Explicit p-adic regulators on K_2 of modular curves

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Construction of p-adic L-functions by interpolation

- $\zeta_p(s), L_p(\chi, s)$ (Kubota-Leopoldt)
- $L_p(E, s), L_p(f, s)$ (Manin, Vishik, Amice-Vélu)
- Deligne-Ribet, Katz, Hida, Panchishkin…

Another possibility is to define p-adic L- functions using compatible systems of global objects (e.g. Coleman series, Euler systems).

Perrin-Riou has given a very general conjectural definition of p-adic L-functions associated to motives, as well as conjectures on their special values.

→ What is the p-adic analogue of Beilinson conjectures on special values of L-functions ?

In this talk, we will consider the special case of curves.

$X/\mathbf{Q}$ smooth projective curve of genus g
$T_X = H^1_{\text{ét}}(X_{\overline{\mathbf{Q}}}, \mathbf{Z}_p) \qquad V_X = T_X \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$.
V_X is a (global) p-adic representation, $\dim_{\mathbf{Q}_p} V_X = 2g$.

- $T_X(1) \cong T_p(J)$ with $J = \mathrm{Jac}(X)$
- Perfect pairing $T_X \times T_X \to \mathbf{Z}_p(-1)$
- T_X is unramified at any prime $\ell \neq p$ such that J has good reduction at ℓ.

Fontaine has shown that V_X is de Rham at p.
$D_{\mathrm{dR}}(V_X) := (V_X \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}})^{G_{\mathbf{Q}_p}} \cong H^1_{\mathrm{dR}}(X/\mathbf{Q}_p)$.

$$\mathrm{Fil}^i D_{\mathrm{dR}}(V_X) = \begin{cases} H^1_{\mathrm{dR}}(X/\mathbf{Q}_p) & \text{if } i \leq 0 \\ \Omega^1(X/\mathbf{Q}_p) & \text{if } i = 1 \\ \{0\} & \text{if } i \geq 2. \end{cases}$$

Theorem (Coleman-Iovita, Breuil)

V_X crystalline at $p \Leftrightarrow J$ has good reduction at p.
V_X semi-stable at $p \Leftrightarrow J$ has semi-stable reduction at p.

Let $D_{\mathrm{cris}}(V_X) := (V_X \otimes B_{\mathrm{cris}})^{G_{\mathbf{Q}_p}} \hookrightarrow D_{\mathrm{dR}}(V_X)$.
$D_{\mathrm{cris}}(V_X)$ is a filtered ϕ-module.
If J has good reduction at p, we have (Katz-Messing)

$$\det(1 - T\phi | D_{\mathrm{cris}}(V_X)) = P_{X,p}(T) \in \mathbf{Z}[T]$$

where $P_{X,p}$ is the classical Euler factor of $L(X,s)$ at p.

Iwasawa cohomology (local case)

$$G_n = \mathrm{Gal}(\mathbf{Q}_p(\zeta_{p^n})/\mathbf{Q}_p) \qquad (n \in \mathbf{N} \cup \{\infty\})$$
$$\Lambda = \mathbf{Z}_p[[G_\infty]] := \varprojlim_{n\geq 0} \mathbf{Z}_p[G_n].$$
$$H^i_{\mathrm{Iw}}(\mathbf{Q}_p, T_X) := \varprojlim_{n\geq 0} H^i(\mathbf{Q}_p(\zeta_{p^n}), T_X) = \text{fin. gen. } \Lambda\text{-module}$$
$$H^i_{\mathrm{Iw}}(\mathbf{Q}_p, V_X) := H^i_{\mathrm{Iw}}(\mathbf{Q}_p, T_X) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$$

- $H^i_{\mathrm{Iw}} = 0$ if $i \notin \{1, 2\}$
- $H^2_{\mathrm{Iw}}(\mathbf{Q}_p, T_X)$ is a torsion Λ-module
- $\mathrm{rk}_\Lambda H^1_{\mathrm{Iw}}(\mathbf{Q}_p, T_X) = \mathrm{rk}_{\mathbf{Z}_p}(T_X) = 2g$
- Compatibility with Tate twists :

$$H^i_{\mathrm{Iw}}(\mathbf{Q}_p, T_X) \xrightarrow{\cong} H^i_{\mathrm{Iw}}(\mathbf{Q}_p, T_X(k)) \qquad (k \in \mathbf{Z})$$
$$(c_n)_{n\geq 0} \mapsto (c_n \otimes \zeta_{p^n}^k)_{n\geq 0}$$

Iwasawa cohomology (global cae)

$$G_\infty \cong \mathrm{Gal}(\mathbf{Q}(\zeta_{p^\infty})/\mathbf{Q})$$
$$H^i_{\mathrm{Iw}}(\mathbf{Z}[\tfrac{1}{p}], T_X) := \varprojlim_{n \geq 0} H^i(\mathbf{Z}[\zeta_{p^n}, \tfrac{1}{p}], T_X) = \text{fin. gen. } \Lambda\text{-module}$$
$$H^i_{\mathrm{Iw}}(\mathbf{Z}[\tfrac{1}{p}], V_X) := H^i_{\mathrm{Iw}}(\mathbf{Z}[\tfrac{1}{p}], T_X) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$$

- Localization maps $H^i_{\mathrm{Iw}}(\mathbf{Z}[\frac{1}{p}], T_X) \to H^i_{\mathrm{Iw}}(\mathbf{Q}_p, T_X)$
- Weak Leopoldt conjecture : $H^2_{\mathrm{Iw}}(\mathbf{Z}[\frac{1}{p}], T_X)$ is a torsion Λ-module. This implies $\mathrm{rk}_\Lambda H^1_{\mathrm{Iw}}(\mathbf{Z}[\frac{1}{p}], T_X) = \mathrm{rk}_{\mathbf{Z}_p}(T_X^-) = g$.

Theorem (Kato)

If X is covered by a modular curve then WL is true for T_X.

The proof uses Euler systems.

Definition

A *Euler system* for $T = T_X(2)$ is a collection of classes

$$z_m \in H^1(\mathbf{Z}[\zeta_m, \frac{1}{p}], T)$$

satisfying the corestriction conditions

$$\mathrm{cores}(z_{m'}) = \prod_{\substack{\ell \mid m' \\ \ell \nmid m}} P_{X,\ell}(\sigma_\ell^{-1}) \cdot z_m \qquad (m \mid m')$$

where $\sigma_\ell \in \mathrm{Gal}(\mathbf{Q}(\zeta_m)/\mathbf{Q})$ is the arithmetic Frobenius.

Remarks

- The set of Euler systems is a Λ-module.
- If $z = (z_m)$ is a Euler system then $(z_{p^n})_{n\geq 1} \in H^1_{\mathrm{Iw}}(\mathbf{Z}[\frac{1}{p}], T)$.

Definition

Let Z be the Λ-submodule of $H^1_{\mathrm{Iw}}(\mathbf{Z}[\frac{1}{p}], T)$ consisting of Euler systems.

Optimistic guess : Z is Λ-cotorsion in $H^1_{\mathrm{Iw}}(\mathbf{Z}[\frac{1}{p}], T)$.

Assuming this, and assuming V_X is crystalline, we explain, following Perrin-Riou, how to construct the « full » p-adic L-function of X.

$\Lambda \subset \mathcal{H}$ = large Iwasawa algebra $\subset \mathbf{Q}_p[[G_\infty]]$.
Perrin-Riou has constructed a « logarithme élargi »

$$\mathcal{L} : H^1_{\mathrm{Iw}}(\mathbf{Q}_p, V_X(2)) \to \mathcal{H} \otimes_{\mathbf{Q}_p} D_{\mathrm{cris}}(V_X)^\vee$$

by interpolation of the Bloch-Kato exponential maps for $V_X(k)$.
$\mathcal{L}$ is a map of Λ-modules and it is known that $\mathcal{L} \otimes \mathrm{Frac}(\mathcal{H})$ is an isomorphism, hence $\mathrm{rk}_\Lambda \mathcal{L}(Z) = g$.

Definition

The module of p-adic L-functions of X is $L_X = \det_\Lambda \mathcal{L}(Z)$.

Remark

L_X is a Λ-line in $\mathcal{H} \otimes_{\mathbf{Q}_p} \wedge^g_{\mathbf{Q}_p} D_{\mathrm{cris}}(V_X)^\vee$.

This defines the p-adic L-function of X up to a unit of Λ.
We normalize it using an interpolation property.
Let $\omega_1, \ldots, \omega_g$ be a $\mathbf{Q}$-basis of $\Omega^1(X)$.
Let $\eta_1, \ldots, \eta_g \in D_{\mathrm{cris}}(V_X) \otimes \overline{\mathbf{Q}_p}$ such that

- $\phi(\eta_i) = \alpha_i \eta_i$ with $v_p(\alpha_i) < 1$ for all $1 \leq i \leq g$;
- $\det\langle \omega_i, \eta_j \rangle = 1$.

Definition

The full p-adic L-function of X (with values in $\bigwedge^g D^\vee$) is

$$L_p(X, s) = \int_{\mathbf{Z}_p^\times} \langle x \rangle^{s-1} \mu_X(x) \qquad (s \in \mathbf{Z}_p)$$

where μ_X is the unique generator of $L_X \otimes \mathbf{Q}_p$ such that

$$\int_{\mathbf{Z}_p^\times} \chi \cdot \mu_X(\eta_1 \wedge \ldots \wedge \eta_g) = \frac{\tau(\chi)^g}{\alpha_1^n \cdots \alpha_g^n} \cdot \frac{L(J, \overline{\chi}, 1)}{\Omega_J^{\chi(-1)}}$$

for every primitive character $\chi : (\mathbf{Z}/p^n\mathbf{Z})^\times \to \overline{\mathbf{Q}}^\times$.
$\tau(\chi)$ = Gauss sum of χ
$\Omega_J^\pm$ = periods of J with respect to $\omega_1 \wedge \ldots \wedge \omega_g$.

Remarks

- This definition is completely conjectural... In fact the analytic continuation of $L(J, s)$ and algebraicity of $L(J, 1)/\Omega_J^+$ is not known in general as soon as $g \geq 2$.
- In the case $X = E$ is an elliptic curve, one recovers the usual p-adic L-function by projecting to the φ-eigenspace.

Let $X/\mathbf{Q}_p$ be any smooth curve.
Let $\mathrm{ch}_X : K_2(X) \to H^2_{\text{ét}}(X, \mathbf{Z}_p(2))$ be the Chern class map.

Lemma

$H^2_{\acute{e}t}(X, \mathbf{Z}_p(2)) \cong H^1(\mathbf{Q}_p, H^1_{\acute{e}t}(X_{\overline{\mathbf{Q}_p}}, \mathbf{Z}_p(2)))$

So we get a regulator map

$$K_2(X) \to H^1(\mathbf{Q}_p, V_X(2))$$

Bloch and Kato conjectured that the integral subspace of $K_2(X)$ is mapped into $H^1_f(\mathbf{Q}_p, V_X(2))$ (proved in special cases by Nekovar and Niziol).

Lemma

The Bloch-Kato exponential map $D_{dR}(V_X(2)) \to H^1_f(\mathbf{Q}_p, V_X(2))$ is an isomorphism.

In this way one gets a regulator map

$$\mathrm{reg}_X : K_2(X)_{\mathbf{Z}} \to D_{\mathrm{dR}}(V_X) \cong H^1_{\mathrm{dR}}(X/\mathbf{Q}_p).$$

Remark

Other definitions of the p-adic regulator map on $K_2(X)$:

- Coleman-de Shalit (using Coleman integrals)
- Besser (syntomic regulator)

Besser has proved that all these constructions coincide (in the good reduction case).

Let $X/\mathbf{Q}$ be a smooth projective curve of genus g, with good reduction at p.

Conjecture (p-adic Beilinson for X at $s = 0$)

- *$K_2(X)_{\mathbf{Z}} \otimes \mathbf{Q}$ has dimension g.*
- *The image of $\mathrm{reg}_X \otimes \mathbf{Z}_p$ is a free $\mathbf{Z}_p$-module of rank g.*
- *Up to standard identifications, we have*

$$\det \mathrm{reg}_X(K_2(X)_{\mathbf{Z}}) \sim L_p(X, \omega^{-1}, 0)$$

where $\sim$ indicates equality up to a nonzero rational factor and $\omega : \mathbf{Z}_p^\times \to \mathbf{Z}_p^\times$ is the Teichmüller character.

Remark

Using the syntomic regulator, one could also formulate a p-adic Beilinson conjecture at any non-critical integer (at least in the good reduction case).
For any $m \geq 0$, the value of $L_p(X,s)$ at $s = -m$ should be linked with the syntomic regulator on $K_{2m+2}^{(m+2)}(X)$.

Theorem (Coleman-de Shalit)

If $E/\mathbf{Q}$ is a CM elliptic curve with good ordinary reduction at p, then $L_{p,\alpha}(E,0)$ can be expressed as the regulator of an explicit element of $K_2(E)$.

Aim of the talk : investigate the case of non CM elliptic curves using the deep results of Kato, Perrin-Riou, Colmez.

$E/\mathbf{Q}$ elliptic curve of conductor N, without CM.
$L(E,s) = \sum_{n\geq 1} a_n/n^s$ complex L-function of E.
$f = \sum_{n\geq 1} a_n q^n \in S_2(\Gamma_0(N))$
$\varphi : X_1(N) \to E$ modular parametrization
p prime number not dividing $2N$.

We have $V_E \xrightarrow{\varphi^*} V_{X_1(N)} \hookrightarrow V_{Y_1(N)}$ which induces an isomorphism $V_E \cong V_{Y_1(N)}/\langle T_n - a_n ; n \geq 1\rangle$.
$\eta_\alpha \in D_{\mathrm{cris}}(V_E) \otimes \overline{\mathbf{Q}_p}$ such that $\varphi(\eta_\alpha) = \alpha\eta_\alpha$ with $v_p(\alpha) < 1$ and $\langle \omega_f, \eta_\alpha \rangle = 1$.
On $Y(N)$ we have the Siegel modular units $g_{a,b}$ $(a, b \in \mathbf{Z}/N\mathbf{Z})$.
So we can form the cup product $\{g_{a,b}, g_{c,d}\} \in K_2(Y(N)) \otimes \mathbf{Q}$.

We consider the following regulator map

$$\begin{tikzcd} K_2(Y(N)) \arrow[r, "\mathrm{trace}"] \arrow[rrr, bend left, "\mathrm{reg}"] & K_2(Y_1(N)) \arrow[r, "\mathrm{reg}_{Y_1(N)}"] & D_{\mathrm{dR}}(V_{Y_1(N)}) \arrow[r] & D_{\mathrm{dR}}(V_E). \end{tikzcd}$$

Theorem (B.)

$\langle \mathrm{reg}\{g_{1,0}, g_{0,1}\}, \eta_\alpha \rangle =$
$(\prod_{\ell | N} 1 - a_\ell) \frac{L(E,1)\Omega_E^-}{i\langle f, f\rangle} [(1 - \frac{p}{\alpha})(1 - \frac{1}{p\alpha})]^{-1} L_{p,\alpha}(E, \omega^{-1}, 0)$

A similar formula for higher weight modular forms, at all non-critical integers, was obtained by Gealy (unpublished).

$$\langle \mathrm{reg}\{g_{1,0}, g_{0,1}\}, \eta_\alpha\rangle =$$
$$\left(\prod_{\ell|N} 1 - a_\ell\right)\frac{L(E,1)\Omega_E^-}{i\langle f,f\rangle}\left[(1-\tfrac{p}{\alpha})(1-\tfrac{1}{p\alpha})\right]^{-1} L_{p,\alpha}(E,\omega^{-1},0)$$

Remarks

- The formula is similar to the formula for the *complex* regulator of $\{g_{1,0}, g_{0,1}\}$ (Beilinson, Kato).
- The presence of $\left[(1-\frac{p}{\alpha})(1-\frac{1}{p\alpha})\right]^{-1}$ is a well-known phenomenon (extra Euler factor at p).
- The formula is not optimal (the RHS can be zero).

$$\langle \mathrm{reg}\{g_{1,0}, g_{0,1}\}, \eta_\alpha \rangle = \\ (\prod_{\ell | N} 1 - a_\ell) \frac{L(E,1)\Omega_E^-}{i\langle f, f\rangle} [(1-\frac{p}{\alpha})(1-\frac{1}{p\alpha})]^{-1} L_{p,\alpha}(E, \omega^{-1}, 0)$$

Remarks

- We can also compute $\mathrm{reg}\{g_{a,b}, g_{c,d}\}$ for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z})$ ($\to$ modular symbol in the RHS).
- The method doesn't seem to work for non-invertible matrices (such matrices would be needed in order to prove the surjectivity of the regulator map).
- We can also treat the case where E has non-split multiplicative reduction at p.

Ingredients of the proof :

1. Kato's Euler system
2. Perrin-Riou's « logarithme élargi »
3. Explicit reciprocity law

1. *Idea* : there is a commutative diagram

$$\begin{tikzcd} K_2(Y(Np^{n+1})) \arrow[r] \arrow[d, "\mathrm{trace}"'] & H^1(\mathbf{Q}(\zeta_{p^{n+1}}), V_{Y_1(N)}(2)) \arrow[d, "\mathrm{cores}"] \\ K_2(Y(Np^n)) \arrow[r] & H^1(\mathbf{Q}(\zeta_{p^n}), V_{Y_1(N)}(2)) \end{tikzcd}$$

The elements $z_{Np^n} = \{g_{1,0}, g_{0,1}\} \in K_2(Y(Np^n)) \otimes \mathbf{Q}$ are compatible for the trace when $n \geq 1$.

By considering the image of $(z_{Np^n})_{n\geq 1}$ under the regulator map and projecting to the elliptic curve E, one gets

$$z_E(2) \in H^1_{\mathrm{Iw}}(\mathbf{Q}, V_E(2)).$$

Kato's Euler system is defined by $z_{\mathrm{Kato}}(2) = \lambda^{-1} z_E(2)$ for some explicit $\lambda \in \Lambda$ (which explains the bad Euler factors $1 - a_\ell$ in the final formula).

2. Perrin-Riou has constructed an exponential map

$$\Omega_{V_E} : D_{\mathrm{cris}}(V_E) \to \left(H^1_{\mathrm{Iw}}(\mathbf{Q}_p, V_E)/V_E^{G_{\mathbf{Q}_p(\zeta_{p^\infty})}}\right) \otimes \mathcal{H}$$

interpolating the Bloch-Kato exponential maps for $V_E(k)$, $k \geq 1$.

Dualizing, we get $\mathcal{L} : H^1_{\mathrm{Iw}}(\mathbf{Q}_p, V_E(2)) \to \mathcal{H} \otimes D_{\mathrm{cris}}(V_E)^\vee$.

Theorem (Kato)

$\mathcal{L}(z_{\mathrm{Kato}}(2))(\eta_\alpha) = L_{p,\alpha}(E)$

Evaluating at the trivial character of G_∞, we get

$$\begin{aligned} L_{p,\alpha}(E, \omega^{-1}, 0) &= \langle \pi_0(z_{\mathrm{Kato}}(2)), \pi_0(\Omega_{V_E}(\eta_\alpha)) \rangle \\ &= \langle \log \pi_0(z_{\mathrm{Kato}}(2)), \exp^* \pi_0(\Omega_{V_E}(\eta_\alpha)) \rangle \end{aligned}$$

Note that $\pi_0(z_E(2)) = \pi_0(\lambda)\pi_0(z_{\mathrm{Kato}}(2))$ and that $\log \pi_0(z_E(2))$ is equal to $\mathrm{reg}(z_N)$ up to a simple rational factor (involving taking the trace from $Y(Np)$ to $Y(N)$). Thus it remains to compute $\exp^* \pi_0(\Omega_{V_E}(\eta_\alpha))$.

3. We use the explicit reciprocity law (proved by Colmez, Kato-Kurihara-Tsuji, Benois). It states that Ω_{V_E}, although defined in terms of the positive twists of V_E, is also linked with the negative twists $V_E(k)$, $k \leq 0$. More precisely if $\exp^* : H^1(\mathbf{Q}_p, V_E) \to D_{\mathrm{dR}}(V_E)$ denotes Bloch-Kato's dual exponential map, we have

$$\exp^* \pi_0(\Omega_{V_E}(\eta_\alpha)) = (1 - \frac{1}{p\phi})(1-\phi)^{-1}\eta_\alpha.$$

This accounts for the extra Euler factor at p.
Combining all these ingredients, we get the formula.

Some questions

- Extensions to higher weight modular forms
- Is it possible to prove that the image of the p-adic regulator map has the correct rank ? (assuming the p-adic L-values are nonzero)
- In the case of split multiplicative reduction, does there exist a link between the regulator and the $\mathcal{L}$-invariant ?
- Formulate p-adic Beilinson for more general varieties