Multi-dimensional shock interaction for a Chaplygin gas

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Abstract

A Chaplygin gas is an inviscid, compressible fluid in which the acoustic fields are linearly degenerate. We analyse the multi-dimensional shocks, which turn out to be sonic. Two shocks in general position interact rather simply. We investigate several two-dimensional Riemann problems and prove the existence of a unique solution. Among them is the supersonic reflection of a planar shock against a wedge: we remark that the solution cannot be a Mach Reflection, contrary to what happens for other gases, and that there always exists a solution in the form of a Regular Reflection.

Plan of the paper ; main results

We define a Chaplygin gas in the first section, where we prove that the pressure (also called acoustic) waves are characteristic. In the sequel, such waves are called 'shocks', despite the fact that they are contact discontinuities for such a gas. We also observe that the entropy remains constant across shocks, contrary to what happens in other gases. This explains why the solution must be isentropic when the initial data is so.

The local analysis of multidimensional shocks is made at Section 2. For steady flows, we show the lack of vorticity generation, even across a curved shock (Theorem 2.1). We give a five-lines proof of the non-existence of a three-shocks pattern (Paragraph 2.2), a result that holds true in a much more general context, see [12]. We then show that there are a lot of four-shocks patterns, and that they provide the interaction of two incoming shocks given in rather general position (Paragraphs 2.3 and 2.4). We also describe the shock reflection against an infinite wall (Paragraph 2.5). When the shock strength is large enough, we observe a concentration phenomenon of the mass along the wall; this is due to the boundedness of the pressure at inifinite density.

Section 3 is devoted to self-similar flows, which obey to a system that differs from the steady Euler equations by zero-order terms only. We prove again the lack of vorticity generation. This explains why the solution must be irrotational when the self-similar initial data is so. We show that shock curves bounded by a constant state are *a priori* known, because they are

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characteristic. We then describe two specific 2-D Riemann problems, that we call respectively the *saddle* and the *vortex*. They occur when the initial data is contant in each quadrant of the plane, and each initial discontinuity is resolved by one shock only. We also describe the problem of shock reflection past a wedge.

In Section 4, we reduce the solution of the various self-similar problem to the resolution of a quasi-linear elliptic PDE (22) in an *a priori* known subsonic domain, with a Dirichlet boundary condition on the sonic boundary. In case there is a physical boundary, then we have a Neumann condition on it. This PDE displays two difficulties: – it degenerates along the sonic line, – it is non-uniformly elliptic. For the saddle and the vortex, the subsonic domain is convex and bounded by four arcs of circles. At a formal level, the PDE is the Euler-Lagrange equation of a functional. This is however useless because this functional is infinite for functions satisfying the Dirichlet condition.

We prove in Section 5 the existence and uniqueness of the Dirichlet boundary-value problem in an arbitrary compact convex domain of which the curvature and the curvature radius are bounded. Sub-/super-solutions play an important role in the proof of Lipschitz estimates. This is the more technical part of the paper, though not the least interesting. We observe the strange property that the derivatives of the solution at all order can be computed explicitly along the boundary, in terms of the curvature and its arc-length derivatives. Our general result, Theorem 5.1 allows us to construct the solution to Riemann problems that are much more general than those described in Section 3 ; see Theorems 6.1 and 6.2. It is remarkable that the exterior, hyperbolic boundary-value problem admits solutions given by an explicit formula.

The last section is devoted to the shock reflection past a wedge. We distinguish three regimes, according to the strength of the incoming shock. We point out that the Mach reflection is irrelevant for a Chaplygin gas. The subsonic regime is the one that has received most attention in the literature for ideal gases. It displays a singularity at the tip of the wedge. We postpone the proof of the existence of a solution to a further work. The supersonic regime may display or not a concentration phenomenon. In the former case, the solution is extremely simple, since the shock does not reflect at all. The intermediate regime is more interesting and we prove the existence of a unique solution to the reflection problem. We get rid of the Neumann boundary condtion with the help of a symmetry, and they make use of Theorem 5.1.

1 What is a Chaplygin gas ?

Let us consider the Euler equations of an inviscid adiabatic compressible fluid of material velocitiy field u

$$\partial_t\rho + \operatorname{div}(\rho u) = 0, \tag{1}$$

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p(\rho, s) = 0, \tag{2}$$

$$\partial_t\left(\frac{1}{2}\rho|u|^2 + \rho e\right) + \operatorname{div}\left(\left(\frac{1}{2}\rho|u|^2 + \rho e + p\right)u\right) = 0. \tag{3}$$

The internal state variables ρ, s, p, e denote respectively the mass density, the specific entropy, the pressure and the specific energy. The last two are given positive functions of (ρ, s), which satisfy the thermodynamical constraint

$$\theta \mathrm{d}s = \mathrm{d}e + p\mathrm{d}\frac{1}{\rho}\,, \tag{4}$$

where $\theta = \theta(\rho, s)$ is another positive function, called temperature.

The state (ρ, s) is observable only when (1,2,3) is hyperbolic, which amounts to saying that the derivative $\partial p/\partial \rho$ is positive. Then the wave velocities in the direction $\xi \in S^2$ (the unit sphere) are given by $\lambda_0 = u \cdot \xi$ and $\lambda_\pm = u \cdot \xi \pm c(\rho, s)$ with

$$c := \sqrt{\frac{\partial p}{\partial \rho}}\,.$$

As it is well-known, λ_0 corresponds to entropy and vorticity waves. The corresponding characteristic field is linearly degenerate. The linearity of λ_0 in ξ suggests that it be associated to tranport phenomena at velocity u, that is alongthe particle paths. For instance, the entropy obeys $(\partial_t + u \cdot \nabla_x)s = 0$. Besides, $\lambda_\pm$ are associated to pressure waves (shocks and rarefaction waves), whose fields are usually genuinely nonlinear. This is the case for instance if $p(\rho, s) = \rho^\gamma f(s)$ with $f > 0$ and $\gamma > 1$ a constant (polytropic gases).

We are interested here in the special case where even the pressure fields are linearly degenerate. This amounts to writing $\partial(\rho c)/\partial \rho = 0$. Such fluids therefore have an equation of state of the Chaplygin form

$$p(\rho, s) = g(s) - \frac{1}{\rho} f(s), \text{ with } c(\rho, s) = \frac{1}{\rho}\sqrt{f(s)}\,.$$

We can describe more accurately the functions e and θ, using (4). We have

$$\left(\theta - \frac{1}{2\rho^2} f'(s) + \frac{1}{\rho} g'(s)\right) \mathrm{d}s = \mathrm{d}\left(e + \frac{1}{\rho} g(s) - \frac{1}{2\rho^2} f(s)\right),$$

from which we deduce that there exists a function $s \mapsto h(s)$ such that

$$\theta = \frac{1}{2\rho^2} f'(s) - \frac{1}{\rho} g'(s) + h'(s), \qquad e = \frac{1}{2\rho^2} f(s) - \frac{1}{\rho} g(s) + h(s).$$

Range of validity. A Chaplygin is an academic material. However, it is a good approximation of a real gas in a neighbourhood of a curve across which the sign of $\partial(\rho c)/\partial \rho$ changes. It should not be used on the whole range $\rho > 0$, since the pressure would become negative for small densities ! The advantage of working with a Chaplygin gas is that a lot of analysis can be done more or less explicitly. It thus gives hints for what could happen for more real gases.

1.1 The variation of the entropy across a pressure wave

Since the entropy s is a Riemann invariant for the λ_+-field, it remains constant in an associated rarefaction wave. Because of the linear degeneracy, this remains true in a discontinuity associated to λ_+, provided that the states $U_{l,r}$ on each side belong to the same connected component of the Hugoniot curve. Additionally, such a discontinuity should travel at a normal speed $\sigma = \lambda_+(U_l) = \lambda_+(U_r)$. We prove this directly from the Rankine-Hugoniot relations for arbitrary large shocks, under quite a reasonable assumption: the positivity of θ tells that the polynomial $X \mapsto \frac{1}{2}X^2 f'(s) - Xg'(s) + h'(s)$ takes positive values when X runs over $(0, +\infty)$. This implies $f' \geq 0, h' \geq 0$ and $g'^2 \leq f'h'$ unless one (and only one) of f', h' being zero. We shall assume a little bit more, that $f' > 0$ and $g' \leq 0 \leq h'$.

It is enough, because of Galilean invariance, to consider steady shocks. From (1), we may introduce the mass flux across the shock $j := (\rho u \cdot \nu)_{l,r}$, where ν is the unit normal to the shock. The fact that the shock be associated to λ_+ is the assumption that j is negative. Equation (2) yields

$$j[u] + [p]\nu = 0,$$

from which we have immediately the continuity of the tangential component (using $j \neq 0$): $[u \times \nu] = 0$. There remains

$$j^2 \left[\frac{1}{\rho} \right] + [p] = 0. \tag{5}$$

At last, a combination with the jump relation coming from (3) yields the well-known Hugoniot relation (using again $j \neq 0$)

$$\left[e + \frac{p}{\rho} \right] = \left\langle \frac{1}{\rho} \right\rangle [p],$$

where the brackets denote the arithmetic mean of the left and right values. So far, all the identities are valid for an arbitrary fluid. From now on, we make use of the special equation of state, and we obtain the identity

$$[h(s)] - \left\langle \frac{1}{\rho} \right\rangle [g(s)] + \frac{1}{2\rho_r \rho_l}[f(s)] = 0.$$

From our assumption, each of the three terms in the left-hand side is of the sign of $[s]$. Thus each one vanishes. In particular $[f(s)] = 0$ gives $[s] = 0$, which was our first claim. There remains to prove that $\lambda_+(U_{r,l}) = 0$. To do so, we recall

$$j^2 \left[\frac{1}{\rho} \right] = -[p] = \left[\frac{1}{\rho} f(s) - g(s) \right].$$

Because s is constant and $(\rho c)^2 = f$, this becomes

$$j^2 \left[\frac{1}{\rho} \right] = (\rho c)^2_{r,l} \left[\frac{1}{\rho} \right].$$

We notice that $[1/\rho] \neq 0$, for otherwise there would be no discontinuity at all. We thus have $j = \pm(\rho c)_{r,l}$, that is $j = -(\rho c)_{r,l}$ because of the sign of j, and this rewrites $u_{r,l} \cdot \nu = -c_{r,l}$, that is $\sigma = 0 = \lambda_+(U_r) = \lambda_+(U_l)$.

The previous analysis, plus the fact that s is transported along the particle paths ($(\partial_t + u \cdot \nabla_x)s = 0$), which do not cross the discontinuities associated to λ_0, show that piecewise smooth flows for a Chaplygin gas satisfy the following identities in the sense of distributions, for every smooth function ϕ:

$$\partial_t(\rho\phi(s)) + \mathrm{div}(\rho\phi(s)u) = 0. \tag{6}$$

We thus add the identities (6) to the standard requirements for admissible flows.

In particular, taking $\phi(s) := (s - \bar{s})^2$ and integrating, we find that if the flow is initially at constant entropy $\bar{s}$, then it remains so for every time. This justifies rigorously the isentropic model, in the case of a Chaplygin gas. This remark applies to the situations studied in the sections 4 and 7, where we consider two-dimensional Riemann problems and the reflection of a planar shock wave against a straight wedge.

2 Shocks in an isentropic Chaplygin gas

We thus restrict from now on to an isentropic Chaplygin gas. It has the special equation of state ($a > 0$)

$$p(\rho) = a^2 \left(\frac{1}{\rho_0} - \frac{1}{\rho} \right).$$

Since the flow is isentropic, it is enough to consider the Euler equations with only conservation of mass and momentum:

$$\partial_t \rho + \mathrm{div}(\rho u) = 0, \tag{7}$$

$$\partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla p(\rho) = 0. \tag{8}$$

The domain is either $\mathbb{R}^d_x \times (0, T)$ or an open subset of it. The density is *a priori* bounded below by ρ_0, but of course this limit value can be taken as small as needed.

The sound speed is $c(\rho) = \sqrt{p'(\rho)} = a/\rho$. As shown in Section 1, the wave velocities in direction $\xi \in S^{d-1}$ are $u \cdot \xi$ and $u \cdot \xi \pm a/\rho$ and all shocks are sonic. Recall that the former is associated to slip lines ($d = 2$) or vortex sheets ($d = 3$), across which the mass flux vanishes and the pressure is continuous: $[\rho] = 0$. The pressure waves are called *shocks*, despite the fact that the linear degeneracy leaves room for reversibility. We showed above that across a shock of normal speed σ between two states $U_\pm$, we have

$$(\rho(u \cdot \nu - \sigma))_- = (\rho(u \cdot \nu - \sigma))_+ = \pm a. \tag{9}$$

Of course, one always may take the plus sign in the right-hand side, up to an appropriate orientation of ν. We point out that (9) tells that the shock travels exactly at the normal velocity $u \cdot \nu \pm a/\rho$, which is the velocity of infinitesimal sound waves. This coincidence is typical of linear degeneracy.

Remark. The relation (9) has the amazing consequence that the mass flux across a shock is fixed. It cannot vary, contrary to polytropic gase for instance. For a gas with a general equation of state, the mass flux approaches $\rho c(\rho)$ as the shock strength $[\rho]$ tends to zero, but in a Chaplygin gas, the mass flux equals this value on both sides, regardless the strength.

Proposition 2.1 *Across a steady two-dimensional shock of a Chaplygin gas, there holds*

$$[|u|^2 - c^2] = 0,$$

where $c := a/\rho$ is the sound speed.

Proof
Just write $|u|^2 = (u \cdot \nu)^2 + (u \times \nu)^2$ and the fact that $u \cdot \nu = \pm c$ on both sides, while the jump of $u \times \nu$ is zero.

∎

2.1 The lack of vorticity generation

An other remarquable fact concerns steady shocks in two space dimensions. We define as usual the vorticity by $\omega := \partial_1 u_2 - \partial_2 u_1$.

Theorem 2.1 *Across a steady two-dimensional shock of a Chaplygin gas, there holds*

$$\left[\frac{\omega}{\rho}\right] = 0.$$

Proof
Away from the shock, one has $(u \cdot \nabla)u + \rho^{-1}\nabla p(\rho) = 0$. This can be rewritten as

$$\frac{1}{2}\nabla\left(|u|^2 - \frac{a^2}{\rho^2}\right) = \omega\begin{pmatrix} u_2 \\ -u_1 \end{pmatrix}.$$

Let τ be a unit tangent vector to Σ (here a curve). Then taking the scalar product, we have along the shock:

$$\omega(u \cdot \nu) = \frac{1}{2}\tau \cdot \nabla\left(|u|^2 - \frac{a^2}{\rho^2}\right).$$

On the one hand, the left-hand side is nothing but $j\omega/\rho$. On the other hand, the right-hand side is the derivative of $|u|^2 - a^2/\rho^2$ along the shock curve. The theorem follows from the fact that this quantity is actually the same on both sides of Σ, as stated in Proposition 2.1.

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Remark. The theorem above gives a counter-example to the Theorem 2.3 of [12]. It turns out that this statement is incorrect and should be replaced by the following:

> *Let the equation of state be $\rho \mapsto p(\rho)$ with $p'(\rho) > 0$. Consider a two-dimensional steady shock across a smooth curve Γ, such that the flow is uniform (equal to U_-) on one side of Γ and is irrotational on the other side. Then either Γ is a straight line, or the function*
>
> $$\rho \mapsto \int_{\rho_-}^{\rho} \frac{p'(s)}{s} ds + \frac{1}{2}\left(\frac{1}{\rho} + \frac{1}{\rho_-}\right)(p(\rho_-) - p(\rho))$$
>
> *is constant on some interval.*

This constancy is typical of a Chaplygin gas.

As a by-product, we have:

Corollary 2.1 *Let us consider a steady two-dimensional shock of an isentropic Chaplygin gas. Assume that on one side of the shock, the flow is irrotational ($\omega \equiv 0$). Then it is irrotational on the other side.*

More generally, if ω/ρ is a constant on one side of the shock, it equals the same constant on the other side.

Proof

The corollary is immediate as far as we focus on the points along the shock itself. To show that ω/ρ is locally constant, we use the fact that ω/ρ propagates along the particle trajectories, the latter being transversal to the shock. The transversality follows from $j \neq 0$, thus $u \cdot \nu \neq 0$. The propagation follows from the following computation. Since $\rho^{-1}\nabla p(\rho)$ is a gradient, we have $\mathrm{curl}((u \cdot \nabla)u) = 0$, that is $u \cdot \nabla\omega + \omega \mathrm{div} u = 0$. In other words, $\mathrm{div}(\omega u) = 0$. Combining with $\mathrm{div}(\rho u) = 0$, we obtain $u \cdot \nabla(\omega/\rho) = 0$. This amounts to saying that ω/ρ is constant along the curves $\dot{x} = u$.

∎

Time-dependent flows. We now show that irrotationality is compatible with the Euler equations of an isentropic Chaplygin gas. To begin with, we may consider a gas with an arbitrary equation of state and ask whether a potential flow $(\rho, u = \nabla\phi)$ may admit shocks. The potential equations are

$$\partial_t\phi + \frac{1}{2}|\nabla\phi|^2 + h(\rho) = 0, \tag{10}$$

$$\partial_t\rho + \mathrm{div}(\rho\nabla\phi) = 0., \tag{11}$$

where the enthalpy h is given by $h'(\rho) = p'(\rho)/\rho$. We ask that ϕ be continous, piecewise $\mathcal{C}^1$, and that a shock propagates along a hypersurface Σ. The unit normal to Σ_t, the section at time t, is ν and the normal velocity of $t \mapsto \Sigma_t$ is σ. Since $[\phi] = 0$ across Σ, the tangential derivatives are continuous too across the shock. Thus $[\nabla\phi \times \nu] = 0$, and also

$$0 = [(\partial_t + \sigma\nu \cdot \nabla)\phi].$$

Because of (10) and $|\nabla\phi|^2 = (u \cdot \nu)^2 + |\nabla\phi \times \nu|^2$, this rewrites

$$\left[\frac{1}{2}(u \cdot \nu - \sigma)^2 + h(\rho)\right] = 0.$$

However, the Rankine-Hugoniot condition for (11) writes $[\rho(u\cdot\nu - \sigma)] = 0$ as usual. Introducing the *net flux* $j := \rho(u \cdot \nu - \sigma)$, we deduce

$$[h(\rho)] + j^2 \left[\frac{1}{2\rho^2}\right] = 0. \tag{12}$$

For a general gas, the relation (12) is incompatible with the Rankine-Hugoniot condition

$$[p(\rho)] + j^2 \left[\frac{1}{\rho}\right] = 0,$$

which comes from the Euler equations, where the values $\rho^{r,l} > 0$ can be arbitrary. This is the reason why for a general gas, and even for an ideal one, the irrotationality does not persist beyond shock formation. The compatibility of both conditions holds true if, and only if, we have

$$h(\beta) - h(\alpha) = \frac{1}{2}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)(p(\beta) - p(\alpha))$$

for every relevant $\alpha, \beta > 0$. This exactly says that the gas is a Chaplygin gas.

We therefore conclude that, for a Chaplygin gas, a fluid initially isentropic and irrotational stays so forever, at least if the flow is piecewise smooth. We shall feel free in the sequel to look an irrotational flow whenever the initial data is isentropic and irrotational.

2.2 Non-existence of a three-shock pattern

When working about the Manhattan project, J. von Neumann conjectured that a three-shock pattern (a so-called *triple point*) cannot exist for a reasonnable equation of state; he justified his claim by the gain of entropy of the flow when it crosses either a big shock or two smaller ones, see [13]. R. Courant & K. Friedrichs [5] gave an entropy-free proof for the ideal gas law (when $p(\rho, e) = (\gamma - 1)\rho e$). L. Henderson & R. Menikoff [8] gave an entropy-based proof for a rather broad class of equation of states. Finally, Serre [12] (Theorem 2.3, p. 60) found a proof that does not involve at all neither the equation of state, nor the second principle of thermodynamics. In other words, the impossibility of a so-called "triple point" is of purely kinematical nature. The rather short proof involves a geometrical argument (inversion in the plane).

In the case of a Chaplygin gas, an even shorter proof is available. We start with the observation, which will be useful in several subsequent sections, that a shock line between two constant states $U_\pm$, being characteristic on both sides must be tangent to the sonic circles $C_\pm$ of centers the velocities $u_\pm$ and of radii the sound speeds $a/\rho_\pm$. Since the normal components $u_\pm \cdot \nu$ have the same sign, both circles lie on the same half-plane defined by the shock. Besides, since the tangential component $u_\pm \times \nu$ does not jump, the centers $u_\pm$ lie on the same perpendicular line to the shock. In other words, C_- and C_+ are tangent, one surrounding the other (we say that they are tangent *interiorly*), and their common tangent is the shock line.

If three shocks meet at a point P, separating three constant states U_1, U_2, U_3, we have three circles C_j and each pair (C_i, C_j) with $j \neq i$ has an interior tangency. In particular, these circles are ordered for the inclusion, say that $C_1 \subset C_2 \subset C_3$. But the fact that C_1 and C_3 are tangent at a point z implies that all three circles are tangent at z. Thus all three shocks are supported by the same line (the common tangent), which means that we actually have only one shock and only two states.

2.3 Four-shock patterns

The impossibility of a triple-shock structure does not preclude the existence of four-shock patterns. Actually, W. Bleakney & A. Taub [1] constructed such patterns. They started from a regular reflection along an infinite ramp, across which they performed a symmetry. Their solutions are therefore symmetric with respect to a line. We suggested in [12] that general four-shock patterns between constant states should be symmetric. We show below that this claim is erroneous in the case of Chaplygin gas.

It will be sufficient to build a collection of four circles C_j $(1 \leq j \leq 4)$ with the properties

- $C_1 \subset C_{2,3} \subset C_4$, where each of these inclusions is an interior tangency (though this will not be the case for the inclusion $C_1 \subset C_4$).
- The tangent lines $L_{12}, L_{13}, L_{24}, L_{34}$ intersect at a common point P.

This construction is detailed in Figure 1. These four lines support the four shocks. The states U_1 and U_4 are in opposite sectors, as are U_2 and U_3. The circles are the sonic circles of each state.

We may choose arbitrarily[1] a circle C_4 and two circles C_2, C_3, interiorly tangent to C_4, large enough so that they intersect in two points. The tangents L_{24} and L_{34} intersect at some point P outside C_4. Since L_{24} is tangent to C_2, there is exactly one other tangent to C_2 passing through P, which we denote L_{12}. Likewise, there is an other tangent from P to C_3, denoted by L_{13}. Using the fact that given a point Q outside a circle C, the tangent segments from Q to C have equal lengths, we find that all the tangent segments along L_{12}, L_{13}, L_{24} and L_{34}, between P and the tangency points, have an equal length. In particular, setting $A_{2,3}$ the tangency points of L_{12} with C_2 and of L_{13} with C_3 respectively, we see that there exists a unique circle C_1 tangent to L_{12} at A_2, and to L_{13} at A_3. This circle is interiorly tangent to C_2 at A_2 and to C_3 at A_3. Our construction is complete.

[1]We could equally start from the circles C_j, $j = 1, 2, 3$. The construction is similar. Four choices are possible.

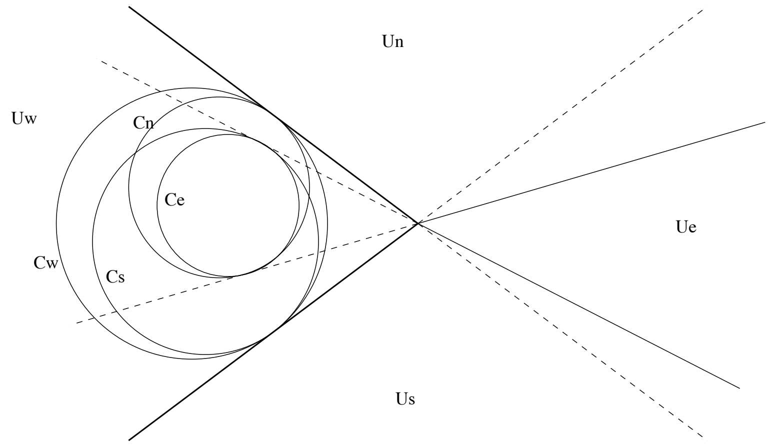

Figure 1: The standard interaction of two planar shocks for a Chaplygin gas.

Comments. – It is clear that our construction yields a non-symmetric four-shock pattern, provided C_2 and C_3 have distinct radii. – It is unclear whether such non-symmetric patterns may persist when some amount of genuine nonlinearity is put in the equation of state. We leave this question open.

2.4 Simple interaction of two shocks

The previous paragraph provides a geometrical construction of the interaction of two planar shocks. Let us give ourselves two shocks separating respectively the states U_r and U_m, and the states U_m and U_l. We thus have three sonic shocks $C_{l,m,r}$, with C_m tangent interiorly to both $C_{l,r}$. There are three possibilities, with either C_m the larger, or the smaller, or in between C_l and C_r. Each of these cases corresponds to a different choice of three circles among the four of Figure 1. Of course, the common tangents support the incident shocks and intersect at point P. From P, we draw the remaining tangents to C_l and C_r. The contact points determine a fourth circle C_b, whose center and radius determine a state U_b. The interaction is a four-shock pattern where the reflected shocks are supported by the new tangents. This is what we call the *simple interaction* of two shocks. It will be used is the solution of some two-dimensional Riemann problem.

We warn the reader that this simple interaction may not exist when C_m is either the larger or the smaller of the three given circles. When it is the larger, it may happen that C_l and C_r do not meet, and then the fourth circle is imaginary. When C_m is the smaller, it may happen that P belongs to the convex hull of $C_l \cup C_r$, and then the fourth circle is tangent exteriorly to C_l and C_r. Therefore the result of the interaction of two planar shocks depends upon the angle between them and on their respective strengths. However, the interaction is always simple when the densities are close to each other.

Let us examine in greater details the limit when the data tends to criticality. We begin with the case when C_m is the larger circle and the intersection of C_r and C_l shrinks to a point. Then the radius of C_b tends to zero, which means that the density tends to infinity. If all the other data, in particular the velocities, are kept bounded, then the mass contained in a bounded region between the transmitted shocks (at right in Figure 1) is proportional to $(\ell_2^2 - \ell_1^2)c_b\rho_b = a(\ell_2^2 - \ell_1^2)$ where $\ell_{1,2}$ are the inner and outer radius of the region under consideration. The zone between the transmitted shocks tends to a half-line, which becomes in the limit the support of the singular part of the density ρ. Its density per unit length is proportional to the distance from P. If we now decrease the radii of C_r or C_l, in such a way that these circles do not intersect at all, what happens is that the gas accumulates along a half-line L originated from P and perpendicular to the segment $[u_r, u_l]$. The density admits a singular part carried by L, where the matter travels along the line, with velocity the common component of $u_{r,l}$ in this direction. This flow is a solution of the Euler system in the distributional sense. The concentration of mass is typical of a gas with an equation of state $p = p(\rho)$ such that $p(+\infty)$ is bounded. It was already observed for one-dimensional flows of a Chaplygin gas by Y. Brenier [2]. A well-known example is that of a pressureless gas.

The opposite situation is when C_m is the smaller circle and P approaches the boundary of the convex hull of $C_r \cup C_l$. Then the radius and the center of C_b tend to infinity, which means that the state U_b after the transmitted shocks tend to vacuum, while the fluid velocity increases to infinity.

2.5 Reflection of a shock along an infinite wall

A classical problem in shock reflection is that of a planar shock along an infinite wall; see Figure 2. Thanks to the Galilean invariance, we may assume that the incident shock is steady. Usually, the fluid is assumed to flow along the wall into the shock (here from right). The upstream and downstream states (denoted hereafter U_1 and U_2), as well as the angle α between the shock and the wall are given, and we have to determine the angle made by the reflected shock and the state behind it. The boundary condition $u \cdot \nu = 0$ along the wall expresses the fact that the fluid does not flow across the rigid boundary. We shall use the analysis of Paragraphs 2.3 and 2.4.

The problem is equivalent to that of the interaction between the incident shock and the one obtained by a symmetry with respect to the wall. To fix the notations, the wall becomes the horizontal axis and the interaction point is the origin. Notice that U_1 is already symmetric and that U_2 is not. We denote by U_3 the state symmetric of U_2. The sonic circles $C_{1,2,3}$ belong to the left quadrant defined by the incident shocks. The problem amounts to finding the unique sonic circle C_4 being interiorly tangent to C_2 and C_3 along their tangent lines passing through the origin. Its construction has been described in Paragraphs 2.3 and 2.4. However, its existence is subjected to the following constraints:

- The circle C_2 must be large enough that it meets the horizontal axis (the wall) non-tangentially.
- The circle C_2 must be small enough that it does not meet the vertical line passing through P.

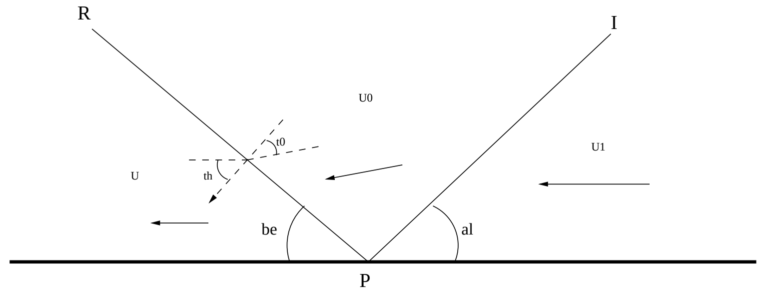

Figure 2: Regular Reflection (RR) along a planar wall.

Let α' denote the angle between the reflected shock and the wall. A rather simple calculus gives

$$\frac{\rho_1}{\rho_2} = \frac{c_2}{c_1} = \frac{\tan\frac{\alpha+\alpha'}{2}}{\tan\alpha}.$$

Denoting

$$t := \tan\frac{\alpha}{2}, \qquad t' := \tan\frac{\alpha'}{2},$$

we find the homographic equation in the unknown t'

$$\frac{t+t'}{1-tt'}\,\frac{1-t^2}{2t} = \frac{\rho_1}{\rho_2}. \tag{13}$$

Contrary to real gases, or to the ideal gas (in which $p = (\gamma-1)\rho e$, $\gamma > 1$ a constant), this equation admits only one solution t'. The admissibility conditions above are equivalent to $\alpha' \in (0,\pi/2)$, that is $t' \in (0,1)$. Whence the requirement

$$\frac{2t}{(1+t)^2} < \frac{\rho_2}{\rho_1} < \frac{2}{1-t^2}, \qquad t := \tan\frac{\alpha}{2}, \tag{14}$$

where we remark that $t \in (0,1)$ since $\alpha \in (0,\pi/2)$. We summarize these calculations in the following statement.

Theorem 2.2 *A necessary and sufficient condition for the existence of a Regular Reflection of an incident shock along a wall is given by the inequalities (14).*

The reflection angle α' is given by $2\tan^{-1}t'$, where t' is the unique solution of (13).

Remark. In the limit case where $\alpha \to 0^+$ (glancing shock), the condition becomes $\rho_2/\rho_1 \in (0,2)$. In the opposite case where $\alpha \to \pi/2$ (normal shock), it becomes $\rho_2/\rho_1 \in (1/2,+\infty)$. A ratio ρ_2/ρ_1 is admissible for every angle of incidence if, and only if, it belongs to $[1/2,2]$. This is qualitatively quite different from the case of an ideal gas, where at given ratio $\rho_2/\rho_1 \neq 1$, the Regular Reflection is no longer possible when α becomes small.

Concentration. When ρ_2/ρ_1 exceeds $2/(1-t^2)$, a concentration phnomenon occurs, which is similar to that described in the previous section. There is no reflected shock. The gas arriving on the wall at velocity u_2, whose vertical component is negative, accumulates along the boundary. The matter concentrated along the wall travels with a speed equal to the horizontal component of u_2. Once again, the concentration phenomenon is due to the boundedness of the pressure at infinite density. This reveals one of the drawbacks of the model, as far as gases are concerned.

2.6 Summary: features of the Chaplygin gas

So far, we have encountered several features which make easier the analysis of the flow in a Chaplygin gas:

- The pressure waves are sonic shocks (contact discontinuities), across which the entropy remains constant.
- In particular, a fluid initially at a constant entropy remains so. The isentropic model is a rigorous one, not an approximation.
- There is no vorticity generation across a steady shock.The irrotational model is a rigorous one when the flow is irrotational at infinity; it is not an approximation.
- More generally, the potential system (10,11) is compatible with the Euler system. It makes sense to look for an irrotational flow when the fluid is initially isentropic and irrotational.
- The interaction of two planar shocks yields a simple four-shocks pattern, except for rather extreme data. This pattern has a geometrical construction.

We have also observe two flaws of the model:

- At low density, the pressure is negative ; undoubtedly a non-physical fact.
- At high density, the pressure saturates and there occur concentration phenomena, as it happens for a pressureless gas.

3 Self-similar flows

We are now interested in two-dimensional ($d = 2$) self-similar flows. Self-similarity refers to the fact that the Euler system (7,8) is invariant under the rescaling $(x, t) \mapsto (\mu x, \mu t)$. Thus it makes sense to consider solutions depending only on the self-similar variable $y := x/t \in \mathbb{R}^2$. Under this assumptions, the equations to solve become

$$\begin{aligned} \operatorname{div}(\rho u) &= y \cdot \nabla \rho, \\ \operatorname{div}(\rho u \otimes u) + \nabla p(\rho) &= (y \cdot \nabla)(\rho u), \end{aligned}$$

where the derivatives act with respect to the variable $y = (y_1, y_2)$. A consequence of the Galilean invariance of the Euler equations is that the system above can be rewritten in a form with constant coefficients. Tho this end we introduce of the *pseudo-velocity* $v(y) := u(y) - y$. The equations become

$$\begin{aligned} \text{div}(\rho v) + 2\rho &= 0, && (15) \\ \text{div}(\rho v \otimes v) + 3v + \nabla p(\rho) &= 0. && (16) \end{aligned}$$

The system differs from the steady Euler equations only by the zero-order terms. But the principal part remains the same, up to the replacement of u by v. In particular, the Rankine-Hugoniot relations remain unchanged. For instance, $[\rho v \cdot \nu] = 0$ allows us to define a *pseudo-flux* $j := \rho v \cdot \nu$. Then a shock satisfies $j \neq 0$ (by definition), $[v \times \nu] = 0$ (which is nothing but $[u \times \nu] = 0$) and $j = \pm a$.

These calculation show that the counterpart of Proposition 2.1 is valid:

Proposition 3.1 *Across a shock of a two-dimensional self-similar flow of a Chaplygin gas, there holds*

$$[|v|^2 - c^2] = 0,$$

where $c := a/\rho$ is the sound speed.

Remark that a combination of (15) and (16) gives

$$(v \cdot \nabla)v + v + \frac{1}{\rho}\nabla p(\rho) = 0, \tag{17}$$

whenever the flow is smooth enough.

Let us examine the self-similar counterparts of the results stated in the previous section in the steady case. To begin with, Theorem 2.1 remains true in the self-similar case:

Theorem 3.1 *Across a shock of a two-dimensional self-similar flow of a Chaplygin gas, there holds*

$$\left[\frac{\omega}{\rho}\right] = 0.$$

Proof

The only difference with the steady case is that we have

$$j\frac{\omega}{\rho} = \frac{1}{2}\tau \cdot \nabla\left(|v|^2 - \frac{a^2}{\rho^2}\right) + v \cdot \tau.$$

Since $[v \cdot \tau] = 0$, there follows

$$j\left[\frac{\omega}{\rho}\right] = \frac{1}{2}\tau \cdot \nabla\left[|v|^2 - \frac{a^2}{\rho^2}\right].$$

The right-hand side is zero, according to Proposition 3.1.

∎

Next, (17) implies $\mathrm{curl}((v\cdot\nabla)v+v)=0$, that is $\mathrm{div}(\omega v)+\omega=0$. Elimination with (15) gives

$$v\cdot\nabla\frac{\omega}{\rho}=\frac{\omega}{\rho}. \tag{18}$$

In particular, the constancy of ω/ρ does not propagate any more along particle paths, but its vanishing does. Therefore, Corollary 2.1 remains true.

Corollary 3.1 *If the fluid is irrotational on one side of a shock in a two-dimensional self-similar flow of a Chaplygin gas, then it is irrotational on the other side.*

3.1 Subsonic/supersonic flows

The system for steady flows consists in three equations in three unknowns ρ, u_1, u_2. It may be hyperbolic in some direction, or it may be of mixed type hyperbolic-elliptic. The former situation occurs when the flow is supersonic: $|u|>a/\rho$. The latter corresponds to the subsonic case $|u|<a/\rho$. The situation is almost the same for self-similar flows since the principal part is the same, except that u must be replaced by v in the definition. We thus say that the flow is *(pseudo-)subsonic* if $|v|<a/\rho$ (where the system is of mixed type) and *(pseudo-)supersonic* if $|v|>a/\rho$ (where the system is hyperbolic in the direction of the flow).

In a region where (ρ,u) is a constant state (we speak of a *uniform flow*), the flow is pseudo-subsonic when $|y-u|<a/\rho$, that is in a disk $D(u;c)$ of center u and radius the sound speed, the *sonic disk*.

The boundary of a region where (ρ,u) is constant is pseudo-characteristic (because even the shocks are characteristic). Say that a part Γ of this boundary satisfies $v\cdot\nu=a/\rho=:c$. Since ν is the unit vector orthogonal to the tangent, this rewrites

$$(u-y)\times\dot{y}=c, \tag{19}$$

where $s\mapsto y(s)$ is an arc-length parametrization of Γ. The general solution of the ODE (19) is an arc of the sonic circle $C=C(u;c)$, continued at its extremities by the tangents to C. Of course, in a local solution, one may have only an arc of C, or only a piece of a tangent to C, or an arc together with a tangent.

3.2 Two-dimensional Riemann problems

Because of the scaling invariance of the Euler equations, a positively homogeneous initial data of degree zero ($U(\mu x)=U(x)$ for every $\mu>0$) yields a self-similar solution, provided that the Cauchy problem is uniquely solvable. Of course, we do not know so far whether this unique solvability holds, but it makes sense to look for self-similar solutions for radially invariant data. In one space-dimension, this is the well-known Riemann problem. Thus one has accostumate to speak of the *two-dimensional Riemann problem* if the data is given in $\mathbb{R}^2$ instead.

In full generality, the data is an arbitrary function U of the angle $\arctan x_2/x_1$, which makes the problem rather hard for a complete analysis. Therefore people use to restrict to the case

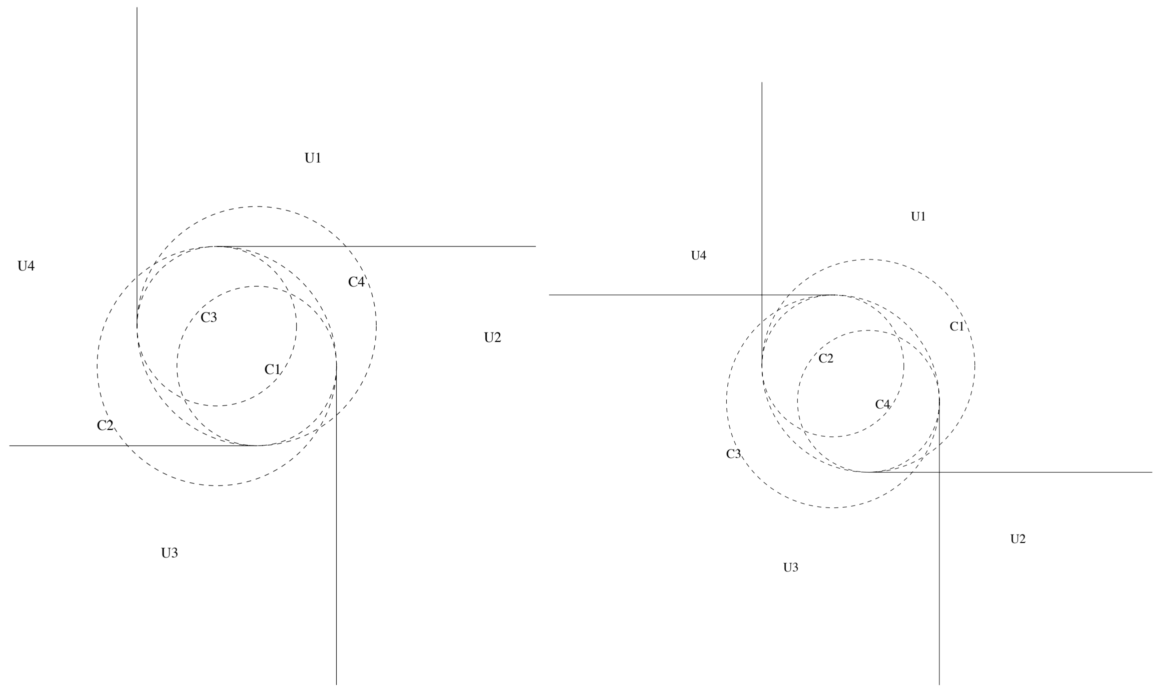

Figure 3: Four-shock Riemann problems. Left: vortex data. Right: compression-expansion (saddle). Shocks shown after motion. Interaction has not been taken in account. Dashed : the sonic circles of the data.

where U is piecewise constant, like in the one-D problem. Typically, U is constant in each of the quarters defined by the coordinate axes. Like in one space-dimension, one expects that this problem is the building block from which every solution could be approximated through either some numerical scheme or a front-tracking algorithm.

Even this four-state Riemann problem is rather complex, because each of the four initial discontinuities yields a 1-D Riemann problem, whose solution consists in general of three waves: a backward and a forward pressure wave, and a slip line. Since these waves do not agree in the interaction zone, the neighbourhood of the origin, there is a more complicated, genuinely two-dimensional pattern there.

The interaction of twelve distinct waves is presumably a very complicated thing. For this reason, one may decide to restrict once more the analysis, by chosing the four constant states in such a way that each of the initial discontinuity is a simple wave, either a pressure wave or a slip line. This reduces the number of simple waves from twelve to only four. It is the framework within which most of the literature belongs. For a general gas, there are nineteen possible initial patterns for the full Euler system and twenty-two for the isentropic one[2], taking in account that

[2]The fact that there are more configurations in the isentropic case might look strange. But there is no contradiction, because for a general gas, the isentropic model is not a special case of the full Euler system. In a Chaplygin one, it is and therefore a complete classification yields less isentropic configurations than non-isentropic ones. This means that in a Chaplygin gas, some configurations, either isentropic or not, do not

a pressure wave may be either a shock or a rarefaction. They have been classified in [14], see also [11] and [15].

In a Chaplygin gas, a few simplifications occur. On the one hand, pressure waves are always sonic shocks. Thus the classification is shorter. On the other hand, the entropy being constant across a shock, the choice of four shocks yields a constant entropy at initial time. In this case, we have seen in Paragraph 1.1 that the entropy remains constant forever (not true for a general gas). We thus face the isentropic Euler equations.

In a first instance, we therefore limit ourselves to the cases where the four states $U_1, \dots, U_4$ (the label are arranged clockwise, starting form the upper-right quadrant) are separated by shocks along the four semi-axes. Instead of speaking of backward/forward shocks, a notion that is not easy to manipulate in two space dimensions, we shall speak of *clockwise/counterclockwise* shocks. At the abstract level, there could be three combinations of clockwise/counterclockwise shocks, up to symmetries. But using the fact that two neighbour states are the centers of two interiorly tangent sonic circles, with horizontal or vertical joint tangent, we find easily the following properties:

Proposition 3.2 *In a four-shocks Riemann problem of a Chaplygin gas, the velocities u_j ($j = 1, \dots, 4$) form a square, each arrow corresponding to a shock. The sound speeds of opposite states are equal: $c_1 = c_3$ and $c_2 = c_4$.*

Depending on whether $u_1, \dots, u_4$ are arranged clockwise (like their corresponding quadrants) or counterclockwise, we have two qualitatively different cases, depicted in Figure 3. The first one could be called a *vortex* problem, in which the four shocks propagate the same way, say counterclockwise. The second one corresponds to a *compression-expansion* or *saddle* problem, where the shocks propagate clockwise and counterclockwise, alternatively. The data are shown in Figure 3, in self-similar coordinates x/t. Mind that the shocks coincide with the semi-axes only at initial time. At time $t = 1$, they are carried by tangents to the sonic circle $C_j = C(u_j, c_j)$. We choosed arbitrarily to draw the shocks lines up to their contact points with the corresponding sonic circles, though this could not be correct in the solutions we are looking for. In the saddle (the picture at right), the interaction of two approaching shocks must give rise to a simple interaction, with new shocks that will not be horizontal or vertical.

Remark. If we compare our classification with that of Zheng & al. (see for instance [15] page 199), we see that what he call a *four forward shocks* configuration does not exist here (the sonic circles would coincide, thus there would be only a constant data), and that our vortex configuration is not present in his classification, presumably because it is structurally unstable when one add some nonlinearity to the equation of state, or because the corresponding waves are rarefactions.

3.3 Shock reflection past a wedge

We are interested in the reflection of a planar shock against a solid wedge, see Figure 4. At time $t = 0$, the state (ρ, u) is uniform on both sides of an incident shock, which has reached the

occur.

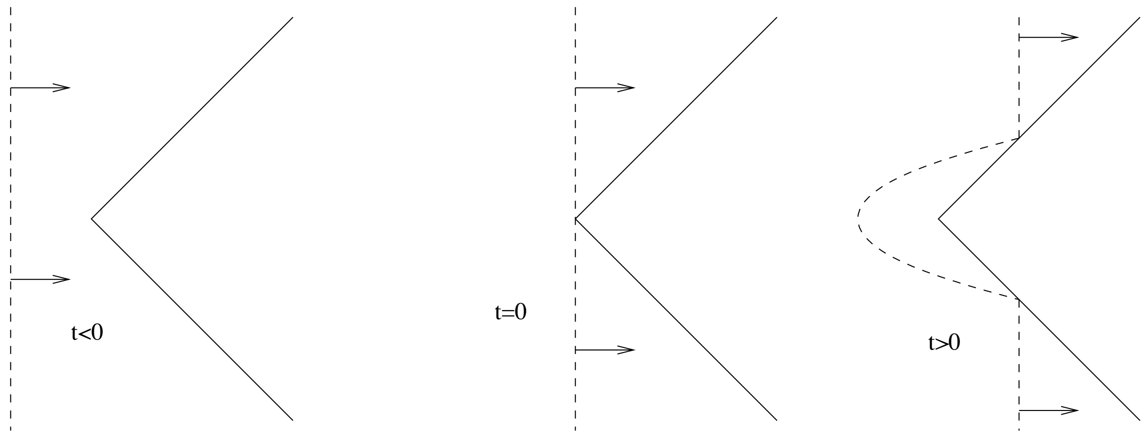

Figure 4: The symmetric reflection past a wedge. Left: A planar shock comes from the left. Middle: At time $t = 0$, the shock hits the tip of the wedge. Right: The interaction takes place with a subsonic bubble (here a subsonic regular Reflection). The flow is uniform in each region but within the bubble.

tip of the wedge. The upstream state U_0 is at rest (ρ_0 constant, $u_0 \equiv 0$). The downstream data is denoted by $U_1 = (\rho_1, u_1)$. We assume a symmetric pattern: the shock front is perpendicular to the axis of the obstacle. The domain and the initial data are invariant under the scaling $x \mapsto \mu x$. Once again, uniqueness implies self-similarity. We thus are led to solve the system (15,16), with appropriate boundary conditions:

- Along the ramp, u is parallel to the boundary, $u \cdot \nu = 0$ where ν is the unit normal.
- Along the symmetry axis, u is parallel to the axis, which therefore plays the role of a rigid wall.
- On the far side ($|y| \gg a/\rho_{0,1}$), the flow is not influenced by the tip. It is thus equal to the flow U_R of the reflection of the incident shock against an infinite ramp. In the regime of *Regular Reflection*, which has been described in Paragraph 2.5, U_R is piecewise constant, with an undeformed incident shock, reaching the wall at the point P, and a reflected shock along a straight line. The flow U_R consists in three constant states, two of them being the data U_0, U_1 ; the third one U_2 is given by explicit algebraic computations, known for general gases since the work of von Neumann [13]. See Section 3.1 of the monograph [12] for a detailed description.

A common belief, based on numerical and physical experiments is that if a Regular Reflection occurs for this problem, then it must have the following form:

- The reflected shock remains straight and the flow between it and the wall remains constant, up to the pseudo-sonic circle C_2 of equation $|u_2 - y| = c_2$ (here equal to a/ρ_2).

- It is then bent, until it reaches the horizontal axis, with a vertical tangent. This part of the reflected shock is usually unknown (a *free boundary*). We shall see that it can be determined analytically in the case of a Chaplygin gas.

- Within the domain bounded by the reflected shock, the symmetry axis, the wall and C_2, the flow is pseudo-subsonic, meaning that $|v| < c(\rho)$.

The problem is therefore to prove the existence of the free boundary and of a solution of an appropriate boundary-value problem in the subsonic domain.

Warning. Two additional phenomena may happen when the sonic circle C_1 of the upstream state U_1 does not enclose the origin. This happens precisely when $c_1 < |u_1|$ (supersonic state U_1).

- A first one is familiar to people working in shock reflection: the Regular Reflection along the infinite ramp can just not exist. This is the case if C_1 is contained in the solid wedge, that is if the aperture α is large enough. According to what happens for a real gas, one expects that a kind of Mach Reflection takes place instead; see for instance Majda's paper [9], which describes the various kinds of reflections that are observed in physical and in numerical experiments. For a Chaplygin gas, it can be shown that such a pattern *does not exist*. As a matter of fact, the boundary of the rest zone where $U \equiv U_0$ is formed of the vertical line tangent at left to C_0 (the incident shock), possibly followed by an arc of C_0. But since the tangency point is on the symmetry axis, thus within the solid wedge, such an arc may not be present. Thus I must be continued up to the ramp, although we cannot build a reflection along it.

  We shall see in Paragraph 7.1 that a concentration phenomenon resolves this paradox.

- In an intermediate regime corresponding to $\sin\alpha < c_1/|u_1|(< 1)$, C_1 is not contained in the solid wedge. Since the subsonic region is too short to enclose O, we expect that a third shock, originated at O, makes a transition between the tip and the subsonic zone. Between this shock and the ramp, the flow should be uniform, equal to some U_3, and the transition between U_3 and the subsonic zone would take place along the sonic circle C_3. In this regime, the flow near the tip is both steady and self-similar, a very simple and well-known one.

4 The solution for the 2-D Riemann problems

We solve here the Riemann problems with four shock data, namely the vortex and the saddle. Before giving details on each one, we make a few observations that are relevant for both. We are looking for a self-similar solution, which is thus a solution of the pseudo-steady Euler equations (15,16). We recall that since the initial entropy is constant, we expect that the solution be isentropic. Because of the finite propagation velocity, the flow must be constant in each quadrant of the form $\{\pm x > A, \pm y > A\}$ for some $A > 0$ large enough. There remains to identify the flow in stripes along the axes and in a neighbourhood $\mathcal{U}$ of the origin.

The same argument tells that in a strip, say that between two states U_j and U_k, away from $\mathcal{U}$, the solution coincides with that of the one-dimension Riemann problem between them. By assumption, this solution is nothing but the shock data. We therefore have a full description of the flow away from $\mathcal{U}$.

To go further, we recall that in a first-order system like (15,16), an open domain where the solution is constant (here we have $(\rho, u) \equiv U_j$ in mind, which means that (ρ, v) is non-constant) must be bounded by a curve which is either characteristic for this state, or a discontinuity. Since for a Chaplygin gas, the shocks are characteristic, we expect that these boundaries are characteristic curves, that is satisfy the equation $|(y-u_j)\cdot\nu| = c_j := a/\rho_j$. We recall that such a curve is a semi-tangent to the sonic circle C_j of U_j, continued by an arc of this circle and possibly by another semi-tangent to C_j.

Our last remark is that since there is no vorticity generation across shocks (Corollary 3.1), and since vorticity is transported along the flow according to (18), we expect that the flow be irrotational everywhere in the plane. We thus look for two scalar functions, the density ρ and the stream function ϕ, the latter defined up to a constant by $v = \nabla\phi$.

Our strategy is the following:

1. We begin by identifying the domain Ω where U is not piecewise constant. We check that ϕ is constant along the boundary of Ω; we may choose $\phi \equiv 0$ there.

2. We look for a smooth solution in Ω, which turns out to be the subsonic domain, where $|v| < c(\rho)$. We show that the whole system reduces to a single second-order partial differential equation of elliptic type in the unknown ϕ, together with the Dirichlet boundary condition.

3. We solve this equation.

Let us move forward to point 2. Since the flow is potential, equation (17) reduces to

$$\nabla\left(\frac{1}{2}|v|^2 + \phi - \frac{a^2}{2\rho^2}\right) = 0.$$

Integrating, we deduce that

$$\frac{1}{2}|v|^2 + \phi - \frac{a^2}{2\rho^2} \equiv \mathrm{cst} \qquad \text{in } \mathbb{R}^2. \tag{20}$$

However, the boundary of Ω is made of characteristic curves, on which we know that $|v|^2 \equiv a^2/\rho^2$. Thus ϕ is constant along $\partial\Omega$. Without loss of generality, we set this constant to zero.

We use (20) to eliminate the density:

$$\rho = \frac{a}{\sqrt{2\phi + |\nabla\phi|^2}}. \tag{21}$$

Putting (21) into the mass equation (15), we obtain the second-order PDE

$$\mathcal{N}[\phi] := \mathrm{div}\frac{\nabla\phi}{\sqrt{2\phi + |\nabla\phi|^2}} + \frac{2}{\sqrt{2\phi + |\nabla\phi|^2}} = 0. \tag{22}$$

This equation holds everywher in $\mathbb{R}^2$, in the distributional sense. But since we do know the flow outside of Ω, we need only to solve it within Ω. The matching with the outer solution requires that ϕ be continous, and that $[\rho v \cdot \nu] = 0$. The latter is a trivial consequence of the fact that $\partial\Omega$ is characteristic. The former gives us a boundary condition

$$\phi = 0 \qquad \text{on } \partial\Omega. \tag{23}$$

A difficulty in the treatment of this boundary-value problem arises from the fact that the type of (22) changes with the sign of ϕ. However, since ϕ vanishes along the boundary, we may expect that it be positive in Ω (this is compatible with the maximum principle), where the equation is now elliptic. There will remain a difficulty however, in that the boundary is characteristic.

Remark that (23) implies that $\nabla\phi \parallel \nu$ and therefore $\rho v = \pm a\nu$ on each side of $\partial\Omega$. The sign $\pm$ is the same on both sides because we expect ϕ to be positive inside and negative outside. Therefore we have the stronger property that $[\rho v] = 0$.

4.1 The vortex problem

We recall the data in the left of Figure 3. We begin by determining the domain where U is none of the U_j, $j = 1, \dots, 4$. This amounts to finding the boundaries of the domains D_j defined by $U \equiv U_j$. As said above, the boundary is either characteristic for U_j or a shock. After the discussion of Paragraph 3.1, it is natural to look for the continuation of the straight shocks, which are tangent to the sonic circle C_j, by an arc of C_j itself. The solution of this geometric problem is obvious and is shown in Figure 5. Each domain D_j is bounded by a pair of two curves:

- The shock between U_j and U_{j-1}, which is the full semi-tangent to C_j and C_{j-1} (with $C_0 := C_4$ for $j = 0$),
- The shock between U_j and U_{j+1}, which is the full semi-tangent to C_j and C_{j+1} (with $C_5 := C_1$ for $j = 4$),
- A quarter of C_j, which we call below γ_j.

We deduce that the subsonic domain Ω is bounded by four quarters of circles, which form an oval. We point out that the curvature of $\partial\Omega$ is piecewise constant and strictly positive.

We notice that the curve γ_j may be a shock between U_j and the solution in Ω. The boundary conditions for the subsonic flow will be given by the Rankine–Hugoniot equations.

There remains to construct the solution in the subsonic domain. We look for a Lipschitz continuous solution ϕ of (22) within Ω.

Since we expect that the flow remains pseudo-subsonic throughout $\mathcal{D}$, (22) has to be elliptic, which means that ϕ remains positive. The fact that ϕ is positive in Ω is consistant with the maximum principle, since the equation provides

$$\operatorname{div}\left(\frac{\nabla\phi}{\sqrt{2\phi + |\nabla\phi|^2}}\right) \leq 0.$$

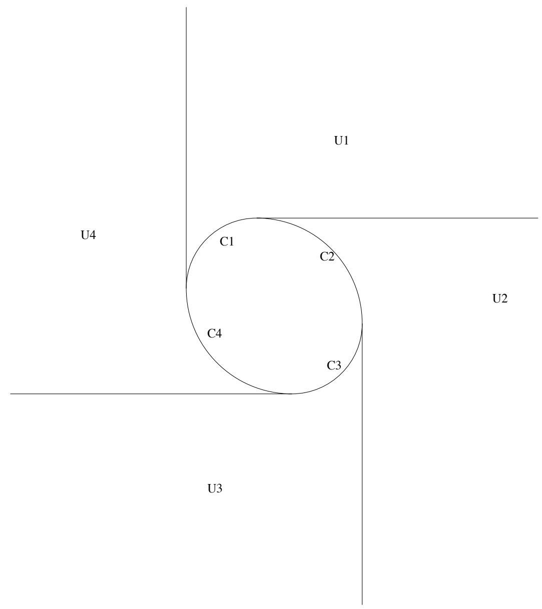

Figure 5: Wave pattern for the vortex Riemann problem.

When this equation is elliptic, the maximum principle tells that the minimum of ϕ is reached along the boundary. From (23), we get the positivity of ϕ.

Remark. The equation (22) is the Euler–Lagrange equation of the functional

$$\mathcal{L}[\phi] = \int_\Omega L(\phi, \nabla\phi)\, dy := \int_\Omega \phi^{-3/2}\sqrt{2\phi + |\nabla\phi|^2}\, dy.$$

This fact is however of little help in our Dirichlet boundary-value problem, because $\mathcal{L}[\phi]$ is infinite for every test function vanishing on the boundary. Minimizing $\mathcal{L}$ is therefore not a good idea.

The rest of the analysis consists in solving the boundary-value problem (22,23). The fact that it provides a solution to the self-similar Euler equations is immediate. Besides the nonlinearity of (22), we observe that the density $L(u, p)$ is asymptotically linear in p and that means that its domain of definition, even for uniformly positive functions, is made of BV functions. This reveals the fact that (22) is not uniformly elliptic.

4.2 The saddle

The situation is a little bit more involved for the compression-expansion (saddle) data. It is tempting to retain the full semi-tangents to C_j and C_{j-1} as shocks separating U_j and U_{j-1}; let us call them $S_{j-1/2}$. However, we observe that these four shocks display two pairwise interactions before reaching the subsonic domain. For instance, $S_{3/2}$ and $S_{5/2}$ interact, see Figure 6. Thus

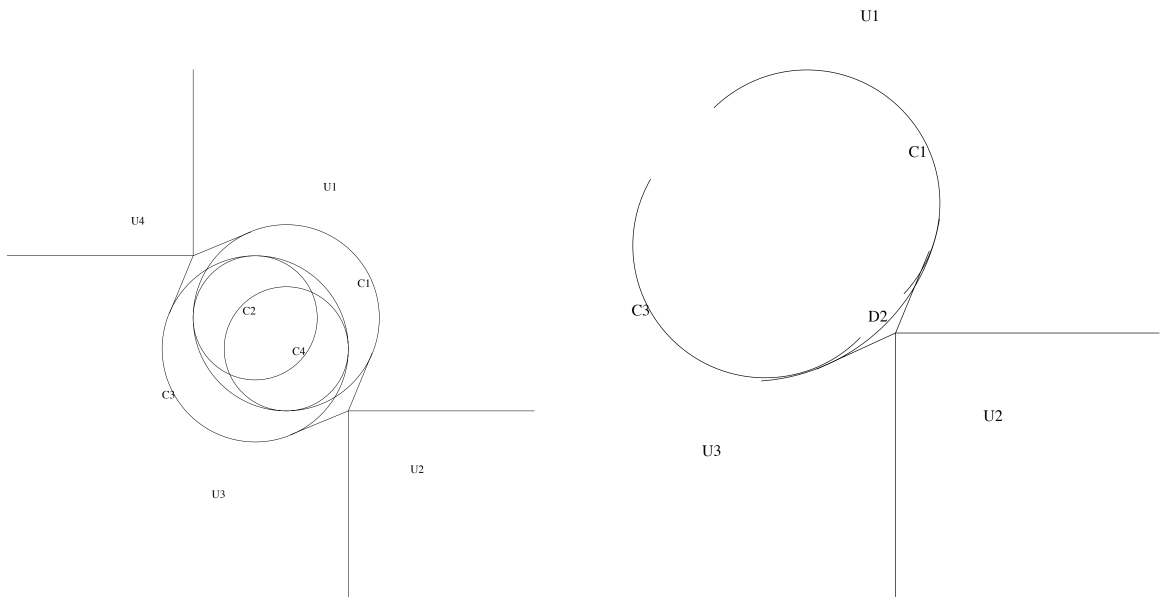

Figure 6: Wave pattern for the saddle. Right: detail showing the sonic circle D_2 of the reflected state U_2'. This circle is a part of the sonic boundary.

the state U_2 is bounded by two straight shocks only, and similarly for the state U_4. The interaction of $S_{3/2}$ and $S_{5/2}$ is resolved as in Paragraph 2.4: the outgoing shocks are straight lines $L_{1,3}$ tangent to C_1 and C_3 respectively (see Figure 6, left). They stop at the tangency points. The flow is uniform, equal to a constant state U_2', behind the interaction, up to the sonic circle of U_2', which we call D_2. Since L_1 separates U_1 and U_2', we find that D_2 is tangent to C_1 and L_1 at the same point, and similarly for D_2, C_3 and L_3. Therefore D_2 is explicitly known. A similar and symmetric picture occurs at the interaction between U_1, U_3 and U_4. The subsonic domain is therefore surounded by four arcs or the circles $C_{1,3}$ and $D_{2,4}$. The constant state U_1 is bounded by the arc of C_1 and two tangents from C_1 to the interaction points.

As in the vortex case, there remains to solve the boundary-value problem (22,23) in the subsonic domain. This is done in the next section.

5 Well-posedness to the elliptic BVP (22,23)

What is the data ? We emphasize that neither the PDE (22) nor the boundary condition (23) involve a data. We should rather consider that the data is the convex domain Ω itself. To each domain, we expect to associate a unique positive solution, provided the convexity is strict enough (see below). We thus consider below the Dirichlet boundary-value problem for a general convex domain, with smooth or piecewise smooth boundary, the minimal regularity being piecewise $\mathcal{C}^2$. We shall also make the fundamental assumption that both the curvature κ of the boundary $\partial\Omega$ and the radius of curvature $R := 1/\kappa$ are bounded. In other words, κ is bounded and bounded away from zero.

Since we are searching for a positive solution of (22), we set $w := \sqrt{2\phi}$, that is $\phi = \frac{1}{2}w^2$ with $w > 0$. The equation becomes

$$\operatorname{div}\frac{\nabla w}{\sqrt{1+|\nabla w|^2}} + \frac{2}{w\sqrt{1+|\nabla w|^2}} = 0, \tag{24}$$

and we keep the same Dirichlet boundary condition

$$w = 0, \qquad \text{along } \partial\Omega. \tag{25}$$

Our result is

Theorem 5.1 *Let Ω be a piecewise $\mathcal{C}^2$ convex domain, with the curvature of $\partial\Omega$ being bounded and bounded away from zero. Then the Dirichlet boundary-value problem (24,25) admits a unique positive viscosity solution (in the sense of [6]). This solution belongs to $C(\overline{\Omega}) \cap C^\infty(\Omega)$.*

If the boundary is of class $\mathcal{C}^{m+1,\alpha}$, then the solution ϕ belongs to $\mathcal{C}^{m,\alpha}(\overline{\Omega})$.

Remarks.

- Once again, we observe that (24) is the Euler-Lagrange equation of the functional

$$w \mapsto \int_\Omega w^{-2}\sqrt{1+|\nabla w|^2}\,dx,$$

  though this is useless since this integral is infinite under the condition (25).

- The divergence term is nothing but twice the mean curvature for the graph $\mathcal{G}_w$ given by $y_3 = w(y_1, y_2)$. The equation (24) actually has the following geometrical interpretation. Let us form a hypersurface G_w in $\mathbb{R}^4$ having a symmetry of revolution with respect to the codimension two subspace $y_3 = y_4 = 0$, by rotating $\mathcal{G}_w$ around this 'axis'. The equation of G_w is

$$\sqrt{y_3^2 + y_4^2} = w(y_1, y_2).$$

  This hypersurface, being of dimension three, has three principal curvatures κ_1, κ_2 and κ_3. The first two are nothing but the curvatures of $\mathcal{G}_w$. As mentionned above, they have the property that

$$\kappa_1 + \kappa_2 = \operatorname{div}\frac{\nabla w}{\sqrt{1+|\nabla w|^2}}.$$

  The third one is the distance from G_w to the 'axis', in the direction of the normal. We immediately see that

$$\kappa_3 = \frac{1}{w\sqrt{1+|\nabla w|^2}}.$$

  In conclusion, (24) can be rewritten as

$$\kappa_3 = \frac{1}{2}(\kappa_1 + \kappa_2). \tag{26}$$

- For general graphs in $\mathbb{R}^{3+1}$, the equation (26) is a PDE of hyperbolic type. It is only because we restrict to graphs of revolution around $x_3 = x_4 = 0$ that this equation becomes elliptic. This is the same phenomenon as a wave equation (hyperbolic) becomes the Helmoltz equation (elliptic) when we fix the time frequency ω.

Strategy of the proof. Equation (24) mixes two difficulties. On the one hand, there is a singularity at the boundary, since w vanishes there. On the other hand, the ellipticity is not uniform, as long as we lack a Lipschitz estimate. This fact has long been encountered in the study of minimal surfaces in non-parametric form.

Uniqueness follows from the comparison principle in the context of viscosity solutions, valid for (24) because the principal part does not depend on w but only on its gradient, while the lower-order term is non-increasing in w. See [6] or [3] for the fundamentals of the theory.

We thus focus on the existence from now on. Since the theory of minimal surfaces is well-established, in particular for convex domains, we shall apply a continuation procedure, by considering the Dirichlet problem with a parameter $\mu \in [0, 2]$:

$$\sqrt{1 + |\nabla w|^2}\,\mathrm{div}\frac{\nabla w}{\sqrt{1 + |\nabla w|^2}} + \frac{\mu}{w} = 0. \tag{27}$$

Besides, we shall replace the homogeneous Dirichlet boundary condition by a non-homogeneous one,

$$w = \epsilon, \qquad \text{along } \partial\Omega, \tag{28}$$

with $\epsilon > 0$.

Our strategy is the following. We first fix $\epsilon > 0$ and show that Problem (27,28) admits a unique smooth solution $w_{\mu,\epsilon}$ for every $\mu \in [0, 2]$. Actually, we show that the set of parameters μ for which there exists a solution is open and closed in $[0, 2]$. Since it contains $\mu = 0$ (solution $u \equiv \epsilon$), it equals $[0, 2]$. In this part of the proof, we establish estimates that are independent of μ. Next, since $w_{2,\epsilon}$ is smooth, we establish estimates (especially Lipschitz ones) that are independent of ϵ. We are thus able to pass to the limit as $\epsilon \to 0+$, showing that there is a smooth solution w. We point out that the theory of viscosity solutions implies uniqueness. Thus w is unique even within the class of viscosity solutions.

The above procedure uses the regularity theory for uniformly elliptic equations in smooth domains. We therefore begin by treating the case where $\partial\Omega$ is $\mathcal{C}^\infty$-smooth. We then show that the result extend to less regular boundaries.

5.1 The continuation procedure

Since $\epsilon > 0$ is fixed along this paragraph, we feel free to denote w_μ the solution, when it exists, of our Dirichlet problem, instead of $w_{\mu,\epsilon}$. This solution must be unique because of the comparison principle.

Let us denote by J_ϵ the set of parameters $\mu \in [0,2]$ for which a smooth solution $w_\mu > \epsilon/2$ exists to (27,28). We show hereafter that there exists a constant C_ϵ independent of μ, such that

$$\epsilon < w_\mu \le C_\epsilon, \qquad |\nabla w_\mu| \le C_\epsilon. \tag{29}$$

J_ϵ **is closed.** From (29), we know that the operator at work in (27)

$$Lv := \Delta v - \frac{\nabla w \otimes \nabla w}{1+|\nabla w|^2} : D^2 v$$

is uniformly elliptic and satisfies the maximum principle. Then standard regularity theory (see for instance [7]) provides bounds in $\mathcal{C}^k(\overline{\Omega})$ for every $k \ge 0$ that are independent of μ. Thus the set of pairs (μ, w_μ) is precompact is $\mathbb{R} \times \mathcal{C}^\infty$ and we can pass to the limit when $\mu_m \to \mu$ with $\mu_m \in J_\epsilon$, up to a subsequence. Thus the limit μ belongs to J_ϵ too.

J_ϵ **is open.** Openess follows from the implicit function theorem, applied in $\mathcal{C}^{2,\alpha}(\overline{\Omega})$. We only have to check that the linearized equation is uniquely solvable. Linearization of (27) writes

$$\left(L + b(\nabla w, D^2 w)\cdot\nabla - \frac{\mu}{w^2}\right) v = f.$$

Since $\mu \ge 0$, w_μ is in $\mathcal{C}^\infty(\overline{\Omega})$ and L is uniformly elliptic at w_μ, the operator in the left-hand side is one-to-one and therefore invertible under the Dirichlet boundary condition. It is thus a matter of routine to apply the implicit function theorem at a point $\mu_0 \in J_\epsilon$, to show that there exists a smooth curve $\mu \mapsto w_\mu$ through w_{μ_0}, such that w_μ is a smooth solution of (27,28). Thus J_ϵ contains a neighbourhood of μ_0. Remark that $\mu \mapsto w_\mu$ is increasing, thanks to the maximum principle applied to

$$\left(L + b(\nabla w, D^2 w)\cdot\nabla - \frac{\mu}{w^2}\right)\frac{\partial w}{\partial \mu} = -\frac{1}{w}, \qquad \left.\frac{\partial w}{\partial \mu}\right|_{\partial\Omega} = 0.$$

L^∞ **estimate.** There remains to prove the estimates (29) for smooth solutions. To begin with, the maximum principle ensures that $w_\mu \ge \epsilon$. Likewise, w_μ is smaller than any super-solution of (27,28). To construct a super-solution, we first observe that the equation (22) has a nice family of exact solutions, namely[3]

$$\phi^{r,m}(y) := \frac{1}{2}(r^2 - |y-m|^2),$$

for every $r > 0$ and $m \in \mathbb{R}^2$. The corresponding $w^{r,m}$ is a solution of (24) and thus a super-solution of (27) for every $\mu \le 2$. We thus define $w_{+,\epsilon}$ as the infimum of all the $w^{r,m}$ that are larger than ϵ over Ω. This amounts to saying that the disk $D(m; \sqrt{r^2-\epsilon^2})$ contains Ω. Since Ω is convex, $w_{+,\epsilon}$ satisfies (28). Since it is also a super-solution of (27), $w_{+,\epsilon}$ is a super-solution of the Dirichlet problem. We thus deduce

$$\epsilon \le w_\mu \le w_{+,\epsilon}. \tag{30}$$

[3]The potentials $\phi^{r,m}$ correspond to uniform flows with velocity m. Since uniform flows are solutions of the Euler system, whatever the equation of state, these functions may be used to build sub-/super-solutions of the potential equation for every equation of state.

Lipschitz estimate. A cumbersome though standard calculation shows that the function $z_\mu := \frac{1}{2}|\nabla w_\mu|^2$ satisfies an equation of the form

$$\left(L - \frac{2\mu}{w_\mu}\right) z_\mu = \mathrm{Tr}\left((D^2u)^2\right) + b_1 \cdot \nabla z_\mu.$$

There follows that z_μ does not have an interior maximum:

$$\sup_\Omega z_\mu = \sup_{\partial\Omega} z_\mu.$$

We deduce that

$$\|\nabla w_\mu\|_\infty \leq \|\nabla w_{+,\epsilon}\|_\infty \leq \frac{1}{\epsilon}\|\nabla \phi_{+,\epsilon}\|_\infty.$$

Since we shall show in the next paragraph that $\phi_{+,\epsilon}$ is Lipschitzian (even uniformly in terms of ϵ), this is the expected Lipschitz estimate.

5.2 The limit as $\epsilon \to 0+$

We now fix $\mu = 2$ and denote w^ϵ instead of $w_{2,\epsilon}$. Recall that this is the unique positive solution of (24,28) and that it belongs to $C^\infty(\overline{\Omega})$. By the maximum principle, we know that $\epsilon \mapsto w^\epsilon$ is monotonous increasing, so that we may define the pointwise limit

$$w(y) := \lim_{\epsilon \to 0+} w^\epsilon(y).$$

When looking at the limit as $\epsilon \to 0+$, the Lipschitz estimate of the previous paragraph is useless because

$$\lim_{\epsilon \to 0+} \|\nabla w_{+,\epsilon}\|_\infty = +\infty.$$

As a matter of fact, we do not expect that the limit w be Lipschitzian in Ω. The reason for that is that we have a sub-solution $w_- = \sqrt{2\phi_-}$, where ϕ_- is defined as the supremum of all the $\phi_{r,m}$ for which the disk $D(m;r)$ is contained in Ω. Each w^ϵ, and thus w itself, is bounded below by w_-. Since ϕ_- is Lipschitz, non-constant, and vanishes along $\partial\Omega$, w_- has a square-root singularity at the boundary, and w must be as bad as w_-, at least.

We thus establish in the sequel a uniform estimate for ϕ^ϵ instead. Since w_- is positive in Ω, this will translate into a Lipschitz estimate for w^ϵ, uniform on every compact subset of Ω. By the classical regularity theory, the sequence will be bounded in $C^\infty(\Omega)$, though not in $C^\infty(\overline{\Omega})$. It will thus converge, in this topology, towards a smooth w, which will thus be a solution of (24). Since $w_- \leq w^\epsilon \leq w_{+,\epsilon}$, we have in the limit

$$w_- \leq w \leq w_+,$$

where $w_+ = \sqrt{2\phi_+}$ and ϕ_+ is the infimum of the $\phi_{r,m}$ for which the disk $D(m;r)$ contains Ω. Since $w_\pm$ vanish on the boundary, w satisfies (25). Thus our boundary-value problem admits a positive smooth solution.

For the completion of the proof of Theorem 5.1 in the case of a smooth boundary, we thus have to establish the uniform Lipschitz estimate for ϕ^ϵ. We proceed in two step.

Lipschitz estimate at the boundary. Since $\epsilon < \phi_\epsilon < \phi_{+,\epsilon}$ in Ω and equalities hold on the boundary, we have immediately that

$$\|\nabla\phi_\epsilon\|_{L^\infty(\partial\Omega)} \leq \|\nabla\phi_{+,\epsilon}\|_{L^\infty(\partial\Omega)}.$$

It is thus enough to find a bound of the right-hand side above. Let y_0 be a boundary point. Since the curvature of $\partial\Omega$ is bounded away from zero, say $\kappa \geq 1/R$, there exists a disk $D(m;R)$ containing Ω, such that $|y_0 - m| = R$. Then $\phi^{r(\epsilon),m}$ is a super-solution, where $r(\epsilon) := \sqrt{\epsilon^2 + r^2}$. Thus $\phi_{+,\epsilon} \leq \phi^{r(\epsilon),m}$ and, since they agree at y_0, we have

$$|\nabla\phi_{+,\epsilon}(y_0)| \leq |\nabla\phi^{r(\epsilon),m}(y_0)| = |y_0 - m| = R.$$

We thus deduce the uniform bound

$$\|\nabla\phi_\epsilon\|_{L^\infty(\partial\Omega)} \leq R.$$

Interior Lipschitz estimate. Let us introduce $z^\epsilon := \frac{1}{2}|\nabla\phi^\epsilon|^2$. After a cumbersome calculation, we find the elliptic equation in z^ϵ

$$(31)\quad L_\epsilon z^\epsilon - 2(\phi^\epsilon + z^\epsilon)\mathrm{Tr}\left((D^2\phi^\epsilon)^2\right) + 2(2z^\epsilon + \nabla\phi^\epsilon\cdot\nabla z^\epsilon)\Delta\phi^\epsilon - |\nabla z^\epsilon|^2 + 8z^\epsilon + 2\nabla\phi^\epsilon\cdot\nabla z^\epsilon = 0,$$

where $L_\epsilon := (2\phi^\epsilon + |\nabla\phi^\epsilon|^2)\Delta - \nabla\phi^\epsilon\otimes\nabla\phi^\epsilon : D^2$ is the second-order operator involved in (22). At this level, it does not seem possible to find an upper bound for z^ϵ by using this equation at a maximum of z^ϵ. To overcome the difficulty, we combine (31) with (22), to obtain for every parameter $\alpha \in \mathbb{R}$:

$$(32)\qquad L_\epsilon(z^\epsilon + \alpha\phi^\epsilon) + b(y)\cdot\nabla(z^\epsilon + \alpha\phi^\epsilon) + 2M(y)z^\epsilon = 2\phi^\epsilon(\mathrm{Tr}\left((D^2\phi^\epsilon)^2\right) - 2\alpha) \geq -4\alpha\phi^\epsilon,$$

with

$$M := -\mathrm{Tr}\left((D^2\phi^\epsilon)^2\right) + 2(1-\alpha)\Delta\phi^\epsilon - \alpha^2 - \alpha + 4.$$

Let us assume that $z^\epsilon + \alpha\phi^\epsilon$ reaches a local maximum at some interior point y_0. Then (32) implies

$$M(y_0)z^\epsilon \geq -2\alpha\phi^\epsilon.$$

Using the fact that $D^2\phi^\epsilon$ is a symmetric matrix, thus has a real spectrum $\{\lambda,\mu\}$, and that

$$M = -\lambda^2 - \mu^2 + 2(1-\alpha)(\lambda+\mu) - \alpha^2 - \alpha + 4,$$

we obtain

$$M \leq \alpha^2 - 5\alpha + 6 = (\alpha-2)(\alpha-3).$$

We choose a parameter for which this upper bound is negative, say $\alpha = 5/2$. Then $M(y) \leq -1/4$, and we deduce

$$z^\epsilon(y_0) \leq 20\phi^\epsilon(y_0).$$

We have therefore

$$(z^\epsilon + \alpha\phi^\epsilon)(y_0) \leq \frac{45}{2}\phi^\epsilon(y_0) \leq \frac{45}{2}\phi_{+,\epsilon}(y_0).$$

Finally, there comes

$$\sup_{\Omega}(z^\epsilon+\alpha\phi^\epsilon)\leq \max\left\{\sup_{\partial\Omega}(z^\epsilon+\alpha\phi^\epsilon),\,\frac{45}{2}\sup_{\Omega}\phi_{+,\epsilon}\right\}.$$

Along the boundary, ϕ^ϵ equals $\epsilon^2/2$, while $|\nabla\phi^\epsilon|$ is bounded by the Lipshitz constant L_ϵ of $\phi_{+,\epsilon}$. We therefore deduce

$$\sup_{\Omega} z^\epsilon\leq \max\left\{\frac{L_\epsilon^2+\alpha\epsilon^2}{2},\,\frac{45}{2}\sup_{\Omega}\phi_{+,\epsilon}\right\},$$

or equivalently

$$\|\phi^\epsilon\|_{Lip}\leq \max\left\{\sqrt{L_\epsilon^2+\alpha\epsilon^2}\,,3\sqrt{5\sup_{\Omega}\phi_{+,\epsilon}}\right\}. \tag{33}$$

In this estimate, two simplifications can be made. On the one hand, we have

$$\phi_{+,\epsilon}=\frac{1}{2}\epsilon^2+\phi_+.$$

We point out that in general, the sum of a constant and of a solution of (22) is not a solution itself, but we use the fact that this is true for solutions of the form $\phi_{r,m}$. The same identity implies, on the other hand, that $L_\epsilon =: L_+$ does not depend on ϵ. In conclusion, we really have

$$\|\phi^\epsilon\|_{Lip}\leq \max\left\{\sqrt{L_+^2+\alpha\epsilon^2}\,,3\sqrt{5\left(\frac{\epsilon^2}{2}+\sup_{\Omega}\phi_+\right)}\right\}.$$

The right-hand side has a finite limit when $\epsilon\to 0+$. This shows that equation (24) remains uniformly elliptic on every compact subset of Ω. Since in addition $w^\epsilon\geq w_-$ is uniformly positive on compact subsets, the singularity of the equation is not visible in the interior. By the classical regularity theory (Schauder estimates), we thus have interior estimates for all the derivatives of w^ϵ, that are uniform in ϵ, though not in x as $d(x;\partial\Omega)\to 0$. The limit w therefore belongs to $C^\infty(\Omega)$.

5.3 Domain with a boundary of finite regularity

So far, we have proved Theorem 5.1 when the boundary of Ω is a smooth curve and κ is bounded and bounded away from zero. There remains to consider domains Ω with a piecewise $\mathcal{C}^2$ boundary. Let $\kappa_\pm$ be upper/lower bounds of the piecewise continuous curvature. Let ℓ be the length of $\partial\Omega$. The boundary of Ω is fully determined, up to a displacement, by the arc-length-curvature relation $s\mapsto\kappa(s)$, which is ℓ-periodic.

Let ρ_m be a smoothing sequence, with which we define $\kappa_m:=\rho_m*\kappa$, an ℓ-periodic function with the same bounds $\kappa_\pm$. The Gauss–Bonnet formula is preserved since

$$\int_0^\ell\kappa_m\,ds=\int_0^\ell\kappa\,ds=2\pi.$$

The function κ_m is therefore the curvature of a convex body Ω_m, which tends to Ω in every desirable sense, and at least in the Hausdorff topology of compact sets.

To each $m \in \mathbb{N}$, we associate the unique elliptic solution ϕ_m of (22,23) in Ω_m. From the previous section, we know the bounds

$$\phi_{-,m}(y) \leq \phi_m(y) \leq \phi_{+,m}(y)$$

and

$$\|\phi_m\|_{Lip} \leq \max\{L_{+,m}, 3\sqrt{5 \sup \phi_{+,m}}\},$$

where the bounds can be written in terms of $\kappa_\pm$ only. Therefore the sequence is bounded in the Lipschitz norm. From interior elliptic regularity, we deduce again that the sequence is locally bounded in $\mathcal{C}^\infty$. We can therefore extract a subsequence, converging uniformly on $\overline{\Omega}$, and converging in the $\mathcal{C}^\infty$-topology of Ω. The limit is the solution of our problem.

This completes the proof of Theorem 5.1.

∎

5.4 The regularity across the boundary

The fact that the boundary $\partial\Omega$ is both characteristic and strictly convex yields a rather strange property. It looks like if our boundary value problem was a Cauchy problem ! As a matter of fact, we are able to compute the normal derivative and many others of the unknown ϕ, everywhere ϕ is a smooth function. This requires at least that the boundary be smooth enough. We shall not investigate whether this regularity occurs on the subsonic side of $\partial\Omega$, but we content ourselves to compute the derivatives, assuming enough regularity.

To begin with, the boundary condition implies

$$\nabla \cdot \tau = 0. \tag{34}$$

Thus $\nabla\phi = S\nu$ where $S < 0$ is to be determined. We next use (22) in the form

$$(2\phi + |\nabla\phi|^2)\Delta\phi - D^2\phi(\nabla\phi, \nabla\phi) + 4\phi + |\nabla\phi|^2 = 0. \tag{35}$$

The restriction of (35) to the boundary gives

$$D^2\phi(\tau, \tau) = -1. \tag{36}$$

We now differentiate (34) along $\partial\Omega$:

$$0 = D^2\phi(\tau, \tau) + \nabla\phi \cdot \dot{\tau}.$$

This, together with (36), give

$$\nabla\phi \cdot \nu = -R, \tag{37}$$

with R the curvature radius. Let us differentiate this identity along the boundary:

$$D^2\phi(\nu, \tau) + \nabla\phi \cdot \dot{\nu} = -\dot{R}.$$

This gives

$$D^2\phi(\tau,\nu) = -\dot{R}, \tag{38}$$

If we differentiate (36) instead, we have

$$D^3\phi(\tau,\tau,\tau) + 2D^2\phi(\tau,\dot{\tau}) = 0.$$

This, together with (38), yields

$$D^3\phi(\tau,\tau,\tau) = -2\kappa\dot{R}. \tag{39}$$

To go further we differentiate (35) in the direction of a vector field X. Restricting to $\partial\Omega$, we obtain

$$R^2D^3\phi(X,\tau,\tau) - 2R(X\cdot\nu + D^2\phi(X,\nu))\Delta\phi + 2D^2\phi(D^2\phi X,\nu) - 4RX\cdot\nu - 2RD^2\phi(X,\nu) = 0.$$

Chosing $X = \tau$, several simplifications arise because of the previous results, and we conclude that

$$D^2\phi(\nu,\nu) = -1 + 2\dot{R}^2 - R\ddot{R}. \tag{40}$$

The calculation can be continued *ad libitum*, yielding an explicit formula

$$D^k\phi(\tau^{\otimes l},\nu^{\otimes k-l}) = \kappa^{k-2}P_{l,k-l}(R), \tag{41}$$

where $P_{l,k-l}(R)$ is a differential polynomial in R, homogeneous of degree zero when R has degree one and each derivative d/ds contributes to a degree -1. To see this, we proceede by induction. Let us assume that for a given $k \geq 3$, the expressions (41) are known for every $1 \leq l \leq k$. We also assume that the derivatives of order less than k are known along $\partial\Omega$. Differentiating (41) along the boundary, we obtain

$$\begin{aligned} D^{k+1}\phi(\tau^{\otimes l+1},\nu^{\otimes k-l}) - \kappa l D^k\phi(\tau^{\otimes l-1},\nu^{\otimes k-l+1}) \quad &+ \quad \kappa(k-l)D^k\phi(\tau^{\otimes l+1},\nu^{\otimes k-l-1}) \\ &= \quad \frac{d}{ds}(\kappa^{k-2}P_{l,k-l}(R)). \end{aligned} \tag{42}$$

If $l \geq 2$ or if $l = 0$, this provides the polynomials $P_{l+1,k-l}$.

The next step is to apply $(X\cdot\nabla)^{k-1}$ to (35), where X is a vector field that coincides with ν over $\partial\Omega$. We then specialize to boundary points. The resulting expression contains only known quantities, except for t

$$R^2D^{k+1}(\tau,\tau,\nu^{\otimes k-1}).$$

Let us emphasize that because $k \geq 3$, the derivative $D^k(\nu^{\otimes k})$ does not occur in this idendity. This calculation therefore provides the polynomial $P_{2,k-1}$.

We now go back to (42), but with $l = 1$. Since the first $(D^{k+1}(\tau,\tau,\nu^{\otimes k-1}))$ and the third $(D^k\phi(\tau^{\otimes 2},\nu^{\otimes k-2}))$ terms are already known, we deduce the expression for the second one. Whence the polynomial $P_{0,k}$. This completes the induction.

Regularity across the sonic boundary. The calculations above can be made on either side of the boundary provided the solution is smooth and, above all, that the curvature κ is bounded away from zero.Therefore they provide the same values on the supersonic side as on the subsonic side. This means that ϕ could be at least of class $\mathcal{C}^k$ across $\partial\Omega$, provided that this boundary is of class $\mathcal{C}^{k+2}$. In terms of the flow, we see that (ρ, u) would be of class $\mathcal{C}^{k-1}$. In particular, it should be continuous across every part of $\partial\Omega$ of class $\mathcal{C}^3$. *This is however an illusion.* To see this, we just observe that the flow outside of Ω is not unique: the next paragraph provides at least two explicit solutions that are smooth in the complement of $\overline{\Omega}$. They are not smooth up to $\partial\Omega$, except in zones where the boundary is an arc of circle. Even their difference is not $\mathcal{C}^{3/2}$-smooth. Since the elliptic solution in Ω is unique, we deduce that general solutions may not be smooth across $\partial\Omega$.

The question whether the flow is continuous or not across the sonic boundary in the saddle and the vortex problem, or in the reflection against a wedge, remains open. The exterior solution is piecewise smooth, up to the sonic boundary, which is piecewise circular. Thus the above discussion does not apply. Pro is the fact that the flow is continuous across the sonic line of the reflected state for the reflection of an ideal gas, as shown by Chen & Feldman [4]. Con is the general fact that the reflected shock itself is always a discontinuity. One does not see why it should not be for a Chaplygin gas. At least, our calculations show that some singularity must occur at the contact points of two arcs of circle.

6 Generalized 'vortex' Riemann problem

Theorem 5.1 can be used o solve much more general $2-D$ Riemann problem. To do so, we first choose a convex domain as above, so that we have a solution ϕ of the Dirichlet problem in Ω. We extend ϕ outside Ω with the following construction. To each point $y \notin \Omega$, exactly two tangents to Ω can be drawn from y. Since the curvature is strictly positive, each tangent has only one contact point. Looking at Ω from y, we select the rightmost tangent and denote the tangency point as $x(y)$.

Smoothness. Let us parametrize $\partial\Omega$ by arc-length, say in the trigonometric sense: $s \mapsto m(s)$. The unit tangent and the normal at $m(s)$ are denoted by $\tau(s)$, $\nu(s)$. There holds $\dot\tau = -\kappa\nu$ and $\dot\nu = \kappa\tau$. When y is not on the boundary, $x(y)$ has the form $m(s(y))$ and $(y, s(y))$ satisfies

$$N(y,s) := (y - m(s)) \cdot \nu(s) = 0.$$

Assume that the boundary be of class $\mathcal{C}^{l+1}$. Then the map N is of class $\mathcal{C}^l$ over $\mathbb{R}^2 \times \mathbb{R}/L\mathbb{Z}$, L being the perimeter of Ω. The calculation

$$\frac{\partial N}{\partial s} = -\dot m \cdot \nu + (y - m) \cdot \dot\nu = \kappa(y - m) \cdot \tau = -\kappa|y - m(s)|$$

shows that the s-derivative of N does not vanish outside $\overline{\Omega}$. By the implicit function Theorem, we deduce that the map $y \mapsto s(y)$ is of class $\mathcal{C}^l$. Moreover, one has

$$\kappa|y - x(y)|\nabla s(y) = \nu(s(y)). \tag{43}$$

The function ϕ in the exterior domain We now define

$$\phi(y) := -\frac{1}{2}|y - x(y)|^2 \tag{44}$$

whenever $y \notin \Omega$. Obviously, ϕ is continuous up to $\partial\Omega$ and satisfies the boundary condition $\varphi = 0$. Remark that ϕ being negative, the exterior of Ω wil be a supersonic zone, once we shall now that ϕ is the potential of a self-similar flow.

We now verify that ϕ solves the PDE (22). To begin with, we first compute its gradient:

$$\nabla\phi = -(x(y) - y) \cdot \nabla(x(y) - y) = x(y) - y - |x(y) - y|\tau(s(y)) \cdot \nabla x(y).$$

Since $x(y) = m(s(y))$, we have $\nabla s = \dot{m} \otimes \nabla s = \tau \otimes \nabla s$. Thus

$$\nabla\phi = x(y) - y - |x(y) - y|\nabla s.$$

Using (43), we conclude that

$$\nabla\phi(y) = x(y) - y - R(s(y))\nu(s(y)). \tag{45}$$

From (45), we infer $|\nabla\phi|^2 = |x(y) - y|^2 + R(s(y))^2$. We therefore find

$$\sqrt{2\phi + |\nabla\phi|^2} = R(s(y)).$$

In roder to check the PDE (22), we have thus to show that

$$\mathrm{div}(\kappa(s(y))\nabla\phi) + 2\kappa(s(y)) = 0. \tag{46}$$

Lat A denotes the left-hand side of (46). We have

$$A = \mathrm{div}(\kappa(x(y) - y) - \nu(s(y)) + 2\kappa = \kappa\mathrm{div}m(s(y)) + (x(y) - y) \cdot \nabla\kappa(s(y)) - \mathrm{div}\nu(s(y)).$$

For every vector-valued map $s \mapsto X(s)$, we apply the formula

$$\mathrm{div}X \circ s = \dot{X}(s(y) \cdot \nabla s.$$

The thus find

$$A = \nabla s(y) \cdot (\kappa\dot{m} - \dot{\nu}) + (x(y) - y) \cdot \nabla\kappa(s(y)) = \nabla s(y) \cdot (\kappa\dot{m} - \dot{\nu} + \dot{\kappa}(x(y) - y)).$$

However, ∇s is parallel to ν, according to (43. Its scalar product with tangent vectors at $m(s)$ thus vanishes. Since each of $\dot{m}$, $\dot{\nu}$ and $x(y) - y$ is a tangent vector, we find $A = 0$ and therefore ϕ satifies (22).

Since along $\partial\Omega$, ϕ matches the solution given by Theorem 5.1 in the interior, we have obtain a solution in the full plane. From it, we can define a self-similar flow by setting

$$u(y) := x(y) - R(s(y)), \qquad \rho(y) := a\kappa(s(y)).$$

Notice that $u(y)$ is nothing but the center of curvature for $\partial\Omega$ at point $x(y)$.

The initial data. It is interesting to identitfy which initial data can be treated as above. We begin by identifing the initial value of the flow constructed above; To do so, we reparametrize the boundary by the angle θ that the tangent vector $-\tau(s)$ makes with the horizontal axis:

$$\tau(s) = -\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}.$$

We have $ds/d\theta = R$, or equivalently $d\theta/ds = \kappa$.

Selecting an angle θ, let us consider the semi-tangent originated from $m(s)$. Along this line, (45) tells that $u(y)$ is constant, equal to $m(s) - R(s)\nu(s)$. Since the initial flow can be seen from $U(y)$ by looking towards infinity, we have thus

(47) $$u^0(\theta) = m(s) - R(s)\nu(s), \qquad s = s(\theta).$$

Likewise, we have

(48) $$\rho^0(\theta) = a\kappa(s), \qquad s = s(\theta).$$

We summarize our result in the following statement.

Theorem 6.1 *Let Ω be a bounded convex set of $\mathbb{R}^2$ be given. Assume that its boundary is of class piecewise $\mathcal{C}^2$, with the curvature bounded and bounded away from zero.*

Let ϕ be define on $\mathbb{R}^2$ in the following way. In Ω, it is provided by Theorem 5.1 as the unique solution of the Dirichlet problem for (22). Outside of Ω, ϕ is given by (44).

Then ϕ defines a piecewise smooth self-similar flow of the Chaplygin gas, which is isentropic and irrotational.

Remark. Our choice of the rightmost tangency point in the definition of $x(y)$ was arbitrary. If we select the leftmost tangency point, we obtain a different flow outside of Ω, which corresponds to a vortex rotating in the opposite sense. See the discussion below. Notice that we may not vary our choice of $x(y)$. It has to be the same, either rightmost or leftmost, for every $y \notin \Omega$. Otherwise, the definition (refeq:defphi) does not provide a solution of (22).

Construction from an initial data. We now ask what degree of freedom the formulæ above leave us in the choice of an initial data. For this, we use the polar coordinates for the vector field u^0. Since $e_\theta = \nu(s)$ and $e_r = -\tau(s)$, we have

$$u_r^0 = m(s) \cdot e_r, \qquad u_\theta^0 = m(s) \cdot e_\theta - R(s).$$

Differentiating with respect to θ, we have

$$(u_r^0)' = R(s)\dot{m} \cdot e_r + m(s) \cdot e_\theta = -R(s) + m(s) \cdot e_\theta.$$

Finally, we obtain

(49) $$u_\theta^0 = (u_r^0)'.$$

Let us differentiate once more:

$$(u_r^0)'' = (u_\theta^0)' = R\dot{m} \cdot e_\theta - m \cdot e_r - R' = 0 - u_r^0 - R'.$$

Because of (48), this rewrites as

$$\rho^0 = \frac{a}{f(\theta)}, \qquad f' = -u_r^0 - (u_r^0)''. \tag{50}$$

Interpretation. So far, we have found two differential relations (49) and (50)between the initial data $u_r^0, u_\theta^0, \rho^0$. The former is nothing but irrotationality, which allows us to look for an irrotational flow (see Paragraph 2.6). The latter is more subtle. It ensures that in the Cauchy problem, as long one stays away from the influence domain of the origin, the pressure waves (these are the only one, since the flow is isentropic and irrotational) move only in the trigonometric sense. This is why we call our construction a *generalized vortex*. Without the assumption (50), the waves could move both in trigonometric and clockwise sense, but then the 2-D Riemann problem would be much more difficult.

Remark. By symmetry, we can construct a generalized vortex moving in the clockwise sense instead. It suffices to replace the condition (50) by

$$\rho^0 = \frac{a}{f(\theta)}, \qquad f' = u_r^0 + (u_r^0)''.$$

We are now in position to construct a solution to some 2-D Riemann problems for the Chaplygin gas. Fix first the constant entropy and let $a > 0$ be the constant $a \equiv \rho c$. Then let three bounded functions $\theta \mapsto u_r^0, u_\theta^0$ and f be given, satisfying (49,50), with $f > 0$ in addition. To each angle θ, we associate the line L_θ of equation

$$y \cdot e_\theta = u_r^0(\theta) + \frac{a}{f(\theta)},$$

which we rewrite $H(y, \theta) = 0$, with

$$H := y \cdot e_\theta - u_r^0(\theta) - \frac{a}{f(\theta)}.$$

The one-parameter family $\theta \mapsto L_\theta$ generates an envelop $\theta \mapsto y(\theta)$, given by the equations

$$H(y, \theta) = \frac{\partial H}{\partial \theta}(y, \theta) = 0.$$

This gives $y(\theta) = u_r^0(\theta)e_r + (u_\theta^0 + f)(\theta)e_\theta$. Differentiation gives, with the help of (49) and (50)

$$\frac{dy}{d\theta} = -f e_r.$$

Differentiating again, we find

$$\frac{d^2y}{d\theta^2} \wedge \frac{dy}{d\theta} = f^2 e_\theta \wedge e_r \neq 0.$$

We deduce that the curvature of the curve $\theta \mapsto y(\theta)$ is bounded and bounded away from zero. In particular, the curve enclose a compact subset K. It is not difficult to see that the construction given by Theorem 6.1, starting from Ω, the interior of K, yields the solution of the 2-D Riemann problem with data $(u^0, \rho^0 := a/f)$. Finally, we have

Theorem 6.2 *Let $\theta \mapsto (u^0, \rho^0)$ be given functions with $\rho^0 = a/f$, such that (49) and (50) hold.*

Then the solution of the 2-D Riemann problem for the isentropic Chaplygin gas is given by Theorem 6.1, where the boundary of Ω is the parametrized curve

$$\theta \mapsto u^0(\theta) + f(\theta) e_\theta.$$

7 The Regular Reflection past a wedge

7.1 The three regimes

Our first observation is that the constant state U_0 must be bounded by a characteristic curve of equation $|(u_0 - y) \times \dot{y}| = a/\rho_0$, thus be the union of tangents to the sonic circle C_0 and an arc of it. The incident shock I is such a tangent and the tangency point a_I belongs to the horizontal axis, at abcissa $|u_0| + c_0 = c_0$ (recall that $u_0 = 0$). Since this point lies out of the physical domain, I extends up to the ramp. This rules out the possibility of a Mach Reflection for a Chaplygin gas. We thus look for a Regular Reflection. In the sequel, we denote P the intersection point of the shock I with the ramp.

We point out that C_0 is tangent at left to I, because its center is at the origin. Since I is tangent to C_1 at a_I too, and the circles are interiorly tangent, we find that C_1 is at left of I too.

Since the reflected shock R bounds the constant state U_1, it is the second tangent from P to C_1. This raises a difficulty, when C_1 is so small that it is entirely contained in the solid wedge. Then no reflection is possible. As we have already seen in Section 2.5, a concentration phenomenon takes place. The solution is particularly simple: it is piecewise constant with only two states $U_{0,1}$, except for the presence of a singular layer of the density along OP. We point out that the density along the ramp increases linearly in time, a property consistent with the self-similarity. The gas concentrated on OP travels along the boundary with a velocity equal to the tangential component of u_1.

The true Regular Reflection therefore takes place when C_1 meets the domain occupied by the fluid, which means that $|u_1| \sin\alpha < c_1$, where 2α is the angle formed by the wedge. Let us denote by Q the point of tangency of R with C_1. We have known, since the seminal work by von Neumann, that the flow is uniform (state U_2) between R, the ramp and the sonic circle C_2 of U_2. Since C_1 and C_2 are interiorly tangent at Q, and since u_2 must be parallel to the ramp because of the boundary condition, we find u_2 at the intersection of the segment $u_1 Q$ with the ramp (Figure 8). The density ρ_2 equals a/c_2 where $c_2 = |u_2 Q|$ is the radius of C_2. Since in a

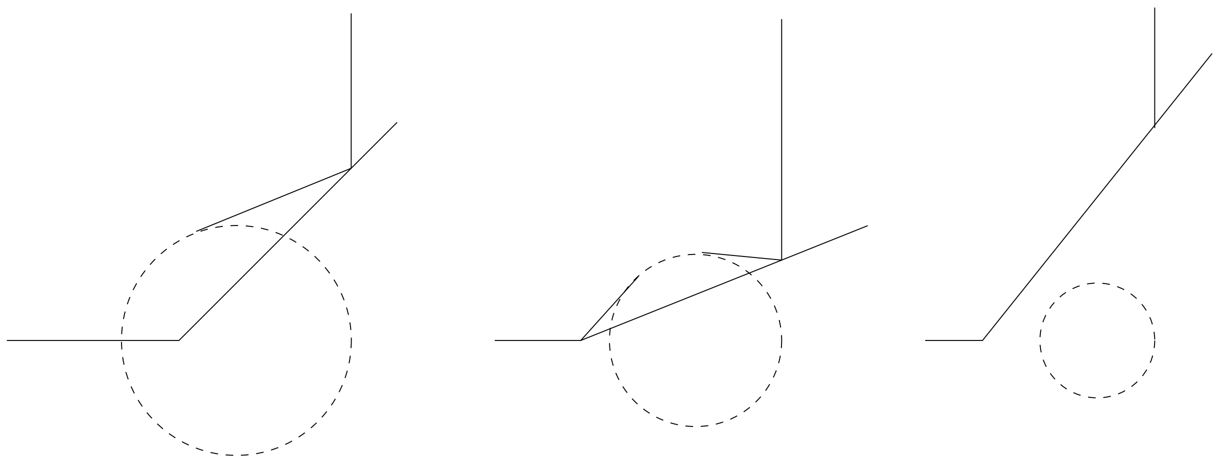

Figure 7: The three cases for the symmetric Regular Reflection. **Left**: U_1 is subsonic. There is one reflected shock. **Center**: U_1 is supersonic but $\sin\alpha|u_1| < c_1$. There is an additional shock emanating from the tip. **Right**: $\sin\alpha|u_1| > c_1$. The gas ultimately concentrates and travels along the ramp.

Chaplygin gas the boundary of the domain $U \equiv U_1$ must be characteristic for U_1, R must be continued beyond Q by following an arc of C_1.

There are two distinct regimes in the Regular Reflection, depending whether C_1 encloses the origin or not. If it does, that is if $|u_1| < c_1$ (we may say that the upstream state is subsonic) then one has the standard picture, where the reflected shock is curved up to the symmetry axis, where its tangent becomes vertical.

In the supersonic regime, which happens when $|u_1|\sin\alpha < c_1 < |u_1|$, the reflected shock must be continued by the tangent to C_1 passing through the origin, see Figure 8. This determines a tangency point T. Between the shock OT and the ramp, the flow is uniform $U \equiv U_3$. Since u_3 has to be parallel to the ramp and the sonic circle C_3 is interiorly tangent to C_1 at T, we find u_3 at the intersection of the ramp with the segment u_1T. The density ρ_3 is given by a/c_3, where $c_3 = |u_3T|$ is the radius of C_3. The flow remains supersonic and thus equal to U_3 up to the sonic circle C_3. The subsonic domain is thus bounded by the segment AB of the ramp, the arc AT of C_3, the arc TQ of C_1 and the arc QB of C_2.

Summary. So far, we have shown that depending on the location of the Mach number $M_1 := |u_1|/c_1$ with respect to 1 and $1/\sin\alpha$, there are three regimes:

- Concentration, without reflection ($M_1 \geq 1/\sin\alpha$),
- Subsonic Regular Reflection ($M_1 < 1$),
- Supersonic Regular Reflection ($1 \leq M_1 < 1/\sin\alpha$).

The first one (concentration) is fully explicit. The subsonic case raises an extra difficulty, with a geometrical singularity at the origin. We leave it for a future work. We prove that the solution

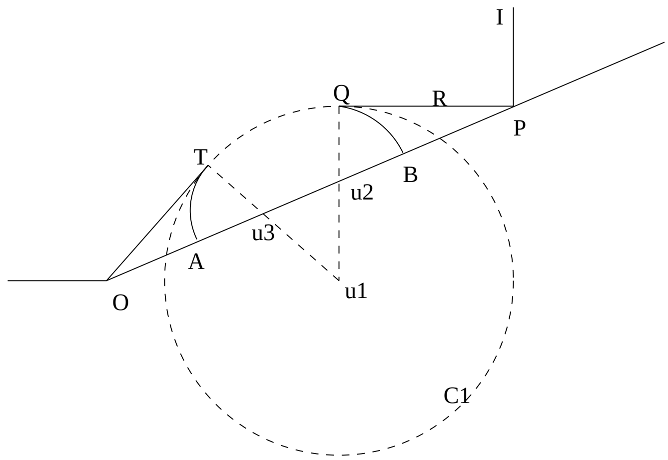

Figure 8: The intermediate case. The reflected shock and that originating at the tip at tangent lines to the sonic circle C_1. The states u_2 and u_3 are at the intersectiing of the ramp with the corresponding radii. The arc QB is centered at u_2, while AT is centered at u_3. The subsonic zone is $ATQB$.

of the supersonic RR does exist. This amounts to proving that a flow exists within the subsonic domain Ω.

7.2 The supersonic Regular Reflection

Since the initial data is isentropic and irrotational, we are allowed to look for an isentropic, potential solution. With ϕ the potential of the pseudo-velocity, we have to solve the PDE (22). Since Ω is subsonic, we look for a positive solution, for which the equation is thus elliptic. The PDE is completed by boundary conditions. We distinguish two parts in the boundary of Ω: we denote by Γ_N the part that is also in the boundary of the whole domain and by Γ_D the part that separates Ω from the supersonic domain. The former is the segment AB of the ramp, while the latter is made of three arcs of circles, the centers of two of them belonging to the ramp.

The boundary condition on Γ_D is the same matching as in the 2-D Riemann problems:

$$\phi = 0 \qquad \text{on } \Gamma_D. \tag{51}$$

On Γ_N, we write the no-flow boundary condition $u \cdot \nu = 0$, which amounts to

$$\frac{\partial \phi}{\partial \nu} = 0 \qquad \text{on } \Gamma_N. \tag{52}$$

We have to solve (22,51,52) in Ω. Since deal with viscosity solutions, we lack of references in the literature: the paper [10] by Milakis and Silvestre does not apply because it deals with equations of the form $F(D^2u, x) = 0$ that are uniformly elliptic. We remark that this domain meets the ramp perpendicularly.

Let S be the symmetry with respect to the ramp. We form a new domain $\Omega' = \Omega \cup (A, B) \cup S\Omega$. This is an oval, whose boundary is $\mathcal{C}^1$ and piecewise $\mathcal{C}^2$, made of four arcs of circles, respectively (TT'), (TQ), $QQ')$ and $(T'Q')$ where $T' := ST$ and $Q' := SQ$. We point out that Ω' is convex, and that its boundary has a piecewise constant curvature, everywhere equal to one of ρ_j/a, $j = 1, 2, 3$.

We now apply Theorem 5.1 to the domain Ω' and find a unique solution Φ of the Dirichlet problem (22,23). Since $y \mapsto \Phi(Sy)$ is a solution too, uniqueness implies $\Phi(Sy) = \Phi(y)$. In particular, the normal derivative along the axis (AB) vanishes. Therefore the restriction ϕ of Φ to Ω solves (22,51,52). This proves that the supersonic Regular Reflection does exist.

Uniqueness follows from a similar argument. If ϕ is a solution of our mixed boundary-value problem in Ω, then one defines a Φ in Ω' by reflection across (AB). This Φ turns out to be a solution of (22,23) in Ω', which is unique.

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