An arithmetic result concerning the centralizers of diffeomorphisms of the half-line

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Abstract

Let f be a smooth diffeomorphism of the half-line fixing only the origin and $\mathcal{Z}_f^r$ its centralizer in the group of $\mathcal{C}^r$ diffeomorphisms. According to well-known results of Szekeres and Kopell, $\mathcal{Z}_f^1$ is always a one-parameter group, naturally identified to $\mathbb{R}$, with $f \cong 1$. On the other hand, $\mathcal{Z}_f^r$, $2 \leq r \leq \infty$, can be smaller: in [Se], Sergeraert constructed an f whose $\mathcal{C}^\infty$ centralizer reduces to the infinite cyclic group generated by f (*i.e* $\mathcal{Z}_f^\infty \cong \mathbb{Z}$). In [E1], we adapted Sergeraert's construction to obtain an f whose $\mathcal{C}^r$ centralizer, for all $2 \leq r \leq \infty$, contains a Cantor set K but is still strictly smaller than $\mathcal{Z}_f^1 \cong \mathbb{R}$. Here, we improve the construction of [E1] to prove that for any Liouville number α, there is an f as above such that, in addition, $\alpha \in K \subset \mathcal{Z}_f^r$.

We want to understand what the $\mathcal{C}^r$ centralizer, $2 \le r \le \infty$, of a smooth diffeomorphism f of $\mathbb{R}_+ = [0, \infty)$ can possibly look like. If $\mathcal{D}^r$ denotes the group of $\mathcal{C}^r$ diffeomorphisms of $\mathbb{R}_+$, $1 \le r \le \infty$, endowed with the usual $\mathcal{C}^r$ (compact-open) topology, the $\mathcal{C}^r$ centralizer $\mathcal{Z}_f^r$ of f is the (closed) subgroup of $\mathcal{D}^r$ made up of all diffeomorphisms commuting with f. Here, we limit ourselves to the diffeomorphisms f which fix only the origin. The $\mathcal{C}^1$ centralizer of such an f is very well understood: well-known theorems by G. Szekeres and N. Kopell [Sz, K] show that $\mathcal{Z}_f^1$ is always a one-parameter subgroup of $\mathcal{D}^1$ (see also [Y, chap. 4] and [N, chap. 4] for complete proofs and more discussion). More precisely, f is the time-1 map of a unique $\mathcal{C}^1$ vector field ν_f on $\mathbb{R}_+$ (we call it the *Szekeres vector field of* f), and $\mathcal{Z}_f^1$ reduces to the flow of ν_f. Hence, there is a natural identification of $\mathcal{Z}_f^1$ to $\mathbb{R}$, with $f \cong 1$. Since $\mathcal{Z}_f^r$ decreases with r and contains the infinite cyclic subgroup generated by f, one has

$$\mathbb{Z} \cong \{f^n,\ n \in \mathbb{Z}\} \quad \subset \quad \mathcal{Z}_f^r \quad \subset \quad \mathcal{Z}_f^1 \cong \mathbb{R}.$$

If ν_f is of class $\mathcal{C}^r$, the inclusion on the right is an equality. According to F. Takens [T], this is always the case if f is not infinitely tangent to the identity at 0. However, this inclusion can also be strict, as Sergeraert shows in [Se], and one can actually check [E2] that in his example, $\mathcal{Z}_f^2 = \mathcal{Z}_f^\infty$ reduces to the group spanned by f, and is hence as small as possible. It is then easy, for any integer $q \ge 1$, to find an f whose $\mathcal{C}^\infty$ centralizer, seen as a subgroup of $\mathbb{R}$, is $\frac{1}{q}\mathbb{Z}$. The next natural question then is whether $\mathcal{Z}_f^\infty$ can be a dense (but still proper) subgroup of $\mathcal{Z}_f^1 \cong \mathbb{R}$. Article [E1] gives a positive answer: $\mathcal{Z}_f^\infty$ can contain a Cantor set K.

In the construction of [E1], based on Sergeraert's techniques and Anosov–Katok-like methods (introduced in [A–K]; see also [F–K] and the references therein), the very good approximation of the elements of K by rational numbers plays a crucial role. This fact urges us to consider $\mathcal{Z}_f^\infty$ not merely from a topological point of view, but from an arithmetic one:

What kind of irrational numbers can $\mathcal{Z}_f^\infty$ contain ?

Here, it seems natural to distinguish between numbers who satisfy a diophantine condition (*i.e* are "badly" approximated by rational numbers) and numbers who don't. Recall that a number α is said to *satisfy a diophantine condition* if there exist constants $c > 0$ and $\gamma \ge 0$ such that

$$\left|\alpha - \frac{p}{q}\right| \ge c q^{-2-\gamma} \tag{1}$$

for every rational number p/q, with $q \ge 1$. An irrational number which satisfies no diophantine condition is called a *Liouville number*. The following result might constitute one half of the answer to the above question.

Theorem A. *For any Liouville number α, there exists a $\mathcal{C}^\infty$ diffeomorphism f of $\mathbb{R}_+$ with a single fixed point at the origin, whose $\mathcal{C}^r$ centralizer, for all $2 \le r \le \infty$, is a proper subgroup of $\mathcal{Z}_f^1 \cong \mathbb{R}$ and contains a Cantor set $K \ni \alpha$.*

The aim of this article is to prove the following equivalent statement.

Theorem A'. *For any Liouville number α, there exists a $\mathcal{C}^1$ vector field ν on $\mathbb{R}_+$ vanishing only at 0 whose time-t map is smooth for every $t \in \{1\} \cup K$, for some Cantor set K containing α, but not $\mathcal{C}^2$ for some other $t \in \mathbb{R}$.*

Half of the question remains open: one would now like to prove that a $\mathcal{C}^1$ vector field on $\mathbb{R}_+$ whose time-1 and α maps are smooth, for some α satisfying a diophantine condition, is necessarily smooth itself, drawing one's inspiration from similar problems in the case of circle diffeomorphisms. This parallel suggests many more questions: can the set of smooth times be dense but countable? Is there some particular arithmetic relation between two irrational smooth times of a nonsmooth flow?...

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1 Overview

The general idea of the construction is the same as in [E1]. We repeat it here for completeness' sake as well as to emphasize the slight (but key) improvements, gathered at the end of the section. All below statements will be made precise and proved in the subsequent sections.

1.1 Sergeraert's construction

We first need to explain how to build a $\mathcal{C}^1$ vector field whose flow is smooth for some times but not $\mathcal{C}^2$ for others. Therefore, we sketch Sergeraert's construction (with some minor modifications). Sergeraert starts with a diffeomorphism f_0 which is the time-1 map of a "well-chosen" smooth vector field ν_0 on $\mathbb{R}_+$ (described later). He subjects it to infinitely many "small" (explicit) perturbations, with disjoint supports, closer and closer to 0, denoted by γ_k, $k \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$, so that

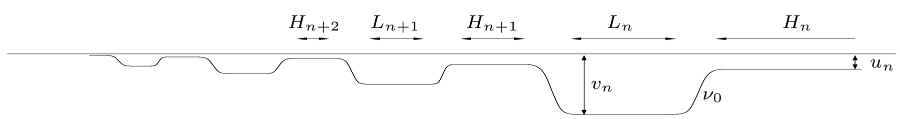
$$f = f_0 + \sum_{k \geq 1} \gamma_k$$

is still a smooth diffeomorphism of $\mathbb{R}_+$ (to ensure this, he only needs to pick the γ_k's so that their sum converges in $\mathcal{C}^\infty$ topology and is $\mathcal{C}^1$-small compared to f_0), but that its Szekeres vector field, on the other hand, is not smooth anymore. More precisely, he makes sure that the time-1/2 map of the resulting vector field is not $\mathcal{C}^2$.

It is not straightforward, even when one knows their expressions, to visualize the effect of the perturbations γ_k on the Szekeres vector field of f_0 and on its time-1/2 map. A way to understand how things work is to interprete Sergeraert's construction in terms of deformation by conjugation. Let us therefore describe the construction all over again, in a different language.

We start with the same smooth vector field ν_0 (Sergeraert's, described below) and this time, we're going to obtain the desired vector field ν (the one with a smooth time-1 map and a non $\mathcal{C}^2$ time-1/2 map) as a limit of a sequence of deformations ν_k, each ν_k being the pull-back $h_k^*\nu_0$ of ν_0 by a smooth diffeomorphism h_k of $\mathbb{R}_+$. The flow f_k^t of ν_k is then related to the flow f_0^t of ν_0 by $f_k^t = h_k^{-1} \circ f_0^t \circ h_k$. The point is to cook up the conjugations h_k so that f_k^1 converges in $\mathcal{C}^\infty$ topology while $f_k^{1/2}$ converges only in $\mathcal{C}^1$ topology (in particular, the h_k must diverge in $\mathcal{C}^2$ topology).

Here, the behaviour of the initial vector field plays a crucial role: it vanishes only at 0, is negative elsewhere, and its graph resembles an undersea landscape consisting of a sequence of alternating lowlands L_n and highlands H_n, accumulating at the origin, whose respective altitudes $-v_n$ and $-u_n$ (measured from the water surface, so that $0 < u_n < v_n$) go to zero very fast when n grows (so that ν_0 is infinitly flat at 0), but "oscillate wildly" in the sense that the ratios v_n/u_n (and actually v_n^k/u_n for all k) tend to infinity. A consequence of this behaviour is that,



if an element f_0^t of the flow takes a segment $S \subset L_n$ (resp. $S \subset H_n$) into L_n, then its restriction to S is the translation $x \mapsto x - tv_n$ (resp. an affine map with big dilation factor v_n/u_n). This follows immediatly from the invariance of ν_0 under its flow: $\nu_0 \circ f_0^t = \nu_0 \times Df_0^t$.

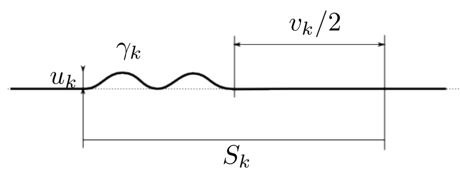
In the light of these remarks, we can move on to the definition of the conjugations h_k. What we actually construct for each k is a diffeomorphism g_k, and we then define h_k as $g_k \circ h_{k-1}$. Hence $\nu_k = h_k^*\nu_0 = h_{k-1}^* g_k^* \nu_0$, so that the flows of ν_k and ν_{k-1} are given by

$$f_k^t = h_{k-1}^{-1} \circ (g_k^{-1} \circ f_0^t \circ g_k) \circ h_{k-1} \quad \text{and}$$
$$f_{k-1}^t = h_{k-1}^{-1} \circ f_0^t \circ h_{k-1}$$

respectively. Thus, intuitively, we want $g_k^{-1} \circ f_0^1 \circ g_k - f_0^1$ to be $\mathcal{C}^k$-small (say less than 2^{-k}) while $g_k^{-1} \circ f_0^{1/2} \circ g_k - f_0^{1/2}$ is $\mathcal{C}^2$-big. To do that, we chose a g_k which

- commutes with f_0^1 everywhere except in a small region: a fondamental interval S_k of f_0^1 lying "in the middle of L_k";
- is $\mathcal{C}^k$ close to the identity in this region.

More precisely, we take g_k equal to the identity near 0 and of the form $\mathrm{id} + \gamma_k$ on S_k, where γ_k is a $\mathcal{C}^k$ small function supported in S_k, of the form:



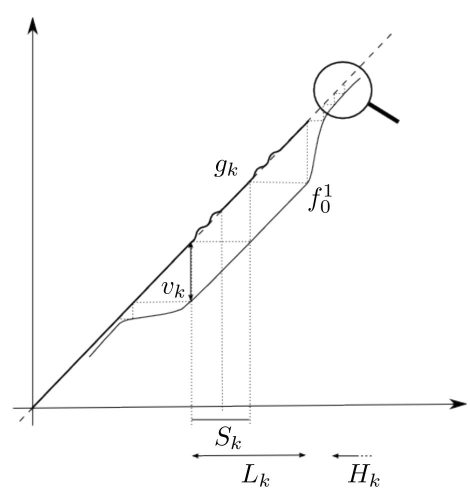
(we will see shortly why this form in particular). One easilly checks that this choice of g_k gives:

$$f_k^1 = f_{k-1}^1 + \gamma_k.$$

(this construction is thus really equivalent to Sergeraert's). The support of $g_k - \mathrm{id}$, on the other hand, is *not* S_k. Indeed, the above information is enough to determine g_k on all of $\mathbb{R}_+$: g_k is the identity on $[0, \min S_k]$, but $[\max S_k + \infty)$ is tiled by segments $S_k^p = f_0^{-p/q_k}(S_k)$, $p \geq 1$, on which

$$g_k \mid_{S_k^p} \quad = \quad f_0^{-p} \circ (g_k \mid_{S_k}) \circ f_0^p \quad = \quad f_0^{-p} \circ (\mathrm{id} + \gamma_k) \circ f_0^p.$$

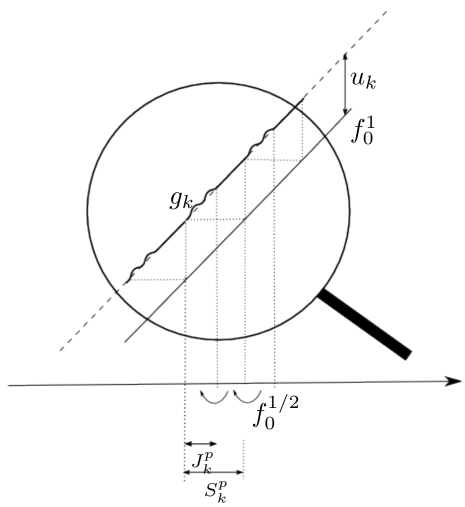
On $[\sup S_k, \sup L_k]$ in particular, f_0^1 coincides with the translation by $-v_k$, so g_k commutes with this translation.



If $S_k^p \subset H_k$ on the other hand, the restriction of f_0^p to S_k^p is an affine map of the form

$$x \in S_k^p \mapsto \frac{v_k}{u_k}\, x + c_k,$$

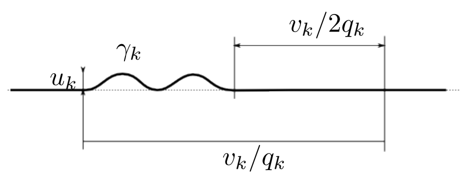
where c_k is a real constant. Hence, $g_k \mid_{S_k^p}$ is conjugate to $g_k \mid_{S_k}$ by an affine map of huge ratio, precisely cooked up to make $g_k \mid_{S_k^p}$ $\mathcal{C}^2$ big (g_k converges towards the identity in $\mathcal{C}^1$ topology, though).



The disymetric behaviour of γ_k had a purpose as well: on one half of the segment S_k^p, one can check that $g_k^{-1} \circ f_0^{1/2} \circ g_k - f_0^{1/2}$ is exactly $g_k - \mathrm{id}$, and hence $\mathcal{C}^2$ big. Superimposing all these perturbations (*i.e* conjugating by $h_k = g_k \circ \ldots \circ g_1$ and taking the $\mathcal{C}^1$ limit) has the desired effect on the time-1/2 map of the limit vector field.

1.2 Combination with Anosov–Katok-type methods

Now let α be an irrational number. We want to modify the above construction so that in the end, both 1 and α are smooth times of the limit vector field. The idea is to pick an approximation of α by rational numbers p_k/q_k, $k \geq 1$, to take an initial vector field ν_0 similar to Sergeraert's, and, this time, to ask g_k to commute almost everywhere not with f_0^1 anymore, but with f_0^{1/q_k} (and thus with both $f_0^{p_k/q_k}$ and $f_0^{q_k/q_k} = f_0^1$). More precisely, g_k is still the identity near 0, but this time, it is of the form $\mathrm{id} + \gamma_k$ on a fondamental interval of f_0^{1/q_k} lying in L_k (and thus of length v_k/q_k). Again, γ_k must be chosen $\mathcal{C}^k$ small. In particular, u_k must be a $o(v_k^k/q_k^k)$.



That way, one can make sure, say, that

$$\left\|f_k^t - f_{k-1}^t\right\|_k = \left\|g_k^{-1} \circ f_0^t \circ g_k - f_0^t\right\|_k = \|\gamma_k\|_k < 2^{-k-1} \quad \text{for } t = p_k/q_k \text{ and } 1$$

(both equalities are direct consequences of the construction). Now if $|\alpha - p_k/q_k|$ is small enough (roughly speaking, $|\alpha - p_k/q_k| = o(\|\nu_l\|_k^{-1})$ for $l = k$ and $k-1$, assuming these "norms" are finite), the above bounds remain true for $t = \alpha$ (replacing 2^{-k-1} by 2^{-k}, say), which ensures the regularity of the limit time-α map. But based on the previous paragraph, the more $u_k = o(1/q_k^k)$ is small, the more $\|g_k\|_k$, $\|h_k\|_k$ and thus $\|\nu_k\|_k$ are big. So, basically, in order for the process to converge, $|\alpha - p_k/q_k|$ must be much smaller than $1/q_k^k$, and hence α must be a Liouville number.

In [E1], we proved the existence of some well-chosen α and q_k for which the process indeed converges. The main contribution of this article is to make all the "rough" estimations above precise, *i.e* to control the size of the perturbations in terms of the initial data q_k, and to deduce from it that *any* Liouville number α has a suitable approximation by rational numbers for which the process converges and provides the desired vector field ν.

2 Notations and toolbox

For any $\mathcal{C}^k$ map g on $\mathbb{R}_+$ we set

$$\|g\|_k = \sup\left\{\left|D^m g(x)\right|,\ 0 \le m \le k,\ x \in \mathbb{R}_+\right\} \in [0, +\infty].$$

For any $g \in \mathcal{D}^2$, we define Lf by

$$Lf = D \log Df = \frac{D^2 f}{Df}.$$

The non-linear differential operator L satisfies the following chain rule:

$$L(h \circ g) = Lh \circ g \cdot Dg + Lg.$$

To compute or control derivatives of products and compositions, we will also use Leibniz rule:

$$D^k(gh) = \sum_{l=0}^{k} \binom{k}{l} D^l h\ D^{k-l} g$$

and Faà di Bruno's formula in the form

$$D^k(h \circ g) = \sum_{\pi \in \Pi_k} \left(D^{|\pi|} h\right) \circ g \cdot \prod_{B \in \pi} D^{|B|} g,$$

where Π_k is the set of all partitions π of $\{1, \cdots, k\}$ and $|X|$, for any finite set X, is the number of its elements.

Finally, let η be a vector field on $\mathbb{R}_+$. Throughout the paper, we will make no difference between η and the function η/∂_x, where x is the underlying coordinate in $\mathbb{R}_+$, and in particular we will identify ∂_x with 1. For $g \in \mathcal{D}^1$, we denote by $g^*\eta$ the pullback of η by g which, viewed as a function, has the following expression:

$$h^*\eta = \frac{\eta \circ h}{Dh}.$$

3 A machine for turning rational approximations into vector fields

What we actually describe in this section is a "manufacturing process" which, to any increasing sequence of positive integers $(q_k)_{k\geq 1}$, associates a specific $\mathcal{C}^1$ vector field ν on $\mathbb{R}_+$, with a smooth time-1 map. Then (in the next sections), we show that any Liouville number α has a suitable approximation by rational numbers $(p_k/q_k)_{k\geq 1}$ such that the vector field ν associated to the q_k's has all the additional properties listed in Theorem A.

Let $(q_k)_{k\geq 1}$ be any increasing sequence of positive integers (fixed until the end of Section 3). In order to produce ν, we must first associate to $(q_k)_{k\geq 1}$ a number of intermediate objects, the main of which being an initial vector field ν_0, smooth on $\mathbb{R}_+$, and a sequence $(g_k)_{k\geq 1}$ of smooth diffeomorphisms of $\mathbb{R}_+$. Those are used to deform ν_0 gradually into new smooth vector fields

$$\nu_k = h_k^* \nu_0 \quad \text{where } h_k = g_k \circ ... \circ g_1,$$

which converge in $\mathcal{C}^1$ topology, and we define ν as their limit.

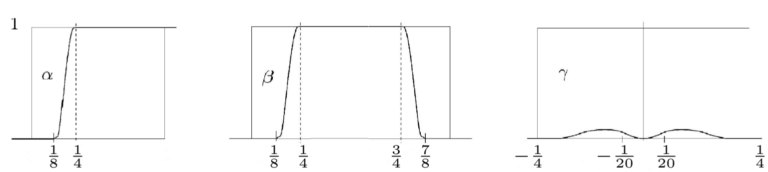
3.1 Common basis

Some material used to construct ν_0 is common to every $(q_k)_{k\geq 1}$, namely the coefficients $(v_n)_{n\geq 1}$ defined by

$$v_n = 2^{-(n+3)^2} \quad \text{for all } n \geq 1,$$

and three smooth functions $\alpha, \beta, \gamma \colon \mathbb{R} \to [0,1]$ satisfying the following conditions:

- α vanishes on $\left(-\infty, \frac{1}{8}\right]$, equals 1 on $\left[\frac{1}{4}, +\infty\right)$, and $\|\alpha\|_1 < 16$;
- β vanishes outside $\left[\frac{1}{8}, \frac{7}{8}\right]$, equals 1 on $\left[\frac{1}{4}, \frac{3}{4}\right]$, and $\|\beta\|_1 < 16$;
- γ vanishes outside $\left[\frac{1}{4}, \frac{3}{4}\right]$, $\gamma(x) = x^2/2$ if $|x| \leq 1/20$, and $\|\gamma\|_1 < 1$.



3.2 Initial vector field and related objects

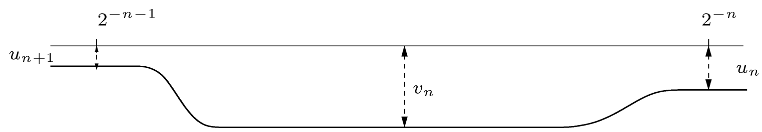
The coefficients $(u_n)_{n\geq 1}$ defined now on the other hand, depend on $(q_k)_k$:

$$u_n = 2^{-n-4}\, q_n^{-n}\, v_n^n\, \|\gamma\|_n^{-1} \quad \text{for all } n \geq 1. \tag{2}$$

The initial vector field ν_0 is then defined by:

$$\nu_0(x) = -u_{n+1} - (u_n - u_{n+1})\, \alpha(2^{n+1}x - 1) - (v_n - u_n)\, \beta(2^{n+1}x - 1) \quad (3)$$

$$\text{for } x \in [2^{-n-1}, 2^{-n}],\ n \geq 1, \quad \nu_0(0) = 0 \quad \text{and} \quad \nu_0(x) = -u_1 \quad \text{for } x \geq 1/2. \quad (4)$$



One easily checks that ν_0 is smooth, infinitely flat at the origin and $\mathcal{C}^1$-bounded — with $0 < \|\nu_0\|_1 < 1$. Furthermore, ν_0 equals $-v_n$ identically on the central part of $[2^{-n-1}, 2^{-n}]$, namely $[2^{-n-1} + 2^{-n-3}, 2^{-n} - 2^{-n-3}]$, and $-u_n$ on $[2^{-n} - 2^{-n-4}, 2^{-n} + 2^{-n-3}]$.

We denote by $\{f_0^t, t \in \mathbb{R}\}$ the flow of ν_0, and fix a forward orbit $\{a_l, l \geq 0\}$ of $f_0 = f_0^1$, where $a_0 = 1$ and $a_l = f_0(a_{l-1})$ for all $l \geq 1$. A simple computation of travel time at constant speed shows that for every $n \geq 1$, there exist integers i and j such that

$$2^{-n} - 2^{-n-4} \leq a_{i+2} < a_{i-1} \leq 2^{-n} + 2^{-n-3} \quad (5)$$

$$\text{and} \quad 2^{-n-1} + 2^{-n-3} \leq a_{j+2} < a_{j-1} \leq 2^{-n} - 2^{-n-3}. \quad (6)$$

We denote by $i(n)$ (resp. $j(n)$) the smallest integer i (resp. j) satisfying (5) (resp. (6)). Thus ν_0 equals $-v_n$ on $[a_{j(n)+2}, a_{j(n)-1}]$, and hence f_0^t induces on $[a_{j(n)+1}, a_{j(n)-1}]$ the translation by $-tv_n$ for $0 \leq t \leq 1$. Similarly, f_0^t induces the translation by $-tu_n$ in a neighbourhood of $a_{i(n)}$.

3.3 Conjugating diffeomorphisms and their properties

For all $k \geq 1$, we define $\gamma_k : \mathbb{R}_+ \to [0, 1]$ by:

$$\gamma_k(x) = u_k \gamma\Big(\frac{q_k}{v_k}\big(x - a_{j(k)}\big)\Big) \qquad \text{for all } x \in \mathbb{R}_+. \quad (7)$$

The map γ_k is supported in $S_k = \Big[a_{j(k)} - \frac{v_k}{4q_k}, a_{j(k)} + \frac{v_k}{4q_k}\Big]$, which is a fundamental interval of $f_0^{1/2q_k}$ since it lies inside $[a_{j(k)+1}, a_{j(k)-1}]$ where the flow f_0^s of ν_0 at time $0 \leq s \leq 1$ coincides with the translation by $-sv_k$. Furthermore, for all $x \in \mathbb{R}_+$ and all $m \in \mathbb{N}$

$$\begin{aligned} D^m \gamma_k(x) &= u_k \left(\frac{q_k}{v_k}\right)^m D^m \gamma\Big(\frac{q_k}{v_k}\big(x - a_{j(k)}\big)\Big) \\ &= 2^{-k-4} \left(\frac{q_k}{v_k}\right)^{m-k} \|\gamma\|_k^{-1}\, D^m \gamma\Big(\frac{q_k}{v_k}\big(x - a_{j(k)}\big)\Big) \end{aligned}$$

by definition (2) of u_k. In particular,

$$\|\gamma_k\|_k = 2^{-k-4}. \tag{8}$$

Now let J_k denote the fundamental interval $\left[a_{j(k)} - \frac{v_k}{4q_k}, a_{j(k)} + \frac{3v_k}{4q_k}\right]$ of f_0^{1/q_k}. We define $g_k \colon \mathbb{R}_+ \to \mathbb{R}_+$ as the unique map satisfying:

- $g_k = \mathrm{id}$ on $\left[0, a_{j(k)} - \frac{v_k}{4q_k}\right]$;
- $g_k = \mathrm{id} + \gamma_k$ on J_k;
- g_k commutes with f_0^{1/q_k} outside J_k, so that

$$g_k = f_0^{-p/q_k} \circ (\mathrm{id} + \gamma_k) \circ f_0^{p/q} \quad \text{ on } f_0^{-p/q_k}(J_k) \text{ for all } p \geq 0. \tag{9}$$

In particular, all segments $f_0^{-p/q_k}(J_k)$, $p \in \mathbb{Z}$, are stable under g_k. We now list some key properties of g_k.

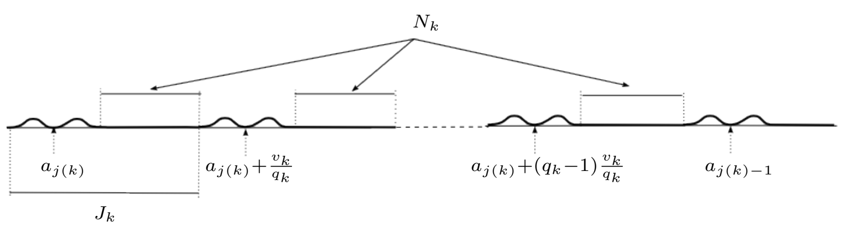
For all $0 \leq p \leq q_k$, f_0^{-p/q_k} and f_0^{p/q_k} coincide with the translations by $\frac{p}{q_k} v_k$ and $-\frac{p}{q_k} v_k$ on J_k and $f_0^{-p/q_k}(J_k)$ respectively, so that (9) becomes:

$$g_k = \mathrm{id} + \gamma_k \circ \left(\mathrm{id} - \frac{p}{q_k} v_k\right) \quad \text{ on } f_0^{-p/q_k}(J_k), \ 0 \leq p \leq q_k. \tag{10}$$

In particular, g_k is the identity on

$$N_k = \bigcup_{p=0}^{q_k - 1} \left(a_{j(k)} + (2p+1)\frac{v_k}{2q_k} + \left[-\frac{v_k}{4q_k}, \frac{v_k}{4q_k} \right] \right), \tag{11}$$

and *a fortiori* on every $f_0^{-p/q_k}(N_k)$, $p \geq 0$. This is also true for $p < 0$ since g_k is the identity on $[0, a_{j(k)} - v_k/4q_k]$.



Note furthermore that since ν_0 is constant equal to $-u_1$ on $[1/2, +\infty)$, f_0^{-1/q_k} coincides with the translation by u_1/q_k on $[1/2, +\infty)$, so g_k commutes with that translation there. *A fortiori*, g_k commutes with the translation by u_1 on $[1, +\infty)$. Furthermore,

$$\begin{aligned} g_k(a_0 = 1) &= f_0^{-j(k)} \circ g_k \circ f_0^{j(k)}(a_0) \\ &= f_0^{-j(k)}(g_k(a_{j(k)})) = f_0^{-j(k)}(a_{j(k)}) = a_0 = 1, \end{aligned}$$

so $[1,+\infty)$ is stable under g_k.

After differentiation, (9) becomes

$$Dg_k = \frac{Df_0^{p/q_k}}{Df_0^{p/q_k}\circ g_k} \times \left(1 + D\gamma_k \circ f_0^{p/q_k}\right) \quad \text{on } f_0^{-p/q_k}(J_k),\ p\geq 0, \tag{12}$$

so g_k is a diffeomorphism since $\|\gamma_k\|_1 < 1$, according to (8). One can actually simplify expression (12). The vector field ν_0 being invariant under the diffeomorphisms of its flow,

$$Df_0^t = \frac{\nu_0\circ f_0^t}{\nu_0} \quad \text{on } \mathbb{R}_+^* = (0,+\infty) \text{ for all } t\in\mathbb{R},$$

so

$$\frac{Df_0^{p/q_k}}{Df_0^{p/q_k}\circ g_k} = \frac{\nu_0\circ f_0^{p/q_k}}{\nu_0} \times \frac{\nu_0\circ g_k}{\nu_0\circ f_0^{p/q_k}\circ g_k}.$$

But for all $x\in f_0^{-p/q_k}(J_k)$,

$$\nu_0\circ f_0^{p/q_k}(x) = \nu_0\circ f_0^{p/q_k}\circ g_k(x) = -v_k$$

so

$$Dg_k = \frac{\nu_0\circ g_k}{\nu_0} \times \left(1 + D\gamma_k\circ f_0^{p/q_k}\right) \quad \text{on } f_0^{-p/q_k}(J_k),\ p\geq 0. \tag{13}$$

We now define for all $k\geq 1$ a smooth diffeomorphism $h_k = g_k\circ ...\circ g_1$ and a smooth vector field $\nu_k = h_k^*\nu_0$. The flow $\{f_k^t,\ t\in\mathbb{R}\}$ of ν_k is well defined and consists of smooth diffeomorphisms of $\mathbb{R}_+$ satisfying $f_k^t = h_k^{-1}\circ f_0^t\circ h_k$. Note that h_k, like g_l for all $l\leq k$, commutes with the translation by u_1 on $[1,+\infty)$. Let us define furthermore the (possibly empty) sets H_{k_0}, for all $k_0\geq 1$, and H by

$$H_{k_0} = \bigcap_{l\geq k_0}\bigcup_{0\leq p<q_l}\left[\frac{2p+1}{2q_l} - \frac{1}{4q_l}, \frac{2p+1}{2q_l} - \frac{1}{4q_l}\right] \tag{14}$$

and

$$H = \bigcup_{k_0\geq 1} H_{k_0}. \tag{15}$$

We will need the following lemma in the proof of Proposition 2 to show that for all $t\in H$, the time-t map of the limit vector field ν is not $\mathcal{C}^2$.

Lemma 1. *Let $t\in H_{k_0}\subset H$ for some $k_0\geq 1$. For all $k\geq k_0$, h_k has the following behaviour on the orbits $\{a_n,\ n\in\mathbb{Z}\}$ and $\{b_n = f_0^{-t}(a_n),\ n\in\mathbb{Z}\}$ of f_0^1:*

1. *h_k is infinitly tangent to the identity at b_n for all $n\geq j(k_0)$;*
2. *h_k is $\mathcal{C}^1$-tangent to the identity on $\{a_n,\ n\in\mathbb{Z}\}$ — i.e $h_k(a_n) = a_n$ and $Dh_k(a_n) = 1$ for all $n\in\mathbb{Z}$;*
3. *$(Lh_k - Lh_{k-1})(a_n)$ equals $\frac{u_k q_k^2}{v_k|\nu_0(a_n)|}$ if $n\leq j(k)$ and 0 otherwise.*

Proof. Let $k \geq k_0$. To prove the first point, we must check that for all $l \geq 1$ and $n \geq j(n_0)$, g_l is the identity near b_n. For $l < k_0$, this is true because $b_n \notin [a_{j(l)} - \frac{v_l}{4q_l}, +\infty)$, which contains the support of g_l. As for $l \geq k_0$, according to (11), we only need to check that $b_n \in f_0^p(N_l)$ for some $p \in \mathbb{N}$. But

$$b_{j(l)} = f_0^{-t}(a_{j(l)}) = a_{j(l)} + tv_l \in N_l$$

by definition of H_{k_0}, so $b_n = f_0^{n-j(l)}(b_{j(l)}) \in f_0^{n-j(l)}(N_l)$ for all $n \in \mathbb{Z}$, which concludes the proof of the first point.

Now $\gamma(0) = D\gamma(0) = 0$, so $\gamma_l(a_{j(l)}) = D\gamma_l(a_{j(l)}) = 0$ for all $l \geq 1$, according to (7), and since $g_l = \mathrm{id} + \gamma_l$ on J_l, g_l is tangent to the identity at $a_{j(l)}$. This is also true at every point $f_0^{-p/q_l}(a_j(l))$, $p \geq 0$, by definition (9) of g_l (in particular at every a_n, $n \leq j(l)$), and at every a_n, $n > j(l)$ since $g_l = \mathrm{id}$ on a neighbourhood of $[0, a_{j(l)+1}] = [0, a_{j(l)} - v_l]$. This, applied to $h_k = g_k \circ ... \circ g_1$, proves the second point.

Let us now apply the chain rule to $h_k = g_k \circ h_{k-1}$:

$$Lh_k = Lg_k \circ h_{k-1} \times Dh_{k-1} + Lh_{k-1}.$$

For all $n \in \mathbb{Z}$, point 2 tells us that $h_{k-1}(a_n) = a_n$ and $Dh_{k-1}(a_n) = 1$, so the above equality gives

$$(Lh_k - Lh_{k-1})(a_n) = Lg_k(a_n).$$

For $n > j(k)$, $Lg_k(a_n) = 0$ since g_k is the identity on a neighbourhood of $\left[0, a_{j(k)+1}\right]$. Suppose now that $n \leq j(k)$ and write $p = j(n_k) - n \geq 0$. According to (9), on a neighbourhood of a_n, g_k is given by:

$$g_k = f_0^{-p} \circ (\mathrm{id} + \gamma_k) \circ f_0^p.$$

Furthermore

$$\mathrm{id} = f_0^{-p} \circ \mathrm{id} \circ f_0^p.$$

The chain rule formula applied to both equalities gives:

$$\begin{aligned} Lg_k = Lg_k - L\mathrm{id} = Lf_0^{-p} \circ (\mathrm{id} + \gamma_k) \circ f_0^p \times D\,(\mathrm{id} + \gamma_k) \circ f_0^p \times Df_0^p \\ + L(\mathrm{id} + \gamma_k) \circ f_0^p \times Df_0^p - Lf_0^{-p} \circ f_0^p \times Df_0^p. \end{aligned}$$

At $a_n = f_0^{-p}(a_{j(k)})$, we get

$$\begin{aligned} Lg_k(a_n) = Lf_0^{-p}\left(a_{j(k)} + \gamma_k(a_{j(k)})\right) \times (1 + D\gamma_k(a_{j(k)})) \times Df_0^p(a_n) \\ - Lf_0^{-p}(a_{j(k)}) \times Df_0^p(a_n) + L(\mathrm{id} + \gamma_k)(a_{j(k)}) \times Df_0^p(a_n). \end{aligned}$$

Since $\gamma_k(a_{j(k)}) = D\gamma_k(a_{j(k)}) = 0$, the first two terms cancel each other. In the end, the invariance relation $\nu_0 \circ f_0^p = Df_0^p \times \nu_0$ applied at a_n and the definition of γ_k give

$$Lg_k(a_n) = L(\mathrm{id} + \gamma_k)(a_{j(k)}) \times \frac{\nu_0(a_{j(k)})}{\nu_0(a_n)} = \frac{u_k q_k^2}{v_k^2} \times \frac{v_k}{|\nu_0(a_n)|}.$$

□

3.4 Convergence of the deformation process and properties of the limit

Proposition 2. *For all $k \geq 1$,*

$$\left\|f_k^t - f_{k-1}^t\right\|_k \leq 2^{-k-4} \quad \textit{for all} \quad t \in \frac{1}{q_k}\mathbb{Z} \cap [0,1]. \tag{i_k}$$

In particular, the time-1 maps f_k^1 converge in $\mathcal{C}^\infty$ topology towards a smooth diffeomorphism f with no other fixed point than 0, whose Szekeres vector field ν is the $\mathcal{C}^1$ limit of the vector fields ν_k. On the other hand, for all t in H, the time-t map f^t of ν is not $\mathcal{C}^2$.

Proof. Let us start with estimate (i_k). Let $\{\varphi_k^t,\ t \in \mathbb{R}\}$ denote the flow of $g_k^* \nu_0$, so that

$$\varphi_k^t = g_k^{-1} \circ f_0^t \circ g_k.$$

Since

$$\nu_k = h_k^* \nu_0 = h_{k-1}^* g_k^* \nu_0 \qquad \text{and} \qquad \nu_{k-1} = h_{k-1}^* \nu_0,$$

the flows of ν_k and ν_{k-1} are given by

$$f_k^t = h_{k-1}^{-1} \circ \varphi_k^t \circ h_{k-1} \quad \text{and} \quad f_{k-1}^t = h_{k-1}^{-1} \circ f_0^t \circ h_{k-1}.$$

By definition, g_k commutes with f_0^{1/q_k} outside J_k. As a consequence, g_k commutes with any iterate f_0^{p/q_k}, $p \geq 1$, outside the interval

$$\bigcup_{q=0}^{p-1} f_0^{-q/q_k}(J_k).$$

Thus, φ_k^{p/q_k} coincides with f_0^{p/q_k} outside this interval. In particular, for $0 \leq p \leq q_k$, since f_0^s coincides with the translation by $-sv_k$ on $[a_{j(k)} - v_k, a_{j(k)} + v_k]$ for all $0 \leq s \leq 1$, φ_k^{p/q_k} coincides with f_0^{p/q_k} outside

$$M_k = \left[a_{j(k)} - \frac{v_k}{4q_k}, a_{j(k)} + v_k - \frac{v_k}{4q_k}\right].$$

Moreover, for all $x \in J_k$,

$$\begin{aligned}
\varphi_k^{1/q_k}(x) &= g_k^{-1} \circ f_0^{1/q_k} \circ g_k(x) \\
&= g_k^{-1}\left(g_k(x) - \frac{v_k}{q_k}\right) \\
&= g_k^{-1}\left(x + \gamma_k(x) - \frac{v_k}{q_k}\right) \quad \text{by definition of } g_k \text{ on } J_k \\
&= x - \frac{v_k}{q_k} + \gamma_k(x) \quad \text{since } x + \gamma_k(x) - \frac{v_k}{q_k} < \min(\operatorname{Supp} g_k^{-1}) \\
&= f_0^{1/q_k}(x) + \gamma_k(x).
\end{aligned}$$

Thus, since φ_k^{1/q_k} coincides with f_0^{1/q_k} outside J_k, $\varphi_k^{1/q_k} - f_0^{1/q_k} = \gamma_k$ on all of $\mathbb{R}_+$. Similarly, for all $0 \le p \le q_k$,

$$\varphi_k^{p/q_k}(x) - f_0^{p/q_k}(x) = \sum_{q=0}^{p-1} \gamma_k \left(x - \frac{q v_k}{q_k} \right) \qquad \text{for all } x \in \mathbb{R}_+, \tag{16}$$

$$\text{so } \left\| \varphi_k^{p/q_k} - f_0^{p/q_k} \right\|_m = \|\gamma_k\|_m \quad \text{for all } m \in \mathbb{N}. \tag{17}$$

But in the region M_k where φ_k^{p/q_k} and f_0^{p/q_k} differ for $0 \le p \le q_k$, the diffeomorphism h_{k-1} is the identity since

$$\operatorname{Supp} h_{k-1} \subset \bigcup_{l \le k-1} \operatorname{Supp} g_l \subset \left[a_{j(k-1)} - \frac{v_{k-1}}{4 q_{k-1}}, +\infty \right).$$

Consequently, for all $0 \le p \le q_k$, the relations

$$f_k^{p/q_k} = h_{k-1}^{-1} \circ \varphi_k^{p/q_k} \circ h_{k-1}$$

and

$$f_{k-1}^{p/q_k} = h_{k-1}^{-1} \circ f_0^{p/q_k} \circ h_{k-1}$$

imply:

$$f_k^{p/q_k} - f_{k-1}^{p/q_k} = \begin{cases} \varphi_k^{p/q_k} - f_0^{p/q_k} & \text{on } M_k \\ 0 \quad \text{outside}, \end{cases} \tag{18}$$

which, together with (17), gives (i_k):

$$\left\| f_k^{p/q_k} - f_{k-1}^{p/q_k} \right\|_k \le \left\| \varphi_k^{p/q_k} - f_0^{p/q_k} \right\|_k = \left\| \gamma_k \right\|_k \le 2^{-k-4}.$$

As a consequence, the time-1 maps $f_k^1 = f_k$ converge towards a smooth diffeomorphism f. Let us note furthermore that

$$\left| \frac{f_k(x) - f_{k-1}(x)}{f_0(x) - x} \right| \le 2^{-k-2} \quad \text{for all } k \ge 1. \tag{19}$$

Indeed, according to (16) and (18),

$$f_k(x) - f_{k-1}(x) = \begin{cases} \displaystyle\sum_{q=0}^{q_k-1} \gamma_k \left(x - \frac{q v_k}{q_k} \right) & \text{on } M_k, \\ \qquad 0 & \text{outside}, \end{cases}$$

so since at most one term of the above sum is nonzero,

$$|f_k(x) - f_{k-1}(x)| \le \|\gamma_k\|_0 \le u_k.$$

But on M_k,

$$|f_0(x) - x| = v_k.$$

The last two remarks imply inequality (19) for $u_k/v_k \leq 2^{-k-2}$. Thus for all $x \in \mathbb{R}_+^*$,

$$\begin{aligned}|f(x) - x| &= \left| f_0(x) - x + \sum_{k\geq 1} \big(f_k(x) - f_{k-1}(x)\big) \right| \\ &\geq |f_0(x) - x| \left(1 - \sum_{k\geq 1} 2^{-k-2}\right) \\ &\geq \frac{|f_0(x) - x|}{2} > 0.\end{aligned}$$

So f has no other fixed point than 0.

We could prove the $\mathcal{C}^1$ convergence of the vector fields ν_k by hand, as in [E1] and [E2]. But since a third similar proof would be of little interest, we choose to invoke a different argument here. In fact, the convergence of the ν_k can be derived directly from the $\mathcal{C}^\infty$ convergence of their time-1 maps, as an immediate consequence of a theorem by J.-C. Yoccoz [Y, chap. 4, Theorem 2.5] asserting the continuous dependence of the Szekeres vector field with respect to its time-1 map (in a more general setting and for suitably defined topologies). We denote by ν the limit of ν_k and by $\{f^t,\ t \in \mathbb{R}\}$ the flow of ν (so that $f = f^1$). For all $t \in \mathbb{R}$, f^t is the limit of f_k^t in $\mathcal{C}^1$ topology.

Now let $t \in H_{k_0}$ for some $k_0 \geq 1$. We want to prove that Lf^t is not continuous at 0. To do that, we compute Lf^t at $b_{i(l)} = f_0^{-t}(a_{i(l)})$ for all $l \geq k_0 + 1$. By invariance of ν under its flow,

$$Df^t = \frac{\nu \circ f^t}{\nu} \quad \text{on } \mathbb{R}_+^*$$

from which one computes

$$Lf^t = \frac{D\nu \circ f^t - D\nu}{\nu}.$$

In particular,

$$Lf^t(b_{i(l)}) = -\frac{D\nu(f^t(b_{i(l)})) - D\nu(b_{i(l)})}{u_l}.$$

But for all $k \geq k_0$,

$$\begin{aligned}f_k^t(b_{i(l)}) &= h_k^{-1} \circ f_0^t \circ h_k(b_{i(l)}) \\ &= h_k^{-1} \circ f_0^t(b_{i(l)}) \quad \text{according to Lemma 1,} \\ &= h_k^{-1}(a_{i(l)}) = a_{i(l)} \quad \text{according to Lemma 1 again,}\end{aligned}$$

so $f^t(b_{i(l)}) = \lim_k f_k^t(b_{i(l)}) = a_{i(l)}$. Besides, the derivative of $\nu_k = h_k^*\nu_0$ is

$$D\nu_k = D\nu_0 \circ h_k - (\nu_0 \circ h_k)\frac{Lh_k}{Dh_k},$$

so for all $k \geq l$, according to points 2 and 3 of Lemma 1,

$$D\nu_k(a_{i(l)}) = D\nu_0(a_{i(l)}) - \nu_0(a_{i(l)})Lh_k(a_{i(l)}) = \sum_{n=l}^{k} \frac{u_n q_n^2}{v_n}, \tag{20}$$

and according to point 1 of the same lemma,

$$D\nu_k(b_{i(l)}) = D\nu_0(b_{i(l)}) - \frac{Lh_k}{Dh_k}(b_{i(l)})\nu_0(b_{i(n_l)}) = 0 - 0 = 0. \tag{21}$$

The vector fields ν_k converge towards ν in $\mathcal{C}^1$ topology on $\mathbb{R}_+$, so Formulae (20) and (21) give

$$D\nu(a_{i(l)}) = \sum_{n \geq l} \frac{u_n q_n^2}{v_n} \quad \text{and} \quad D\nu(b_{i(l)}) = 0.$$

In the end,

$$Lf^t(b_{i(l)}) = -\sum_{n \geq l} \frac{u_n q_n^2}{v_n u_l} < -\frac{q_l^2}{v_l} \to -\infty \quad [l \to \infty]$$

so f^t is not $\mathcal{C}^2$ at 0. □

4 Polynomial control of the manufactured objects

Proposition 3. *There are maps n and $c\colon \mathbb{N}^* \times \mathbb{N} \to \mathbb{R}_+$ such that for any increasing sequence $(q_k)_{k \geq 1}$ of positive integers, the vector fields $(\nu_k)_{k \geq 0}$ built from $(q_k)_{k \geq 1}$ and their flows $\{f_k^t,\ t \in \mathbb{R}\}$ satisfy*

$$\left\| \nu_k \circ f_k^t \right\|_r \leq c(k,r) q_k^{n(k,r)} \quad \textit{for all } (k,r) \in \mathbb{N}^* \times \mathbb{N}. \tag{22}$$

This proposition relies on the following assertions.

Lemma 4. *There are universal bounds on all derivatives of ν_0 and f_0^t, $t \in [0,1]$,* i.e. *bounds which depend neither on $(q_k)_k$ nor on t.*

Lemma 5. *There is a polynomial (in q_k) control on the growth of the derivatives of g_k,* i.e. *there exist universal maps $c, n\colon \mathbb{N}^* \times \mathbb{N} \to \mathbb{R}_+$ such that for any $(q_k)_{k \geq 1}$, the associated $(g_k)_{k \geq 1}$ satisfies*

$$\max\left(\|g_k - \mathrm{id}\|_r, \|g_k^{-1} - \mathrm{id}\|_r \right) < c(k,r) q_k^{n(k,r)} \tag{23}$$

for all $(k,r) \in \mathbb{N}^* \times \mathbb{N}$.

Proof of Proposition 3 using Lemmas 4 and 5. We proceed by induction on k. Step $k = 0$ follows directly from Lemma 4 and Faa di Bruno's Formula. For $k \geq 1$, step k is easily obtained from step $k-1$ and Lemma 5 applying Faa di Bruno's and Leibnitz' derivation formulas to the relations

$$\nu_k = g_k^* \nu_{k-1} = (\nu_{k-1} \circ g_k)(Dg_k^{-1} \circ g_k) \quad \text{and} \quad f_k^t = g_k^{-1} \circ f_{k-1}^t \circ g_k.$$

□

Proof of Lemma 4. It is clear from the definition (3) of ν_0 that its derivatives are bounded independently of the coefficients $(u_n)_n$, and thus of $(q_n)_n$. Similar bounds on the derivatives of the flow (for a compact set of times) are then easilly derived from an appropriate (generalized) version of Gronwall's Lemma. □

Proof of Lemma 5. Let $k \geq 1$. The orders $r = 0$ and $r = 1$ are easily settled using $(*)$, (8) and (13). In particular,

$$\|g_k - \mathrm{id}\|_1 < \frac{1}{2} \quad \text{for all } k. \tag{24}$$

Note that given (24), a polynomial (in q_k) control on the growth of the derivatives of $g_k - \mathrm{id}$ automatically gives one on $g_k^{-1} - \mathrm{id}$. This is because the inverse of any smooth diffeomorphism g satisfies

$$(D^r g^{-1}) \circ g = \frac{P_r(Dg, ..., D^r g)}{(Dg)^{2r+1}}, \tag{25}$$

where P_r is a universal polynomial in r variables (independent of g), and in our case, $Dg = Dg_k$ is bounded below independently of $(q_n)_n$. Formula (25) is obtained by induction on r, starting with the identity $Dg^{-1} \circ g \times Dg = 1$ and using Faa di Bruno's Formula.

We now focus on $g_k - \mathrm{id}$. Recall that

$$g_k = \begin{cases} \mathrm{id} & \text{on } [0, \min J_k] \\ \mathrm{id} + \gamma_k & \text{on } J_k \\ f_0^{-p} \circ (\mathrm{id} + \gamma_k) \circ f_0^p & \text{on } f_0^{-p}(J_k), \text{ for all } p \geq 1. \end{cases} \tag{26}$$

Thus, on $[0, \max J_k]$,

$$|D^r(g_k - \mathrm{id})| = |D^r \gamma_k| \leq u_k \left(\frac{q_k}{v_k}\right)^r \|\gamma\|_r \leq c(r,k) q_k^{n(r,k)},$$

with

$$c(r,k) = \frac{2^{-k-4} \|\gamma\|_r \, v_k^{k-r}}{\|\gamma\|_k} \quad \text{and} \quad n(r,k) = r - k,$$

by definition (2) of u_k. Then, given (26) (and Faa di Bruno's formula again), a uniform (in p) polynomial (in q_k) control on the derivatives of $f_0^p \mid_{f_0^{-p}(J_k)}$ will ensure the desired control on $D^r(g_k - \mathrm{id})$ on the rest of $\mathbb{R}_+$.

The vector field ν_0 being preserved by its own flow,

$$Df_0^p = \frac{\nu_0 \circ f_0^p}{\nu_0} \quad \text{on } \mathbb{R}_+^*.$$

In particular, on $f_0^{-p}(J_k)$,

$$Df_0^p = -\frac{v_k}{\nu_0},$$

and thus, for all $r \geq 1$,

$$D^{r+1} f_0^p = \frac{Q_r(\nu_0, ..., D^r \nu_0)}{\nu_0^{2^r}}, \tag{27}$$

where Q_r is a universal polynomial (independent of ν_0) in $r+1$ variables. According to Lemma 4, for each r, the numerator of (27) is bounded independently of $(q_k)_k$. As for the denominator, $|\nu_0(x)| \geq u_k$ for all $x \in [\max J_k, \infty)$, so by definition (2) of u_k,

$$\frac{1}{\nu_0^{2^r}} \leq \left(2^{k+4} v_k^{-k} \|\gamma\|_k\right)^{2^r} q_k^{2^r(k+1)},$$

which is the kind of control we were looking for (the bound does not depend on p). $\square$

5 Convergence of the time-α maps

Proposition 6. *Let α be a Liouville number. There is a sequence $(p_k/q_k)_{k\geq 1}$ of rational approximations of α such that the vector field ν built from $(q_k)_{k\geq 1}$ has all the properties described in Theorem A.*

Let α be a Liouville number. Then by definition there exists a sequence $(p_k/q_k)_{k\geq 1}$ of rational approximations of α satisfying

$$\left|\alpha - \frac{p_k}{q_k}\right| < \frac{2^{-k-2} c(k,k)^{-1}}{q_k^{n(k,k)}} \quad \text{for all } k, \tag{C_k}$$

where c and n are the maps given by Proposition 3. Note that for such a sequence $(q_k)_k$, the set H defined by (15) is a Cantor set (in particular nonempty). Hence, Proposition 6, and thus Theorem A, follow from Lemma 7 below and Proposition 2.

Lemma 7. *Let α be a Liouville number and $\frac{p_k}{q_k}$, $k \geq 1$, a sequence of rational approximations of α satisfying (C_k) for all $k \geq 1$. Then the vector fields ν_k associated to $(q_k)_{k\geq 1}$ and their flows satisfy*

$$\left\|f_k^\alpha - f_{k-1}^\alpha\right\|_k \leq 2^{-k} \quad \textit{for all } k \geq 1. \tag{28}$$

As a consequence, the time-α map of the limit ν of ν_k is smooth.

Proof. Let $k \geq 1$.

$$\left\|f_k^\alpha - f_{k-1}^\alpha\right\|_k \leq \left\|f_k^\alpha - f_k^{p_k/q_k}\right\|_k + \left\|f_k^{p_k/q_k} - f_{k-1}^{p_k/q_k}\right\|_k + \left\|f_{k-1}^{p_k/q_k} - f_{k-1}^\alpha\right\|_k.$$

According to (i_k) in Proposition 2, the central term is less than 2^{-k-2}. Now

$$D^n\left(f_k^\alpha - f_k^{p_k/q_k}\right) = D^n\left(\int_{p_k/q_k}^\alpha \frac{df_k^t}{dt} dt\right) = \int_{p_k/q_k}^\alpha D^n(\nu_k \circ f_k^t) dt,$$

so

$$\left\|f_k^\alpha - f_k^{p_k/q_k}\right\|_k \le \left|\alpha - \frac{p_k}{q_k}\right| \left\|\nu_k \circ f_k^t\right\|_k \le 2^{-k-2}$$

according to (C_k) and Proposition 3. A similar argument gives

$$\left\|f_{k-1}^{p_k/q_k} - f_{k-1}^\alpha\right\|_k \le 2^{-k-2}$$

and in the end,

$$\left\|f_k^\alpha - f_{k-1}^\alpha\right\|_k \le 2^{-k}. \tag{29}$$

□

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