt} + \nabla p_1 = -\frac{\check{\rho}\hat{\mathbf{z}}}{\mathrm{Fr}} + c\rho \nabla \Delta \rho + \frac{1}{\mathrm{Re}}(\ldots)$$

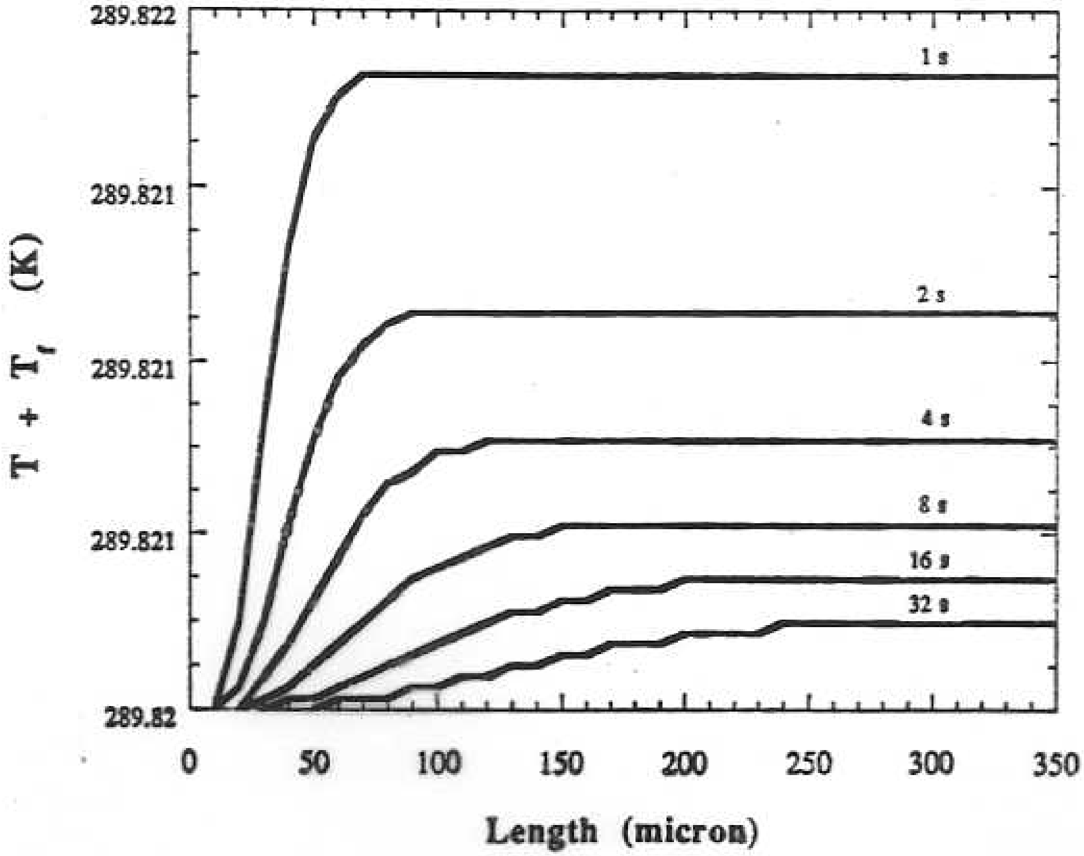
- Density relaxation is driven by heat conduction
- Level sets of density ρ *cannot change topology without heat conduction:* $\rho_t + u \cdot \nabla \rho = -\rho \nabla \cdot u$
- Can capillary stress effects be observed in a real single-phase fluid?

$c \sim L_{cap}^4 \quad L_{cap} \sim .1$ mm

Adiabatic heat transfer via the ‘piston effect’

(cf. Onuki, Ferrell & Hao, Boukari & Gammon)

$t = 0$	Wall temperature reduced from T_i to T_f
$t \sim t_{\partial\text{layer}}$	A cool, *dense* boundary layer forms
$t \sim t_{\text{acoustic}}$	Sound waves *expand* the bulk fluid
$t \sim t_{\text{adiabatic}}$	Pressure drops, until T approaches T_f
$t \sim t_{\text{diffusion}}$	Exponential ‘single mode’ decay



⋆ Internal wave damping

(no heat conduction, in a horizontally periodic layer of fluid with rigid, no-slip walls at $z = \pm L$)

- K. F. Gurski & R.P.,
  Phys. Rev. E 62 (2000) 517
- K. F. Gurski & R.P.,
  Proc. Royal Soc. Edin., to appear.

Linearize at equilibrium and look for normal modes with vertical velocity

$$W(x, y, z, t) = e^{ik_1 x + ik_2 y - \lambda t} w(z)$$

We get a nonlinear eigenvalue problem:

$$\nu(z)(\partial_z^2 - k^2)^2 w + \lambda(\partial_z^2 - \alpha(z)\partial_z - k^2)w = \frac{gk^2}{\lambda}\alpha(z)w$$

$$w = \partial_z w = 0 \quad \text{at } z = \pm L \tag{1}$$

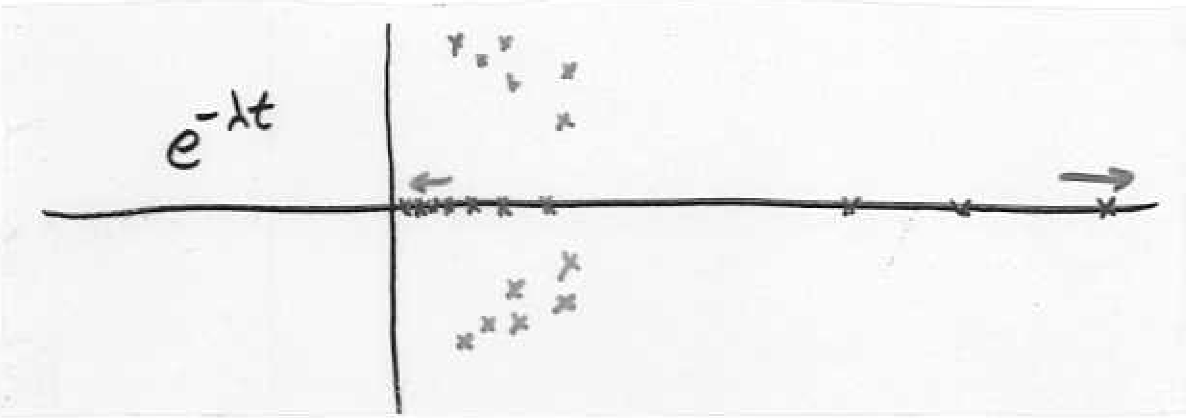
$$\nu(z) = \frac{\mu}{\rho(z)}, \quad \alpha(z) = -\frac{\partial_z \rho(z)}{\rho(z)} > 0$$

Goal: Fix k. Find the slowest decaying mode and study the dependence of damping rate on T near T_c.

Surprise: There is no such mode!

Theorem

The eigenvalues of (1) form a countably infinite set in the half-plane $\Re\lambda > 0$. At most a finite number are non-real. There exists both an infinite number of eigenvalues converging to *zero* and an infinite number converging to ∞, and there is no other accumulation point.



Analysis of the eigenvalue problem in 3D

(Gurski, Kollár, RP, in progress)

$$-\lambda\rho + u\cdot\nabla\rho_0(z) = 0$$
$$-\lambda\rho_0 u - \mu\Delta u + \nabla p + \rho g\hat{z} = 0$$

Solution space: $V = \{u \in H_0^1(\Omega)^3 : \nabla\cdot u = 0\}$

Abstract reformulation in V^* for $f = -\mu\Delta u$:

$$f = \lambda B f + \frac{1}{\lambda} C f$$

B, C are nonnegative, compact, selfadjoint on V^*.

Equivalently, after W. M. Greenlee (1974):

$$1/\nu = \lambda - 1/\lambda, \quad y = (f, f/\lambda)$$

$$\begin{pmatrix} I & -B-C \\ -B-C & I \end{pmatrix}^{-1} \begin{pmatrix} B & 0 \\ 0 & -C \end{pmatrix} y = \nu y$$

Project out kernel, get:

$$\mathcal{A}y = \nu y$$

where $\mathcal{A}$ is 1-1, compact, and “selfadjoint” for an *indefinite inner product space* Π_k (Pontrjagin space).

Structure theorem *There exists an $\mathcal{A}$-invariant decomposition*

$$\Pi_k = M \oplus N,$$

with $\dim M < \infty$ *and* N *uniformly negative.*

$\mathcal{A}$ is compact selfadjoint on the Hilbert space N.

A max-min argument leads to a comparison result for n-th largest eigenvalues $\nu_n(\pm\mathcal{A})$:

Theorem *There exists* $m \in \mathbf{Z}^+$ *such that* $\forall n \geq 1$,

$$\nu_n(B) \geq \nu_n(\mathcal{A}) \geq \nu_{n+m}(B),$$
$$\nu_n(C) \geq \nu_n(-\mathcal{A}) \geq \nu_{n+m}(C).$$

With $\nu_n = \nu_n(\pm\mathcal{A})$, eigenvalues λ include:

$$\lambda_n^{\pm} = \pm\frac{1}{2\nu_n} + \left(\left(\frac{1}{2\nu_n}\right)^2 + 1\right)^{1/2} \to \begin{cases} +\infty & (+) \\ 0 & (-) \end{cases}$$

Numerical Methods for 1D

- 3-term matched asymptotic expansion + Sturm-Liouville solver

$$\begin{aligned}\lambda &= i\omega_0 + \sqrt{\mu}\lambda_1 + \mu\lambda_2 + O(\mu^{\frac{3}{2}}),\\ w(z) &= \psi(z)\sqrt{\rho(-L)/\rho(z)}\\ \psi &= \psi_0 + \sqrt{\mu}\psi_1 + \mu\psi_2 + O(\mu^{\frac{3}{2}})\end{aligned}$$

For a fixed k there is a solution corresponding to any number of *zeros of* ψ_0: $n = 1, 2, 3\ldots$

- Compound matrix shooting method (Ng & Reid 1979)

Idea: Evaluate a Wronskian $W(\lambda)$ from wedge products to stabilize shooting computations. Scale carefully. Find zeros using the argument principle and secant method.

- Parameters model the xenon viscometery experiment of Berg et al., Phys. Fluids 8 (1996)

Figure 1: Decay rate vs. oscillation frequency at $T - T_c = 10$ mK and 1 mK using the RC density profile and CMS method with constant viscosity. Modes are labeled by (m_x, m_y, n).

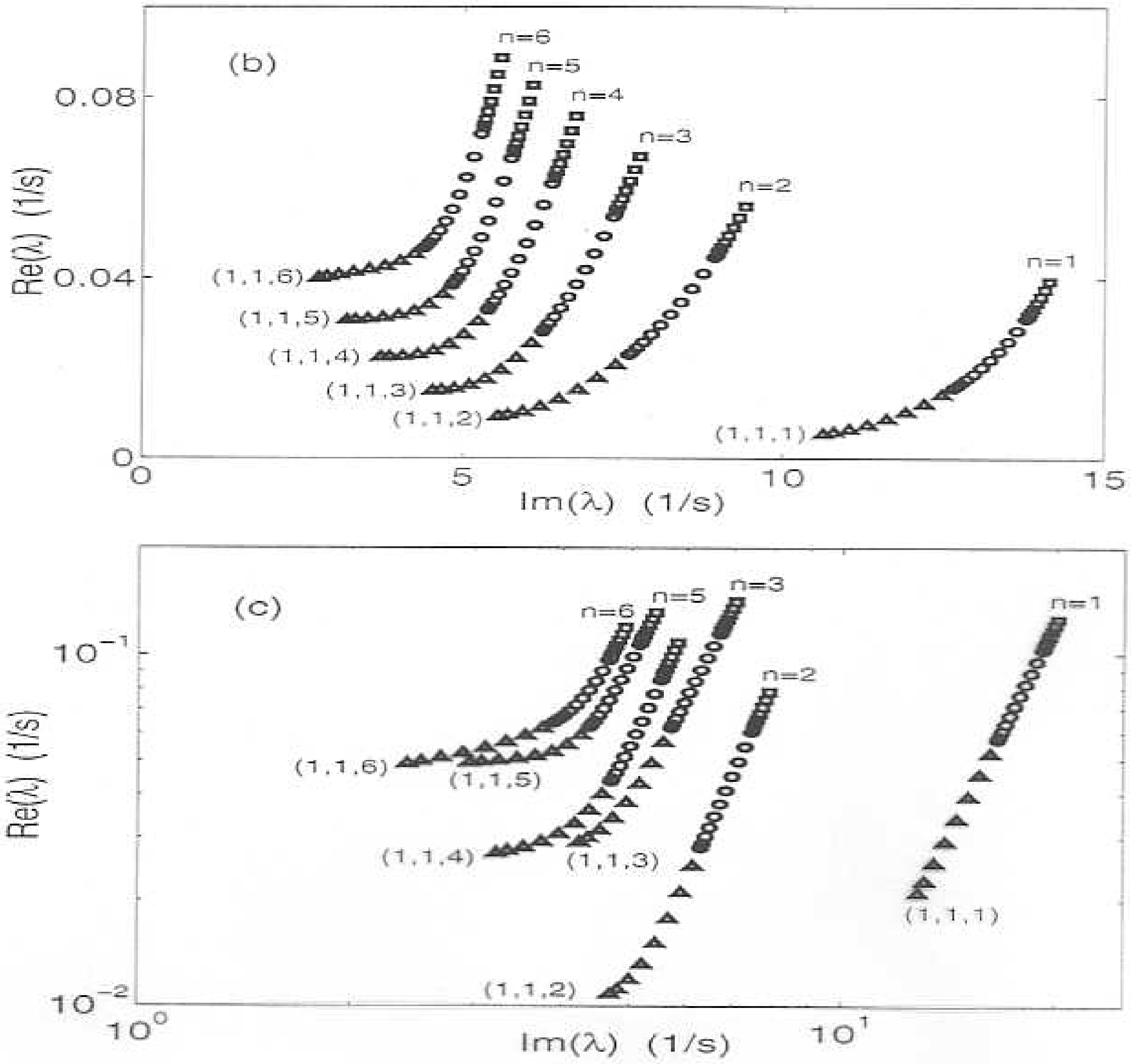
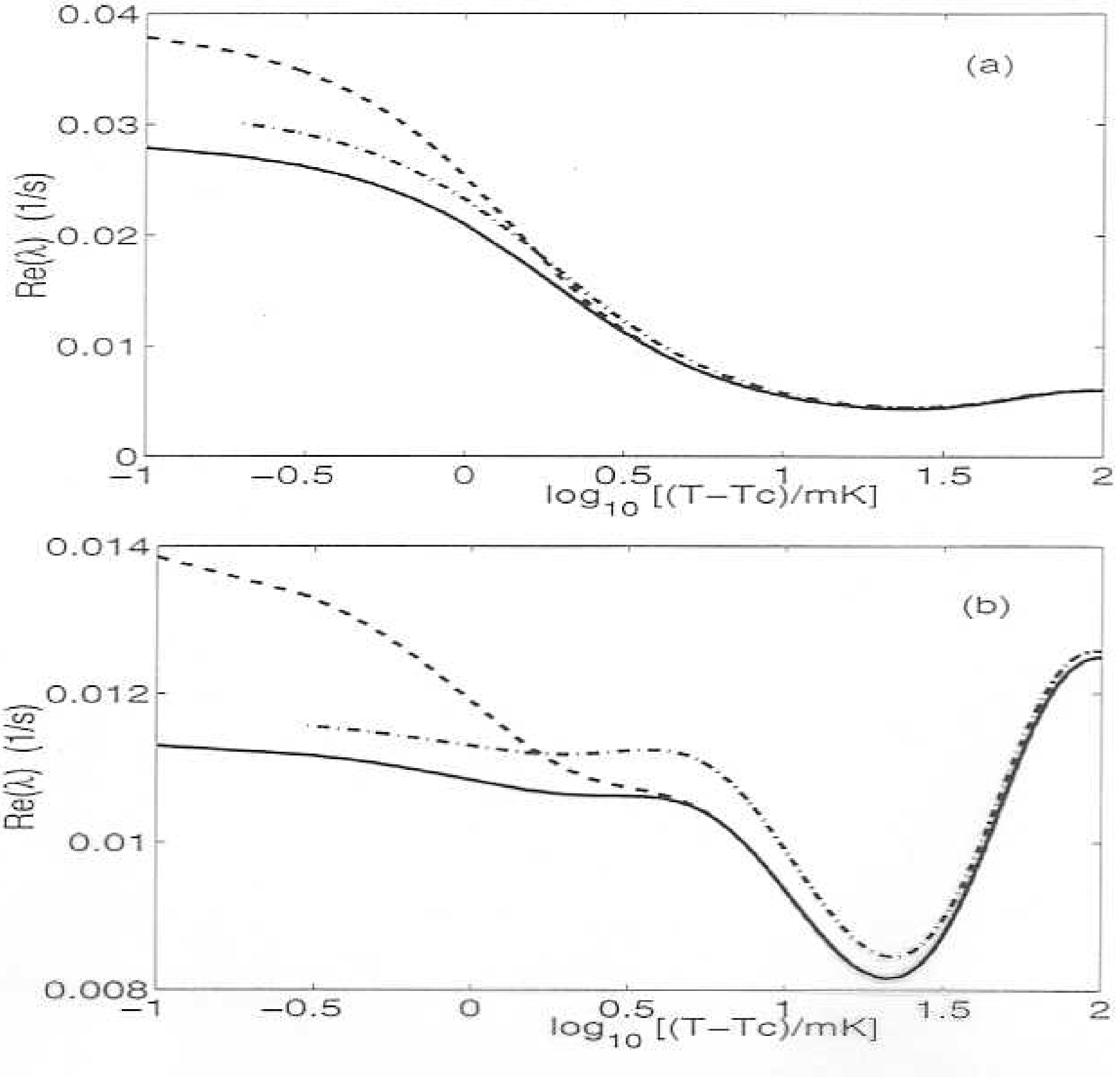


Figure 2: Decay rates vs. temperature for the RC density profile (a) (1,1,1) and (b) (1,1,2) modes.
Solid: CMS method w/ const μ.
Dashed-dot: CMS method variable μ.
Dashed: matched asymptotics w/ const μ.



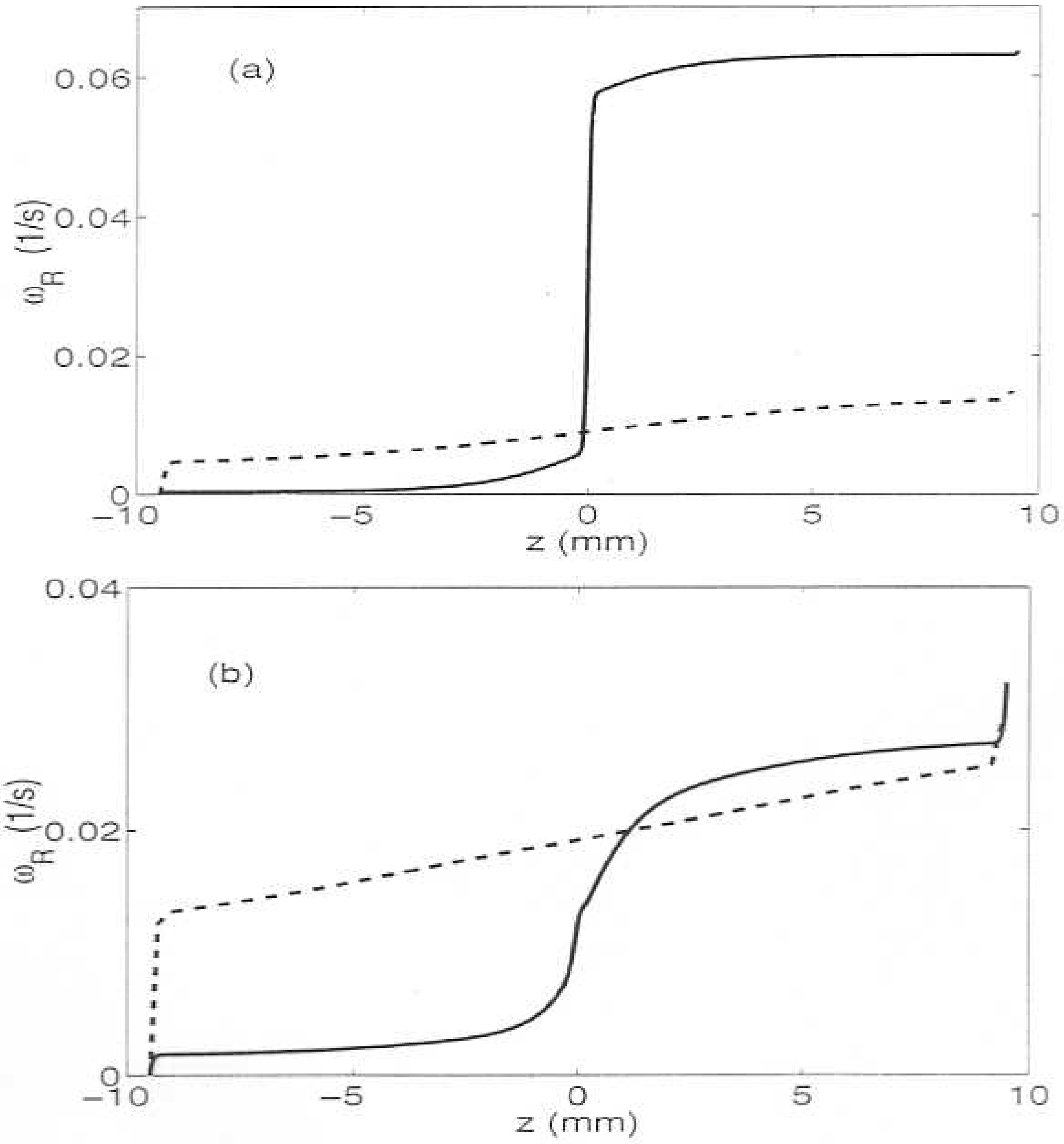
Dissipation: boundary-layer or bulk dominated?

Partially integrated dissipation rate:

$$\lambda_r(\hat{z}) = \int_{z<\hat{z}} \mu |\nabla \mathbf{u}|^2 \, d^3 r / C$$

Horizontal velocity gradients involve $\partial_z^2 w$, which is $O(1/\sqrt{\mu})$ in boundary layers of width $O(\sqrt{\mu})$.

Figure 4: Partially integrated dissipation rate vs z for RC density profile (a) (1,1,1) and (b) (1,1,2) modes.
Solid: $T - T_c = 1$ mK. Dashed: $T - T_c = 100$ mK.



Internal wave damping: summary

- We can prove the existence of non-oscillatory internal wave modes with arbitrarily small damping rates in the presence of viscosity.
- For oscillatory modes, numerics indicate:
  - o For $T >> T_c$ damping of the internal waves is due to both boundary layer effects and volumetric effects.
  - o The boundary layer effect is caused by horizontal shearing layers near the two fixed horizontal boundaries.
  - o As $T \to T_c^+$, an additional horizontal shearing layer forms at the incipient interface/gradient singularity.
  - o For some of the internal wave modes this causes a dramatic increase in the damping rate that dominates the boundary layer effects.