

# On transversely holomorphic flows II

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## 1 Introduction

This paper is a complement to the preceding one by Marco Brunella [2]. Our purpose is to complete the classification of transversely holomorphic flows on closed 3-manifolds, avoiding the "rationality" assumption made in [2]. The main tool is still the existence of a harmonic atlas established by M. Brunella and we shall only add some simple ideas coming from the theory of complex surfaces.

We follow the notations of [2]. As a matter of fact, our main result is the following :

**Theorem 1.1** *Let  $\mathcal{L}$  be an orientable transversely holomorphic foliation on a closed connected 3-manifold  $M$ . Assume that  $H^2(M; \mathcal{O}) \neq 0$  where  $\mathcal{O}$  denotes the sheaf of germs of functions which are constant along the leaves and holomorphic in the transverse direction. Then,  $\mathcal{L}$  is riemannian, i.e. there is a riemannian metric on the normal bundle which is invariant under holonomy.*

Theorem 1 in [2] gives a complete description of the situation on closed 3-manifolds for which  $H^2(M; \mathcal{O}) = 0$ . On the other hand, Y. Carrière obtained in [3] a classification of riemannian foliations in dimension 3. Therefore, the association of theorem 1.1. and Brunella's result gives a classification: the only transversely holomorphic foliations on closed orientable connected 3-manifolds are examples 1) to 6) described in [2].

I would like to thank Yves Carrière: I had the pleasure to make with him the first attempts to classify these objects. As the reader will notice, the main ideas are contained in Brunella's paper; I wish to thank him for communicating these ideas to me.

## 2 General method

We fix a transversely holomorphic orientable foliation  $\mathcal{L}$  on a connected closed 3-manifold  $M$ . We first give a very simple criterium which guarantees that the foliation is riemannian. We say that a differential form of degree 1, with complex values, is a *basic holomorphic 1-form* if it is locally the pullback of a holomorphic 1-form by the projection on a local leaf space.

**Lemma 2.1** *If there exists a non trivial basic holomorphic 1-form, then  $\mathcal{L}$  is riemannian.*

**Proof** Let  $\omega$  be a basic holomorphic 1-form and assume first that  $\omega$  has no singularity. We can define a hermitian metric  $g$  on the normal bundle (of complex dimension 1) in such a way that the length of a vector  $v$  is the modulus of  $\omega(v)$ . Since  $\omega$  is basic, the same is true for  $g$ , *i.e.*  $\mathcal{L}$  is riemannian.

In general, the singular locus of  $\omega$  is transversely isolated, *i.e.* is a finite union of compact leaves  $L_1, \dots, L_n$  of  $\mathcal{L}$  (which are of course circles). Again, we can construct a hermitian metric  $g$  on the normal bundle but  $g$  vanishes along these leaves  $L_i$ . Consider the holonomy  $h_i$  of the leaf  $L_i$ ; this is the germ of a holomorphic diffeomorphism in the neighborhood of a fixed point in a small transverse disc  $D_i$ . By choosing a suitable local coordinate in  $D_i$ , we can assume that the restriction of  $\omega$  to  $D_i$  is  $z^k dz$  for some integer  $k > 0$  in the neighborhood of 0. The invariance condition of  $\omega$  by  $h_i$  means that  $h_i^k(z)h_i'(z) = z^k$  so that  $h_i^{k+1}(z) - z^{k+1}$  is a constant. Evaluating at the origin, we see that this constant vanishes so that  $h_i$  is actually the germ of a rigid rotation of finite order. Therefore,  $\mathcal{L}$  is riemannian in the neighborhood of  $L_i$ , *i.e.* we can find a saturated neighborhood of  $L_i$  in which  $\mathcal{L}$  admits a transverse invariant (non degenerate) metric  $g_i$ . We can now multiply  $g_i$  by a bump function depending only on the modulus of  $z$  in order to obtain a transverse invariant "metric"  $g'_i$  for  $\mathcal{L}$  which is non degenerate in the neighborhood of  $L_i$  but vanishes outside of some other tubular neighborhood of  $L_i$ . The sum of the  $g'_i$  and  $g$  is therefore everywhere non degenerate and is a transverse invariant metric. This shows that  $\mathcal{L}$  is riemannian.  $\square$

Recall that M. Brunella proved the existence of a *harmonic atlas*. This means that there is a covering of  $M$  by a finite number of open sets  $U_j$ , whose intersections are connected and simply connected, equipped with diffeomorphisms  $\Psi_j : U_j \rightarrow V_j = \Psi_j(U_j) \subset \mathbf{D} \times \mathbf{R}$  such that:

- In each  $U_j$ , the foliation  $\mathcal{L}$  is the pull-back by  $\Psi_j$  of the foliation of  $\mathbf{D} \times \mathbf{R}$  whose leaves are the lines  $\{\star\} \times \mathbf{R}$ .
- Changes of coordinates  $\psi_{ij} = \psi_i \circ \psi_j^{-1}$  have the following form on their domain of definition  $V_{ij} = \psi_j(U_i \cap U_j)$ :

$$\psi_{ij} : (z, t) \in V_{ij} \longmapsto (\phi_{ij}(z), t + h_{ij}(z)) \in V_{ji},$$

where  $\phi_{ij}$  is holomorphic and  $h_{ij}$  is *harmonic*.

Let  $H_{ij}$  be a holomorphic function whose real part is  $h_{ij}$ . Define:

$$\widehat{V}_j = V_j \times \mathbf{R} \subset \mathbf{D} \times \mathbf{R} \times \mathbf{R} \simeq \mathbf{D} \times \mathbf{C}$$

$$\widehat{V}_{ij} = V_{ij} \times \mathbf{R} \subset \mathbf{D} \times \mathbf{R} \times \mathbf{R} \simeq \mathbf{D} \times \mathbf{C}.$$

$$\Psi_{ij} : (z, t + i.s) \in \widehat{V}_{ij} \longmapsto (\phi_{ij}(z), t + i.s + H_{ij}(z)) \in \widehat{V}_{ji}.$$

Unfortunately, the  $\Psi_{ij}$  do not necessarily define a cocycle, *i.e.*  $\Psi_{ij} \circ \Psi_{jk}$  does not necessarily coincide with  $\Psi_{ik}$ . Therefore, we cannot in general define a complex surface  $X$  like M. Brunella in the “rational case”. The main idea which will guide our discussion is that when one glues the open sets  $\widehat{V}_j$  together using the  $\Psi_{ij}$ , one gets however some kind of “singular object”  $X$  which projects naturally onto  $M$  and which is not a complex manifold “only in the direction of  $\partial/\partial s$ ”. We shall not try to give a precise meaning to the previous sentence (in a suitable category...). We shall only recall that tensors on the “surface”  $X$  which are invariant under the translations along  $\partial/\partial s$  can be defined with no ambiguity. In the next section, using this heuristic idea, we shall give the precise definitions of these tensors on  $X$  invariant by  $\partial/\partial s$ .

### 3 Some sheaves on $M$

We first define a fibre bundle  $\mathcal{T}$  with fiber  $\mathbf{C}^2$  on  $M$ . Consider the complex tangent bundle of each  $\widehat{V}_j$ . This is a holomorphic vector bundle with fiber  $\mathbf{C}^2$  on which there is a natural action of the translations  $\tau_\sigma$  (for  $\sigma \in \mathbf{R}$ ) that we shall call *vertical*:

$$\tau_\sigma : (z, t + i.s) \in \widehat{V}_j \longmapsto (z, t + i.(s + \sigma)) \in \widehat{V}_j.$$

The quotient of  $\widehat{V}_j$  by the free action of these translations can be canonically identified with  $V_j$  so that we get a natural fiber bundle on  $V_j$ , with fiber  $\mathbf{C}^2$ . Since the  $\Psi_{ij}$  define a cocycle “up to vertical translations”, we get therefore a fiber bundle on  $M$  (which is obtained from the  $V_j$  by gluing with the  $\psi_{ij}$ ). This is the announced bundle  $\mathcal{T}$ .

We shall show that, although  $M$  is not a complex manifold, most properties of the cohomology of complex manifolds can be generalized to  $M$ . We follow the notations of [4] of which we quickly survey §15, and point out the modifications which are necessary in our situation.

We denote by  $\mathbf{T}$  the dual bundle to  $\mathcal{T}$  and by  $\overline{\mathbf{T}}$  the conjugate bundle of  $\mathbf{T}$ . If  $p$  and  $q$  are two integers (smaller than or equal to 2), we consider the vector bundle  $\Lambda^p(\mathbf{T}) \otimes \Lambda^q(\overline{\mathbf{T}})$ , tensor product of exterior powers. Sections of this bundle are called *forms of type  $(p, q)$*  on  $M$ . Local sections define a sheaf denoted by  $\mathcal{A}^{p,q}$ .

Locally, a form  $\omega$  of type  $(p, q)$  defined on  $V_j$  is identified with a form  $\widehat{\omega}$  of type  $(p, q)$ , in the usual sense, on the complex surface  $\widehat{V}_j$ , invariant under vertical translations. Since the decomposition  $d = \partial + \bar{\partial}$  of the exterior differential is of course invariant under vertical translations, and since these operators commute with the biholomorphisms  $\Psi_{ij}$ , we get well defined operators,  $\partial$  and  $\bar{\partial}$  :

$$\partial : \mathcal{A}^{p,q} \rightarrow \mathcal{A}^{p+1,q}$$

$$\bar{\partial} : \mathcal{A}^{p,q} \rightarrow \mathcal{A}^{p,q+1}.$$

The sum of the  $\mathcal{A}^{p,q}$  with  $p + q = r$  is denoted by  $\mathcal{A}^r$ . The kernel of  $\bar{\partial} : \mathcal{A}^{p,0} \rightarrow \mathcal{A}^{p,1}$  is the sheaf  $\Omega(\Lambda^p(\mathbf{T}))$  of germs of *holomorphic p-forms* on  $M$ . Since a holomorphic function on  $\widehat{V}_j$  invariant under vertical translations is in fact a function which depends only on the variable  $z$  and is holomorphic in this variable, the sheaf of germs of *holomorphic 0-forms* is identified with the sheaf  $\mathcal{O}$  of functions on  $M$  which are constant along the leaves and which are transversely holomorphic.

If we consider each  $\widehat{V}_j$  and a version of Dolbeault's theorem which is equivariant under the action of vertical translations, we get the following resolution analogous to the classical one:

$$0 \rightarrow \Omega(\Lambda^p(\mathbf{T})) \rightarrow \mathcal{A}^{p,0} \rightarrow \mathcal{A}^{p,1} \rightarrow \dots \rightarrow \mathcal{A}^{p,q} \rightarrow \dots$$

Hence, Dolbeault's theorem holds in our context. More precisely, the  $q$ -th cohomology group  $H^{p,q}(M)$  of  $M$  with values in the sheaf  $\Omega(\Lambda^p(\mathbf{T}))$  can be identified with:

$$Z^{p,q}/\bar{\partial}(\mathcal{A}^{p,q-1}),$$

where, of course,  $Z^{p,q}$  denotes the global forms of type  $(p, q)$  which are  $\bar{\partial}$ -closed. The dimension of  $H^{p,q}(M)$  will be denoted by  $h^{p,q}$  (we shall see that it is finite). Note that  $H^2(M; \mathcal{O}) = H^{0,2}(M)$ .

We now show how to extend Serre's duality and Hodge's theory in our context. The main point is to define a "fundamental class", *i.e.* to be able to "integrate" a  $(2, 2)$ -form. So, let us consider such a form  $\omega$ . In an open set  $V_j$ , this form corresponds to a  $(2, 2)$ -form  $\widehat{\omega}_j$  in the classical sense on  $\widehat{V}_j$ , invariant under vertical translations. The interior product of  $\widehat{\omega}_j$  by the vector field  $\partial/\partial s$  is a 3-form on  $\widehat{V}_j$ , basic for  $\partial/\partial s$ , *i.e.* which is a pull-back of some 3-form  $\tilde{\omega}_j$  on  $V_j$ . Clearly, these 3-forms  $\tilde{\omega}_j$  are compatible on the intersections of the  $V_j$ , *i.e.* they define a global 3-form  $\tilde{\omega}$  on  $M$ . By convention, we define the integral of  $\omega$ , denoted by  $\int \omega$ , as the integral of  $\tilde{\omega}$  on  $M$ .

The simple (but crucial) observation is that Stokes' theorem can be extended with no difficulty:

**Lemma 3.1** *If  $\alpha$  is a form of type  $(2, 1)$  and if  $\omega = d\alpha (= \bar{\partial}\alpha)$ , then  $\int \omega$  vanishes.*

**Proof** We can assume that  $\alpha$  has a compact support contained in some  $V_j$ . We consider the corresponding 3-form  $\hat{\alpha}_j$  in  $\hat{V}_j$ , invariant under vertical translations, and whose differential is the form  $\hat{\omega}_j$ . Since  $\hat{V}_j = V_j \times \mathbf{R}$ , we can embed  $V_j$  in  $\hat{V}_j$  as  $V_j \times \{s_0\}$ . By definition, the integral of  $d\alpha$  is equal to the integral of the interior product  $i_{\partial/\partial s}\hat{\omega}_j$  on  $V_j \times \{t_0\}$  (which is indeed independent of  $s_0$ ). As  $\hat{\alpha}_j$  is invariant under vertical translations:

$$i_{\partial/\partial s}d\hat{\alpha}_j + di_{\partial/\partial s}\hat{\alpha}_j = 0,$$

so that  $i_{\partial/\partial s}d\hat{\alpha}_j$  is an exact form. On the other hand, the support of the restriction of  $i_{\partial/\partial s}d\hat{\alpha}_j$  à  $V_j \times \{s_0\}$  is compact since  $\alpha$  has a compact support contained in  $V_j$ . Hence, the lemma follows from usual Stokes' theorem.  $\square$

Therefore, the integration of the exterior product defines linear maps:

$$\iota : H^{p,q}(M) \otimes H^{2-p,2-q}(M) \longrightarrow \mathbf{C}.$$

Let us now introduce some hermitian metric on  $\mathbf{T}$ . This enables us to define, as usual, anti-isomorphisms:

$$\begin{aligned} \sharp : \Lambda^p(\mathbf{T}) \otimes \Lambda^q(\overline{\mathbf{T}}) &\rightarrow \Lambda^{2-p}(\mathbf{T}) \otimes \Lambda^{2-q}(\overline{\mathbf{T}}) \\ \sharp : \mathcal{A}^{p,q} &\rightarrow \mathcal{A}^{2-p,2-q} \end{aligned}$$

and the operator:

$$\vartheta = -\sharp\bar{\partial}\sharp : \mathcal{A}^{p,q} \rightarrow \mathcal{A}^{p,q-1}.$$

The operators  $\vartheta$  and  $\bar{\partial}$  are adjoint for the scalar product:

$$(\alpha, \beta) = \int (\alpha \wedge \sharp\beta)$$

because of 3.1 and for the same reason as in the classical case. The Laplace operator  $\square = \vartheta\bar{\partial} + \bar{\partial}\vartheta$  is elliptic since, locally, it coincides with the usual Laplace operator acting on forms which are invariant under vertical translations.

Therefore, we get the finite dimensionality of cohomology groups  $H^{p,q}$  and Hodge decomposition:

$$\mathcal{A}^{p,q} = \bar{\partial}\mathcal{A}^{p,q-1} \oplus \vartheta\mathcal{A}^{p,q+1} \oplus B^{p,q},$$

where  $B^{p,q}$  denotes the space of  $\square$ -harmonic forms, *i.e.* the intersection of the kernels of  $\vartheta$  and  $\bar{\partial}$ . According to Dolbeault's theorem, mentioned above,  $B^{p,q}$  is identified with  $H^{p,q}(M)$ . In the same way we get Serre's duality, *i.e.* the isomorphism between  $H^{p,q}(M)$  and  $H^{2-p,2-q}(M)$ .

## 4 Proof of the theorem

We can now prove the theorem. *We assume now that  $\mathcal{L}$  is not riemannian* and we shall prove that  $H^2(M; \mathcal{O}) = 0$ .

According to Serre's duality, we know that  $h^{0,2} = h^{2,0}$  and therefore it suffices to show that  $h^{2,0} = 0$ , *i.e.* that there is no non trivial holomorphic 2-form on  $M$ .

Let us start by observing that any holomorphic 1-form on  $M$  is closed. This is a well known fact for any holomorphic 1-form in the classical sense on a complex compact surface (see [1] page 115) and the proof only uses Stokes' theorem, for which we have proved the analogous version 3.1.

Let  $\omega$  be a holomorphic 2-form on  $M$  and  $\hat{\omega}_j$  be the corresponding holomorphic 2-form on  $\hat{V}_j$ . By contracting with the vertical holomorphic vector field in  $\hat{V}_j$ , we get a holomorphic 1-form  $\hat{\alpha}_j$  in  $\hat{V}_j$ . In other words, we construct a holomorphic 1-form  $\alpha$  on  $M$ . By construction,  $\partial/\partial s$  is in the kernel of  $\hat{\alpha}_j$ . Since we observed that  $\alpha$  is necessarily closed, the forms  $\hat{\alpha}_j$  are obtained by pull-back of some forms  $\alpha_j$  in  $V_j$  which are basic for  $\mathcal{L}$  (a closed form, vanishing on the leaves of a foliation, is a basic form). Hence, these forms  $\alpha_j$  define a global basic holomorphic form for  $\mathcal{L}$ . Since we assumed that  $\mathcal{L}$  is not riemannian, there is no non trivial form by 2.1. Hence  $\omega$  vanishes.

## References

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