Systems of conservation laws with dissipation

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Systems of conservation laws have the form $\partial_t u + \mathrm{div}_x q = 0$. They describe processes in many situations of the real life, including continuum mechanics. A system is in closed form when the flux q is given in terms of u. An algebraic equation of state $q := f(u)$ yields a quasilinear first-order system. But a more accurate description involves a dissipative mechanism, through various additional terms in $q - f(u)$: viscosity, relaxation, hyperbolic-elliptic coupling. Many well-known equations, as Navier-Stokes and Boltzman, are relevant from this framework.

The goal of this course is to give an overview of the basic properties of such dissipative systems: local and global well-posedness, asymptotic behaviour, singular limit,... Key tools are reduced first-order systems and the Kawashima–Shizuta condition. Even in the the singular limit (for instance, vanishing viscosity), a small amount of dissipation is still present, in the admissibility condition for shock waves.

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At the time when this course was given, I had not had enough time to organize the notes below. They were raw meat, but contained some (even fresh) information. Expectedly, successive versions will be more complete and better presented.

Contents

What is dissipation ?

Roughly speaking, dissipation is the opposite to conservation. More precisely, though not that much, dissipation is something added to a conservative time-dependent differential equation, which has the effect to damp the solutions and to let them tend to some equilibrium manifold.

Let us take a basic example. We start with a linear conservative ODE. The conserved quantity is a positive definite quadratic form that we can take to define a norm. Thus the system writes

$$\dot{u} + Au = 0$$

where $A \in \mathbf{M}_n(\mathbb{R})$ is skew-adjoint: $A^T = -A$. Dissipation will come from an additional term Bu; the new system is

$$\dot{u} + (A + B)u = 0.$$

Of course we may assume that B is symmetric. Otherwise put its skew-symmetric part in A. Let us assume that the norm of u is dissipated, meaning that $t \mapsto |u(t)|^2$ is non-increasing in the evolution. This amounts to saying that B is semi-definite positive.

We have two basic questions. The first one is the time-asymptotics: what happens to a solution when $t \to +\infty$? The second is the relaxation limit, in which we put $\tau^{-1}B$ with $\tau > 0$ instead of B in the equation, and we let the relaxation time τ tend to zero. These questions have rather different answers. The relaxation limit is often richer than the time asymptotics. This shows that the relaxation limit is not uniform in time.

Let us have a look at the time asymptotics. Since a trajectory remains bounded (by dissipation) hence compact, we may introduce its ω-limit set, defined as

$$\Omega := \bigcap_{T>0} \overline{\{u(t)\,;\, t \geq T\}}.$$

This is a non-void compact set, which attracts the trajectory. It has many properties: – it is a connected set, – it is invariant (forward and backward) by the flow of the ODE, – there is no dissipation on Ω (in other words, $\Omega \subset \ker B$), – Ω is contained in level sets of the Liapunov functions (here the square of the norm). The two last properties form what is called the *Lasalle's invariance principle.*

Let E be the vector space spanned by Ω. By linearity, it enjoys the invariance and the lack of dissipation. Thus E is an invariant subspace for A and it is contained in $\ker B$. If $\dim E \geq 1$, then the restriction of A to E must have an eigenvector and this means that there is an eigenvector of A in $\ker B$. Conversely, this latter property allows us to find an A-invariant subspace $E \subset \mathbb{R}^n$ contained in $\ker B$ (take the line spanned by the eigenvector if the eigenvalue is real; if not, take the direct sum of this line with its conjugate). Then every initial data in E yields a solution that remains in E and dissipation is not visible for such solutions.

In conclusion, the time-asymptotics is fully described by the restriction of the ODE to the largest A-invariant subspace included in $\ker B$. Of course it often happens that $\ker B$ does not contain any eigenvector of A and we have thus $\lim_{t\to+\infty} u(t) = 0$ for every solution. This is called strong or *strict dissipation.* In the context of PDEs, this property will be the so-called

Kawashima–Shizuta assumption. Notice that in this finite-dimensional context, the decay to zero must be exponentially fast.

We now turn to the relaxation limit. Let us take an orthonormal basis in which some vectors span $\ker B$ and therefore the others span the range $R(B)$. The system decomposes as

$$\dot{v} + \gamma v + \delta w = 0, \qquad \dot{w} - \delta^T v + \sigma w + \frac{1}{\tau}\rho w = 0, \qquad (\gamma^T = -\gamma,\, \sigma^T = -\sigma),$$

in which ρ is symmetric positive definite. Formally (exercise: prove the claims), the solution u^τ, associated to an initial data u_0 independent of τ, has the property that $w^\tau \to 0$ for every $t > 0$. In particular, there is an initial layer at $t = 0$, since w_0 s non-zero in general. Passing to the limit in the first equation, we obtain that the relaxation limit is governed by the ODE of smaller size

$$\dot{v} + \gamma v = 0.$$

The latter is conservative. We point out that its frequencies, which are the real numbers ω such that $i\omega$ is an eigenvalue of γ, span an interval smaller than that spanned by the frequencies of the full ODE. This is a consequence of a theorem of Horn about the eigenvalues of a Hermitian matrix and of its principal submatrices.

In conclusion, both the time asymptotics and the relaxation limit are governed by a smaller number of variables and an ODE system $\dot{z} + Mz = 0$. In the former case, the parameter space is the larger invariant subspace of A contained in $\ker B$, and M is the restriction of A. In the latter case the space is the orthogonal of the range of B (it will be the kernel of B^T in general) and M (here γ) is not exactly a restriction of A, since the space is not A-invariant.

Once again, what is dissipation ?

Dissipation is like a rose. When you see one, you easily recognize it, but it is rather difficult to give a complete definition of the notion. The above discussion suggests that there is an underlying conservative system, but this will not always be the case. Say that the undissipated system has the property that it is globaly well-posed and that all its trajectories are *bounded*. Well-posedness is often both forward and backward, usually because the unperturbed system is reversible. Then the system is perturbed by the addition of a *dissipative mechanism*. Various examples will be given in the next section. Such a mechanism has a tendency to damp the forward trajectories and to break the reversibility. Sometimes, the backward Cauchy problem becomes ill-posed. But it is crucial that the forward Cauchy problem remains uniformly well-posed: the trajectories remain bounded for positive times. The main analytical issues are

- local (forward) well-posedness of the Cauchy problem,
- global and uniform well-posedness,
- time asymptotics,

- (if relevant) relaxation limit.

Because of the conservation/dissipation property, the second point above is often trivial. We shall see that the three others are not.

Chapter 1

The finite-dimensional situation

We focus in this chapter on ODEs in an open domain $\mathcal{U}$ of $\mathbb{R}^n$:

$$\dot{u} = F(u). \tag{1.1}$$

Hereabove, $F : \mathcal{U} \to \mathbb{R}^n$ is a smooth vector field. Given an initial data $u(0) = u_0 \in \mathcal{U}$, there is a unique maximal solution $(I = [0, T); u)$. We recall that either $T = +\infty$, or the solution escapes every compact subset of $\mathcal{U}$ as $t \to T^-$.

Points $\bar{u} \in \mathcal{U}$ such that $F(\bar{u}) = 0$ are called *critical points*, or *zeroes*, or *rest points*, or *equilibria*. They yield globally defined, constant solutions of (1.1) $u(t) \equiv \bar{u}$. We shall be interested in their stability properties.

Definition 1.0.1 *Assume that $F(\bar{u}) = 0$. We say that $\bar{u}$ is* stable *if for every neighbourhood $\mathcal{V}$ of $\bar{u}$, there exists another one $\mathcal{V}_1 \subset \mathcal{V}$, such that every initial data in $\mathcal{V}_1$ yields a globally defined solution taking its values in $\mathcal{V}$.*

We say that $\bar{u}$ is asymptotically stable *if there exists a neighbourhood $\mathcal{V}_2$ of $\bar{u}$ such that, for every initial data in $\mathcal{V}_2$, the solution of (1.1) is globally defined and tends to $\bar{u}$ as $t \to +\infty$.*

Exercises. – Asymptotic stability implies stability, – Rewrite the definitions with epsilons.

1.1 Lyapunov functions

In this course, we are concerned only with the case where $T = +\infty$, and thus are interested in practical criteria which ensure that solutions are globally defined. A convenient tool is given by *Lyapunov functions.*

Definition 1.1.1 *Let $L : \mathcal{U} \to \mathbb{R}$ be a smooth function. We say that L is a* Lyapunov function *for (1.1) if $R := -\mathrm{d}L \cdot F$ is a non-negative function. Then R is called the* dissipation rate *of L.*

For a Lyapunov function L, there holds

$$\frac{d}{dt} L(u) = -R(u) \leq 0, \tag{1.2}$$

for every solution of (1.1).

Definition 1.1.2 *A function $L:\mathcal{U}\to\mathbb{R}$ is said* coercive *if*

$$L^{-1}(-\infty,c):=\{v\in\mathcal{U}\,;\,L(v)\le c\}$$

is a compact set for every $c\in\mathbb{R}$.

Remark. *Coercive* is a common terminology in analysis. But topologists use to speak of *proper* functions.

When $\mathcal{U}=\mathbb{R}^n$, a function L is coercive if and only if it tends to $+\infty$ when $\|v\|\to+\infty$. If $\mathcal{U}$ is bounded, L is coercive iff $L(v)\to+\infty$ when $d(v;\partial\mathcal{U})\to 0$.

Let L be a Lyapunov function of (1.1) and u be a solution. Then $L(u(t))\le c:=L(u_0)$ for all $t\ge 0$. If moreover L is coercive, this shows that the trajectory remains in a compact set, and therefore the solution is globally defined: $T=+\infty$.

1.1.1 Linear example

Let us consider the case $F(u)=Au$, with $A\in\mathbf{M}_n(\mathbb{R})$ and $\mathcal{U}=\mathbb{R}^n$. Then the solution of the Cauchy problem is given by

$$u(t)=e^{tA}u_0.$$

Let us assume moreover that the eigenvalues of A have negative real parts. Then the eigenvalues of e^A (the exponentials of those of A) have modulus less than one. Consequently, the spectral radius satisfies

$$\rho(e^A)<1.$$

From the Householder Theorem (see [46], Theorem 4.2.1), there exists an induced matrix norm over $\mathbf{M}_n(\mathbb{C})$, such that

$$(0<)\,\theta:=\|e^A\|<1.$$

Let $t\ge 0$ be given, with integer part m. We then have

$$\|e^{tA}\|=\|\left(e^A\right)^m e^{(t-m)A}\|\le\theta^m e^{\|A\|}=e^{m\log\theta+\|A\|}\le Ce^{t\log\theta},$$

with $C:=\theta^{-1}e^{\|A\|}$, since $\log\theta<0$. We deduce that every solution decays exponentially fast to zero, as $t\to+\infty$:

$$\|u(t)\|\le Ce^{-\omega t}\|u_0\|,\qquad\omega:=\log\frac{1}{\theta}>0.$$

The estimates above yield the following deeper result, known as *Lyapunov Theorem.*

Theorem 1.1.1 *Let $A\in\mathbf{M}_n(\mathbb{R})$ have all its eigenvalues of negative real part. Then there exists a quadratic form $q(v)=v^TSv$ with the property that the solutions of the linear ODE satisfy*

$$\frac{d}{dt}q(u)\le-\epsilon q(u),\tag{1.3}$$

for some positive ϵ, independent of u.

In other words, q is a Lyapunov function, and the Gronwall inequality yields

$$q(u(t)) \leq e^{-\epsilon t} q(u_0).$$

Proof
Let K be a positive definite real symmetric matrix. We define

$$S := \int_0^{+\infty} e^{tA^T} K e^{tA} dt,$$

a convergent integral since e^{tA} decays exponentially fast.
An elementary calculation yields

$$A^T S + SA = \int_0^{+\infty} \frac{d}{dt} \left(e^{tA^T} K e^{tA} \right) dt = -K.$$

Every solution of the linear ODE thus satisfies

$$\frac{d}{dt} q(u) = u^T (A^T S + SA) u = -u^T K u.$$

At last, since K is positive definite, there exists a positive ϵ such that $K \geq \epsilon S$.

■

Remark. In the theorem of Householder, θ can be chosen arbitrarily close to $\rho(e^A)$. Therefore ω can be chosen arbitrarily close to the opposite of the larger real part of the eigenvalues of A, to the price of a constant C_ω which could be large in presence of non-trivial Jordan blocks.

Exercise. Theorem 1.1.1 can be recast as a theorem about a matrix equation, the *Lyapunov equation.* Do it.

1.2 The omega-limit set

We consider in this paragraph trajectories of (1.1) which remain in some compact subset of $\mathcal{U}$. This happens for instance if the system admits a coercive Lyapunov function. In this situation, we shall denote by ϕ^s the map

$$\phi^s : u(0) \mapsto u(s).$$

The collection $(\phi^s)_{s \geq 0}$ is a semi-group of smooth maps from $\mathcal{U}$ into itself. Each ϕ^s is injective and a diffeomorphism onto its range. In particular, $\phi^s(\mathcal{U})$ is open.

Definition 1.2.1 *Given an initial data* u_0*, the* omega-limit set *of the associated solution* $(u(t))_{t \geq 0}$ *is defined as*

$$\Omega(u_0) := \bigcap_{T \geq 0} \overline{\{u(t)\,;\, t \geq T\}}.$$

Since each subset

$$K_T := \overline{\{u(t)\,;\, t \geq T\}}$$

is non-void and compact, and since $T \mapsto K_T$ is non-increasing, the intersection is non-void. An alternate definition is that $\Omega(u_0)$ is the set of limits of all converging subsequences $(u(t_m))_{t_m \to +\infty}$ (because of precompactness, there are plenty of such subsequences).

If $u(t)$ is converging as $t \to +\infty$, then $\Omega(u_0)$ reduces to the limit. Conversely, if $\Omega(u_0)$ is a singleton $\{\bar{u}\}$, then every sequence $(u(t_m))_{t_m \to +\infty}$ contains a subsequence converging towards $\bar{u}$; therefore $u(t)$ converges and its limit is $\bar{u}$. We thus have

Proposition 1.2.1 *Let $(u(t))_{t \geq 0}$ be a relatively compact (in $\mathcal{U}$) trajectory of (1.1).*

Its omega-limit set $\Omega(u_0)$ is non-void. Moreover, $u(t)$ converges as $t \to +\infty$ if, and only if, $\Omega(u_0)$ is a singleton. In this case, the limit is the unique element of $\Omega(u_0)$.

The following result is more involved.

Proposition 1.2.2 *The omega-limit set $\Omega(u_0)$ is a connected subset.*

Proof

Let A and B be two disjoint open subsets, such that $\Omega(u_0) \subset A \cup B$. Assume that both intersections $\Omega(u_0) \cap A$ and $\Omega(u_0) \cap B$ are non-void. Then for every T, the set $\{u(t)\,;\, t \geq T\}$ meet both A and B ; since it is connected, it may not be contained in $A \cup B$. Therefore there exists a $t \geq T$ such that $u(t) \notin A \cup B$. This allows us to build a sequence $(u(t_m))_{t_m \to +\infty}$ with $u(t_m) \notin A \cup B$ for all m. Since $A \cup B$ is open, a cluster point of this sequence is not in $A \cup B$. But it belongs to $\Omega(u_0)$. Contradiction.

■

The last statement might be the most important.

Proposition 1.2.3 *The omega-limit set $\Omega(u_0)$ is invariant under the flow of (1.1), both forward and backward:*

1. *If $s \geq 0$ and $a \in \Omega(u_0)$, then $\phi^s(a) \in \Omega(u_0)$.*
2. *If $s \geq 0$ and $a \in \Omega(u_0)$, then $a \in \phi^s(\Omega(u_0))$.*

In particular, ϕ^s is a homeomorphism of $\Omega(u_0)$ onto itself.

Proof

Let s and a be as above, with

$$a = \lim_{t_m \to +\infty} u(t_m).$$

Then

$$\phi^s(a) = \lim u(t_m + s),$$

by continuity of ϕ^s. This proves the first point.

On the other hand, the sequence $(u(t_m - s))_{m \geq 0}$ is well-defined for m large enough and is relatively compact in $\mathcal{U}$. Thus it admits a cluster point b. The latter is an element of $\Omega(u_0)$ and is the limit of some subsequence $\big(u(t_{\chi(k)} - s)\big)_{k \to +\infty}$. The first point of the statement tells then that $a = \phi^s(b)$.

■

Exercise. What is the α-limit set of a trajectory ? Write appropriate assumptions and statements satisfied by this set. Hint: *"I am the α and the ω"*.

1.3 The Lasalle invariance principle

We assume in this section that the non-linear ODE (1.1) is endowed with a coercive Lyapunov function L with dissipation rate R. Then every trajectory remains in some compact subset of $\mathcal{U}$ and we can use the material of the previous section. We notice in addition that because of coercivity and continuity, L is bounded by below.

We first observe that since $t \mapsto L(u(t))$ is non-increasing, it must have a limit ℓ as $t \to +\infty$. Since L is bounded by below, ℓ is finite.

For every sequence $(u(t_m))_{t_m \to +\infty}$, one has $L(u(t_m)) \to \ell$. By continuity of L, this implies that $\Omega(u_0)$ is contained in the level set $L^{-1}(\ell)$. This is the first part of Lasalle's principle.

The second one follows from the first one and Proposition 1.2.3. If $a \in \Omega(u_0)$ and $v(t)$ is the solution associated to the initial data a, then $v(t) \in \Omega(u_0)$ and therefore $L(v(t)) \equiv \ell$. We thus have

$$0 = \left.\frac{d}{dt}\right|_{t=0} L(v) = -R(a).$$

Finally, we can state

Theorem 1.3.1 *Let L be a coercive Lyapunov function of (1.1), with dissipation rate R. Then the omega-limit set of any trajectory satisfies the following:*

1. *the restriction of L to $\Omega(u_0)$ is a constant.*
2. *$\Omega(u_0) \subset \{w \in \mathcal{U}\,;\, R(w) = 0\}$,*

Example. Let us consider the system ($n = 2$)

$$\dot{u}_1 = u_2, \qquad \dot{u}_2 = -u_1 - u_2^3. \tag{1.4}$$

One easily verifies that $L(u) := u_1^2 + u_2^2$ is a Lyapunov function, coercive over $\mathbb{R}^2$, with $R(u) = 2u_2^4$.

Let $u^0 \in \mathbb{R}^2$ be an initial data. By the second part of Lasalle's principle, its omega-limit set is contained in the horizontal axis of the plane.

Let $a \in \Omega(u_0)$ be given and v the associated solution. We know that $v_2 \equiv 0$. Therefore $\dot{v}_2 \equiv 0$, which gives $v_1 + v_2^3 \equiv 0$, that is $v_1 \equiv 0$. In particular, $a_1 = 0$. Finally, $\Omega(u^0) = \{0\}$, meaning that

$$\lim_{t \to +\infty} u(t) = 0.$$

This is the *nonlinear stability* of the origin under the system (1.4).

Remark. In the example above, we did not conclude about the rate of convergence of $u(t)$ as $t \to +\infty$. This cannot be solved by the use of the Lasalle principle, which is a purely qualitative statement.

Notice that the example suggests a generalization of Theorem 1.3.1 in the following way, provided L and F are smooth enough. If a and v are as above, we know that $R(v(t)) \equiv 0$. Differentiating at $t = 0$, we obtain

$$(\mathrm{d}R \cdot F)(a) = 0. \tag{1.5}$$

An obvious iteration of the argument gives the following. Let us define recursively

$$R_0 := R, \qquad R_{k+1} := \mathrm{d}R_k \cdot F, \quad (k \in \mathbb{N}),$$

then $\Omega(u_0)$ is contained in the zero set of R_k for every $k \in \mathbb{N}$.

Mind however that $\mathrm{d}R$ vanishes along $\{R = 0\}$, this because R is non-negative everywhere. Thus R_1 does not help in the determination of $\Omega(u^0)$. It is however possible that the vanishing of subsequent functions $R_2, R_3, \ldots$ shrinks the allowable elements for $\Omega(u^0)$. An other possible technique is to write $\mathrm{d}\sqrt{R} \cdot F = 0$, since $\sqrt{R}$ is a smooth function. In the example above, we actually considered $\mathrm{d}(R^{1/4}) \cdot F = 0$.

1.4 Linear systems

Let

$$\dot{u} = Au \tag{1.6}$$

be a linear system in $\mathbb{R}^n$. The matrix $A \in \mathbf{M}_n(\mathbb{R})$ can be decomposed on a Jordan basis. This procedure has the drawback that some of the base vectors might not be real. However, eigen-elements remain unchanged after complex conjugation. For instance, the conjugate of an eigenvalue is an other one, with same multiplicities. Since we prefer to work within $\mathbb{R}^n$, we can group the complex eigenvalues by conjugate pairs. Doing this, we can decompose $\mathbb{R}^n$ as the direct sum of invariant subspaces E_α:

$$AE_\alpha \subset E_\alpha,$$

with equality if 0 is not an eigenvalue of A over E_α.

Even this level description is too much detailed for our purpose. It will be sufficient to group the eigenvalues of A (the *spectrum* $\sigma(A)$) in three disjoint parts $\sigma_-(A)$, $\sigma_0(A)$ and $\sigma_+(A)$, according to the sign of their real part. Notice that these sets are invariant under conjugacy. These subsets of the spectrum are called *stable, neutral* and *unstable* respectively. The ambient real space splits in a unique way into

$$\mathbb{R}^n = E_- \oplus E_0 \oplus E_+,$$

where each of $E_{\pm,0}$ is A-invariant, and the spectrum of the restriction of A to E_s is precisely $\sigma_s(A)$. These subspaces are called the *stable, neutral* and *unstable* invariant subspaces.

Each of $E_{\pm,0}$ is invariant under the linear ODE. By superposition, we thus have

$$x(0) = x_-^0 + x_0^0 + x_+^0, \qquad x(t) = x_-(t) + x_0(t) + x_+(t), \qquad x_s(t) = \exp\left(tA|_{E_s}\right) x_s^0.$$

Let us examine the behaviour of each term $x_s(t)$ as t increases.

Neutral part. The exponential of $tA|_{E_0}$ is a mixing of trigonometric functions and polynomials. Therefore $x_0(t)$ remains bounded by some polynomial of t, and does not tend to zero as $t \to +\infty$, unless $x_s(\cdot) \equiv 0$.

Stable part. We have seen above that $\exp\left(tA|_{E_-}\right)$ decays exponentially fast. Whence a similar decay of $x_-(t)$ as $t \to +\infty$.

Unstable part. The same result applies to $-A|_{E_+}$:

$$\left(\exp(tA|_{E_+})\right)^{-1} = \exp\left(-tA|_{E_+}\right)$$

decays exponentially fast. This shows that $\|x_+(t)\|$ grows exponentially fast, unless $x_+(\cdot) \equiv 0$.

Gathering these results, we see that the origin is asymptotically stable if, and only if, $\sigma_0(A) = \sigma_+(A) = \emptyset$. In other words, the eigenvalues of A have a negative real parts. They need not be semi-simple.

Projections. The projection P_s over E_s, parallel to the other invariant subspaces, are given by a Cauchy formula. Let Γ be a Jordan curve enclosing $\sigma_s(A)$ and no other eigenvalue of A, being oriented in the trigonometric sense. Then we have (Dunford–Taylor formula)

$$P_s = \frac{1}{2i\pi} \int_\Gamma (zI_n - A)^{-1} dz.$$

Definition 1.4.1 *A matrix $A \in \mathbf{M}_n(\mathbb{R})$ is called* hyperbolic *if $\sigma_0(A) = \emptyset$.*

Exercises.

1. For the origin to be stable, we need only that on the one hand $\sigma_+(A) = \emptyset$ and the eigenvalues on the imaginary axis are semi-simple.

2. Show that an alternative definition of E_- is: The set of vectors $\mathbf{v}$ such that $e^{tA}\mathbf{v} \to 0$ as $t \to +\infty$. Write an analogous definition for E_+.

3. Show that an alternative definition of E_- is: The set of vectors $\mathbf{v}$ such that $e^{tA}\mathbf{v} \to 0$ *exponentially fast* as $t \to +\infty$.

4. Show that the maps $A \mapsto \dim E_\pm$ are continuous at every hyperbolic matrix. Within the set of hyperbolic matrices of given stability indices $\dim E_\pm$, show that $A \mapsto E_\pm$ is continuous, with values in the appropriate Grassmannian manifolds.

1.5 Linearization at critical points; the hyperbolic case

A basic method to analyse the stability properties of a zero of F is to linearize the ODE at it. Assume that $F(\bar{u}) = 0$. We define $A := \mathrm{d}F(\bar{u}) \in \mathbf{M}_n(\mathbb{R})$.

Let us begin with the simple case where all the eigenvalues belong to the same side of the imaginary axis. A basic result is

Proposition 1.5.1 *Assume that the real eigenvalues of A have negative real parts. Then $\bar{u}$ is asymptotically stable.*

Hints for the proof: Apply Lyapunov Theorem to A, write $F(u) = A(u - \bar{u}) + G(u - \bar{u})$ where $G(v) = O(\|v\|^2)$. Finally apply Gronwall inequality.

Dropping the asumption above, and using only the fact that 0 is not an eigenvalue, we deduce form the implicit function theorem that $\bar{u}$ is an isolated critical point.

A deeper result is

Theorem 1.5.1 (Hartman–Grobman.) *We assume that A is hyperbolic. Then there exists a homeomorphism h from a neighbourhood $\mathcal{V}$ of $\bar{u}$ to a ball $B(0;\rho)$, which maps local orbits of (1.1) to local orbits of the linearized ODE (1.6). The homeomorphism preserves the parametrization by time.*

The reader is warned however that h is not always a diffeomorphism.

The Hartman–Grobman theorem provides us with invariant manifolds

$$W^s_{\mathrm{loc}}(\bar{u}) := h^{-1}(E_- \cap B), \qquad W^u_{\mathrm{loc}}(\bar{u}) := h^{-1}(E_+ \cap B),$$

the so-called local *stable* and *unstable* manifolds of $\bar{u}$. By *invariant*, we mean that they are locally invariant under the flow of (1.1): if a belongs to some invariant manifold, then there exists an $\epsilon > 0$ such that $\phi^s(a)$ remains in this manifold for every $s \in (-\epsilon, \epsilon)$.

The stable and unstable manifolds can be defined alternately as

$$\begin{aligned} W^s_{\mathrm{loc}}(\bar{u}) &= \{u_0 \in \mathcal{V}\,;\, u(t) \text{ stays in } \mathcal{V} \text{ for all } t \geq 0, \text{ and } u(t) \to \bar{u} \text{ as } t \to +\infty\}, \\ W^u_{\mathrm{loc}}(\bar{u}) &= \{u_0 \in \mathcal{V}\,;\, u(t) \text{ stays in } \mathcal{V} \text{ for all } t \leq 0, \text{ and } u(t) \to \bar{u} \text{ as } t \to -\infty\}. \end{aligned}$$

A variant of the proof of Proposition 1.5.1 yields the fact that the convergences in the second definition above are exponential in time. This can also be deduced from

Theorem 1.5.2 (Stable manifold) *The local invariant manifolds $W^s_{\mathrm{loc}}(\bar{u})$ and $W^u_{\mathrm{loc}}(\bar{u})$ are as smooth as F is. They are tangent to E_- and E_+, respectively. In particular, they have the same dimensions as these subspaces.*

For instance, consider stable trajectories. Using a chart, we may choose coordinates along $W^s_{\mathrm{loc}}(\bar{u})$. Say that E_- is transversal to $\{0\} \times \mathbb{R}^{n-p}$. Then $z := (u_1, \dots, u_p)$ form a coordinate system over $W^s_{\mathrm{loc}}(\bar{u})$, and the remaining coordinates become smooth functions of them: if $u \in \mathcal{V}$, then

$$(u \in W^s_{\mathrm{loc}}(\bar{u})) \Longleftrightarrow ((u_{p+1}, \dots, u_n) = w(z)).$$

The dynamics along $W^s_{\rm loc}(\bar u)$ is thus fully described by the reduced ODE

$$\dot z = f(z, w(z)) =: g(z), \qquad f := (F_1, \dots, F_p)^T. \tag{1.7}$$

One computes easily that $dg(\bar z)$ is conjugated to $A|_{E_-}$ (exercise), thus have eigenvalues of negative real parts. Then Proposition 1.5.1 applies and yields exponential convergence of trajectories over $W^s_{\rm loc}(\bar u)$.

Global dynamics. We shall not give a detailed account of global dynamics here. The objects to be studied are the stable and unstable 'manifolds':

$$\begin{array}{rcl} W^s(\bar u) & = & \{u_0 \in \mathcal{V}\,;\, u(t) \to \bar u \text{ as } t \to +\infty\}, \\ W^u(\bar u) & = & \{u_0 \in \mathcal{V}\,;\, u(t) \to \bar u \text{ as } t \to -\infty\}. \end{array}$$

On the one hand, these sets might not be closed. For instance, let us assume that (1.1) admits another saddle point u^* and that there is a heteroclinic orbit from u^* to $\bar u$. Then this orbit is contained in $W^s(\bar u)$, but its extremity u^* is not.

On the other hand, these sets might not be smooth. Let us assume for instance that (1.1) admits a homoclinic orbit to $\bar u$. Then $W^s(\bar u)$ contains this orbit, which means that in a neighbourhood of $\bar u$, $W^s(\bar u)$ is the union of $W^s_{\rm loc}(\bar u)$ and a submanifold of $W^u_{\rm loc}(\bar u)$.

In conclusion, the word 'manifold' is a confusing terminology when we consider the global dynamics. Pathologies like those described above arise as soon as $n \geq 2$. But once $n \geq 3$, chaotic dynamics may happen. For instance, a *Smale's horseshoe* can take place, for which $W^s(\bar u)$ folds closer and closer to $\bar u$, infinitely many times.

1.6 The center manifold theorem and applications

The non-hyperbolic case, that is when A admits an eigenvalue on the imaginary axis, is more involved. In particular, uniqueness is partly lost. Once again, we restrict to the local dynamics, in a suitably small neighbourhood $\mathcal{V}$ of $\bar u$.

The correct way to define stable and unstable local invariant manifolds is to exploit exponential convergence:

$$\begin{array}{rcl} W^s_{\rm loc}(\bar u) & = & \{u_0 \in \mathcal{V}\,;\, u(t) \text{ stays in } \mathcal{V} \text{ for all } t \geq 0, \\ & & \text{and } u(t) \to \bar u \text{ as } t \to +\infty, \text{ exponentially fast}\}, \\ W^u_{\rm loc}(\bar u) & = & \{u_0 \in \mathcal{V}\,;\, u(t) \text{ stays in } \mathcal{V} \text{ for all } t \leq 0, \\ & & \text{and } u(t) \to \bar u \text{ as } t \to -\infty, \text{ exponentially fast}\}. \end{array}$$

These sets turn out to be smooth manifolds, tangent respectively to E_- and E_+. They do not exhaust in general the stable or unstable trajectories: there often exist other converging solutions $u(t) \to \bar u$ (either $t \to +\infty$ or $t \to -\infty$), but these converge slowlier, typically like a negative power of the time. For this reason, $W^s_{\rm loc}(\bar u)$ is called the *strongly stable* invariant manifold of (1.1) at $\bar u$, while $W^u_{\rm loc}(\bar u)$ is the *strongly unstable* invariant manifold.

What can happen too, since we do not exclude that 0 be an eigenvalue of A, is that the critical point $\bar{u}$ is not isolated. In this particular case, it is likely that there are special orbits like heteroclinic or homoclinic ones. If A admits a pair of imaginary eigenvalues, one can even encounter periodic orbits in a vicinity of $\bar{u}$. Clearly, none of these can meet $W^s_{\rm loc}(\bar{u})$ or $W^u_{\rm loc}(\bar{u})$. This suggests that there is a third special invariant subset besides the strongly stable and unstable manifolds. This third one is the *center manifold* of (1.1) at $\bar{u}$, denoted by $W^c_{\rm loc}(\bar{u})$. Once again, it is only locally defined. However, contrary to $W^s_{\rm loc}(\bar{u})$ or $W^u_{\rm loc}(\bar{u})$, it is not necessarily unique (it turns out that it is usually not). Its proper definition is

- $W^c_{\rm loc}(\bar{u})$ is a smooth manifold, tangent at $\bar{u}$ to E_0, locally invariant under the flow of (1.1),
- there exists a neighbourhood $\mathcal{V}$ of $\bar{u}$ such that every globally defined trajectory of (1.1) that remains forever in $\mathcal{V}$ is actually contained in $W^c_{\rm loc}(\bar{u})$.

By definition, $W^c_{\rm loc}(\bar{u})$ contains every fixed point, homoclinic orbit, heteroclinic one, periodic one, that are close enough to $\bar{u}$. In other words, the center manifold contains the most interesting part of the dynamics.

What is counter-intuitive, at least at first glance, is that $W^c_{\rm loc}(\bar{u})$ is generally non-unique. However, this fact must be tempered by the following observations:

- By definition, certain special points do belong to the intersection of all the local center manifolds. The set of such points can be big enough so as to form a set of non-void interior, relatively to $W^c_{\rm loc}(\bar{u})$.
- Two center manifolds at $\bar{u}$ are not only tangent (since they are tangent to the same subspace E_0), they are actually tangent at every order. For instance, if F is $\mathcal{C}^\infty$, then the manifolds are $\mathcal{C}^\infty$-tangent at $\bar{u}$. This is consistent with the previous observation.

Example. Let us consider the ODE ($n = 2$)

$$\dot{x} = x + f(x,y), \qquad \dot{y} = g(x,y),$$

with $f, g = O(x^2 + y^2)$ at the origin. Thus

$$u = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad F(u) = \begin{pmatrix} x + f(x,y) \\ g(x,y) \end{pmatrix}.$$

Then

$$\mathrm{d}F(0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

We deduce that $W^s_{\rm loc}(0)$ is trivial, while both $W^{u,c}_{\rm loc}(0)$ are one-dimensional. The center manifold has an equation $x = h(y)$ with $h(0) = h'(0) = 0$. To compute an expansion of h, we write the invariance of W^c: for every trajectory along it, we have $\dot{x} = h'(y)\dot{y}$, that is

$$h(y) + f(h(y), y) = h'(y) g(h(y), y). \tag{1.8}$$

For instance, the second order terms give

$$h''(0) = -\frac{\partial^2 f}{\partial y^2}(0).$$

> As an exercise, compute $h'''(0)$ in terms of the derivatives of f and g at the origin. Notice that (1.8) cannot be written as an explicit differential equation $h' = N(h, y)$. This explains why its solution is non unique, although its jet at 0 is unique.

The dynamics along $W^s(0)$ is obtained by replacing x by $h(y)$ in the second equation of the system:

$$\dot{y} = g(h(y), y) =: G(y).$$

Since the jet of h can be fully determined, that of G can be too. For instance,

$$G(0) = G'(0) = 0, \qquad G''(0) = \frac{\partial^2 g}{\partial y^2}(0).$$

If the latter is non-zero, then 0 is neither an attractor, nor a repellor, along the center manifold.

In general, there is an extensive theory for the computation of the jet and the determination of the dynamics reduced to the center manifold for an arbitrary system. It relies upon the notion of *normal form*, which means reducing the system to a simple one by means of appropriate change of coordinates in $\mathbb{R}^n$.

Exercises.

1. Let $n = 2$ and consider the system

$$\dot{x} = x^2, \qquad \dot{y} = -y.$$

   Calculate $E_{\pm,0}$. Compute in close form every trajectory. Identify the stable/unstable manifolds. What are the center manifolds (there are many of them) ?

2. Let $n = 3$ and consider the system

$$\dot{x} = y, \qquad \dot{y} = -x, \qquad \dot{z} = -z.$$

   Again, calculate $E_{\pm,0}$ and compute the trajectories in close form. Identify the stable/unstable manifolds. Show that the center manifold is unique.

Chapter 2

Various dissipative mechanisms in PDEs

Let us start with a system of conservation laws

$$\partial_t u + \mathrm{div}_x q = 0, \tag{2.1}$$

with $x \in \mathbb{R}^d$, $(x,t) \mapsto u(x,t) \in \mathbb{R}^n$ and $(x,t) \mapsto q(x,t) \in \mathbf{M}_{n\times d}(\mathbb{R})$. The terminology calls u the *conserved quantity* and q the *flux*. The system modelizes some natural phenomenon, like gas dynamics, electromagnetism, MHD, electrophoresis, traffic flow,... (endless list). In itself, (2.1) is useless as long as one doesn't have a rule to determine q, knowing the past evolution of u. We thus need an *equation of state*, or something similar, to close the system.

2.1 Quasi-linear first-order systems

Our building block is a system without dissipation. Its simplest form, which we use everyday, comes with a local, algebraic, equation of state

$$q(x,t) := f(u(x,t)), \tag{2.2}$$

$u \mapsto f$ being a given smooth tensor field that characterizes the physics (or chemistry or whatever) under consideration. This yields a *quasi-linear first-order system*:

$$\partial_t u + \mathrm{div} f(u) = 0. \tag{2.3}$$

In coordinates, (2.3) rewrites

$$\frac{\partial u_j}{\partial t} + \sum_{\alpha=1}^{d} \frac{\partial}{\partial x_\alpha} f_j^\alpha(u) = 0, \qquad j = 1, \dots, n.$$

Let us recall the classical example of gas dynamics, governed by the Euler equations:

$$u =: \begin{pmatrix} \rho \\ \rho v \\ \frac{1}{2}\rho|v|^2 + \rho e \end{pmatrix}, \qquad f(u) := \begin{pmatrix} \rho v^T \\ \rho v \otimes v + p(\rho, e) I_d \\ \left(\frac{1}{2}\rho|v|^2 + \rho e + p(\rho, e)\right) v^T \end{pmatrix}. \tag{2.4}$$

With such a closure, (2.1) is a quasi-linear system of first order PDEs, with $n = d + 2$. Typically, $d = 3$, but the analysis of special situations can justify the choice of $d = 1$ or 2.

Quasilinear first-order systems are *reversible*, meaning that the change of variables $(x, t) \mapsto (-x, -t)$ does not affect the equations, thus operates over the set of solutions. This is in contradiction with the physical experiments, which give an evidence of irreversibility, at least at the level of discontinuous solutions. We thus needs a more accurate description of the physics, which might be a more complicated determination of q.

The general problem is to solve (2.3) under the initial condition

$$u(x, 0) = u^0(x), \tag{2.5}$$

where u^0 is a given field. This is the *Cauchy problem*.

2.2 'Viscous' models

For instance, if we admit that q may depend linearly on the first derivatives, we have

$$q(x, t) := f(u(x, t)) - B(u(x, t))\nabla_x u(x, t). \tag{2.6}$$

In the exemple from gas dynamics, this amounts to take in account Newtonian viscosity and heat diffusion (Fourier law):

$$B(u)\nabla u := \begin{pmatrix} 0 \\ \mathcal{T} := \lambda(\rho, e)(\nabla v + \nabla v)^T + \mu(\rho, e)(\mathrm{div} v)I_d \\ v^T\mathcal{T} + \kappa(\rho, e)\nabla\theta \end{pmatrix} \tag{2.7}$$

The special form taken by these additional terms, called diffusion terms, follows from general principles:

- frame indifference,
- conservation of angular momentum (this implies the symmetry of $\mathcal{T}$),
- second principle of thermodynamics; this specifies the state function $(\rho, e) \mapsto \theta(\rho, e)$, called *temperature*.

Of course, depending on the context, one may wish to retain only one source of diffusion, either the viscosity or the heat diffusion. The properties of the system may vary accordingly.

Models with a response given by (2.6) are called *viscous* models, by extension of the concept of fluid viscosity. Viscous models might be the oldest, perhaps the most popular, dissipative models in systems of conservation laws. They occur in so many aspects of continuum physics that we cannot give a complete list of applications. Let us at least mention viscoelasticity, continuum models of traffic flows and magnetohydrodynamics.

One of the important questions we face is the vanishing diffusion limit, in which B contains a small parameter, say

$$q(x, t) := f(u(x, t)) - \epsilon B(u(x, t))\nabla_x u(x, t)$$

instead of (2.6). Then we examine the limit of the solution of the Cauchy problem as $\epsilon \to 0+$. An other important issue is the time asymptotics, where we want to describe $u(x,t)$ as $t \to +\infty$ at fixed ϵ, say $\epsilon = 1$. Both problems are intimately related, because of the change of scale

$$(x,t) \mapsto (x/\epsilon, t/\epsilon).$$

There are however slight differences between them, due to the fact that the rescaling changes also the initial data.

We notice that in some regimes, the dependence of q upon ϵ may be more complicated, so that the limit as $\epsilon \to +\infty$ makes sense. This is the case when the Newtonnian viscosity is negligible but the heat diffusion coefficient is large. Then we formally obtain the *isothermal* gas dynamics.

2.3 Relaxation models

Viscosity is not the only way to improve the description of the $u \mapsto q$ closure. In some circumstances, a relaxation process can be introduced. For instance, this is the case when one describes gas flows off thermodynamical equilibrium. At equilibrium, say in a uniform steady flow, the pressure p is given as the function $\mathbf{p}(\rho, e)$ considered above, according to (2.2). But in a rapid evolution process, with large density gradients, one should think that the actual pressure has only a tendency towards equilibrium. Namely, p obey some differential equation, of which the unique stable equilibrium is $p = \mathbf{p}(\rho, e)$. The simplest equation is an ODE,

$$\partial_t p = \frac{1}{\tau}(\mathbf{p}(\rho, e) - p), \tag{2.8}$$

where $\tau > 0$ is a relaxation time. The system becomes larger, with an unknown $U := (u,p)^T$. It is not any more a system of conservation laws, but a system of *balance laws*, with $n = d+3$ instead of $d+2$. However, it still contains $d+2$ conservation laws (mass, momentum and energy), but they do not form a closed system. In practice, we are interested in the singular limit as $\tau \to 0+$, as well as in the time asymptotics, both being intimately related, as in viscous models.

It turns out however that the coupling of the Euler equations with the ODE (2.8) generates strong instabilities. We shall explain this fact later. But for the moment, this suggests to replace (2.8) by a PDE, say of the form

$$\partial_t p + a^2 \mathrm{div} v = \frac{1}{\tau}(\mathbf{p}(\rho, e) - p), \tag{2.9}$$

$a^2 > 0$ being a given constant. The fact that this constant be positive is necessary for the well-posedness of the Cauchy problem. We shall find later on that the stability as $t \to +\infty$, or as $\tau \to 0+$ requires a lower bound on a, called the *sub-characteristic condition.*

Besides physically motivated relaxation models, there has been a great interest in an abstract one designed by S. Jin and Z. Xin [22] for the study of inviscid systems of conservation laws.

Given a system $\partial_t u + \partial_x f(u) = 0$ in one space-dimension, a relaxation time τ and a positive number a, they build the expanded system

$$\partial_t u + \partial_x v = 0, \tag{2.10}$$

$$\partial_t v + a^2 \partial_x u = \frac{1}{\tau}(f(u) - v). \tag{2.11}$$

Several convergence results, as $\tau \to 0+$, were obtained for various relaxations of a scalar conservation law (when $u(x,t)$ is scalar). For systems, a (not that much) general convergence result was establish in [44] for the Jin–Xin model. See also [52] for another model in elasticity. Multidimensional versions of the Jin–Xin model are available. As usual, their study is harder than in one space variable.

Exercise. Let us consider the Broadwell system, a kinetic model with a discrete set of velocities ($n = 3$, $d = 1$),

$$\partial_t u + \partial_x u = \frac{1}{\tau}(w^2 - uv), \qquad \partial_t v - \partial_x v = \frac{1}{\tau}(w^2 - uv), \qquad \partial_t w = \frac{1}{\tau}(uv - w^2).$$

Check that there are two conservation laws in this system. Proceed on the formal limit as $\tau \to 0+$. Show that the limit system has the form

$$\partial_t \rho + \partial_x m = 0, \qquad \partial_t m + \partial_x h(\rho, m) = 0,$$

but that it is not a case of Euler equations:

$$h(\rho, m) - \frac{m^2}{\rho}$$

is not a function of ρ alone.

2.4 Hyperbolic-elliptic coupling

For both physical and mathematical reasons, the relevance of heat diffusion and viscosity has been discussed. At least in the radiative regime (say within stars), the heat flux, given above as $Q = -\kappa \nabla \theta$, must be determined instead through an elliptic equation, say

$$-\epsilon \Delta Q + Q + \kappa \nabla \theta = 0, \tag{2.12}$$

with $\epsilon > 0$ a small number. This case, called the Hamer models, is described in [53]. Its scalar, one-dimensional counterpart

$$\partial_t u + \partial_x \left(\frac{1}{2} u^2 \right) = \partial_x q, \qquad -\epsilon \partial_x^2 q + q = \partial_x u \tag{2.13}$$

has been intensively studied by Kawashima, Nikkuni and Nishibata [24, 25, 26, 27], Schochet and Tadmor [41], and by Lattanzio & Marcati [30].

Once again, the limit $\epsilon \to 0+$ is of major interest. But since the physics has two ratio ϵ and κ, several other regimes can be considered. The analysis of shock profiles is non-trivial because one has to deal with systems of ODEs that cannot be put in explicit form. In particular, such profiles are never smooth and may even be discontinuous, as remarked by Kawashima and Nishibata in the scalar case. In the case of systems, shock profiles have been analyzed recently by Coulombel & coll. [6] and by Lattanzio & coll. [31].

2.5 Kinetic model

Although we shall not consider them in the sequel, one should not ignore a more involved class of dissipative models, namely the kinetic ones. Typically, the physics is represented by a *density* $f(x,t;\xi)$, where ξ is the *kinetic variable*. When ξ plays the role of a velocity, it is denoted by v.

The most interesting and physically motivated model is of course the Boltzman equation for gaz dynamics:

$$(\partial_t + v \cdot \nabla_x) f = \frac{1}{\kappa} Q(f,f),$$

where $f = f(x,t,v) : \mathbb{R}^7 \to \mathbb{R}^+$ denotes the density of particles at position x, time t and velocity v. The quadratic functional Q is a complicated integral operator and $0 < \kappa << 1$ is the mean free path of the particles. The density, momentum and total energy per unit volume are recovered through the momenta:

$$\rho(x,t) := \int_{\mathbb{R}^3} f(x,t,v)\,dv, \qquad m(x,t) := \int_{\mathbb{R}^3} f(x,t,v) v\,dv, \qquad \varepsilon(x,t) := \int_{\mathbb{R}^3} f(x,t,v) \frac{|v|^2}{2}\,dv.$$

Of course, we may define a mean velocity $\mathbf{v}$ and an internal energy $\mathbf{e}$ through the identities

$$\rho \mathbf{v} = m, \qquad \frac{1}{2} \rho |\mathbf{v}|^2 + \rho \mathbf{e} = \varepsilon.$$

Cauchy–Schwarz inequality ensures that $\mathbf{e}$ is non-negative.

General kinetic models write

$$\partial_t f + \mathrm{div}_x (A(\xi) f) = \frac{1}{\kappa} \mathcal{Q}[f]$$

where $\mathcal{Q}$ stands for local interaction.The conserved quantity $u(x,t)$ is always recovered through momenta (averaging) of f with respect to ξ. The right-hand side is cancelled by these averaging, so that u satisfies a system of finitely many conservation laws of the form (2.1). This system is not closed however, since its flux cannot be expressed in terms of u alone. But in the limit $\kappa \to 0$, the identity $\mathcal{Q}[f] = 0$, which expresses local equilibrium, allows us to close the system.

Such kinetic models are generalizations of relaxation models. The latter are obtained when ξ ranges over a finite set. Besides physically motivated models (Broadwell,...), one finds abstract ones, as the BGK equations or the kinetic formulations of systems of conservation laws. A reliable reference for all these topics is the book by Perthame [39].

The well-posedness of the Cauchy problem is a highly non-trivial matter. The global well-posedness for the Boltzman equation was one of the works (in collaboration with R. DiPerna) for which P.-L. Lions was awarded a Fields medal. In the proof, the regularization provided by the averaging procedure is a striking phenomenon and a key tool. Amazingly enough, this regularization is not available when ξ runs over a discrete set and therefore the global well-posedness of the Cauchy problem for general discrete velocity models is still an open question. Once again, time-asymptotics and the limit as $\kappa \to 0+$ are important issues, which are difficult because of the lack of uniform estimates. The analysis of shock profiles is hard too because it involves non-local operators.

2.6 Shocks and dissipation

Even without any additional term in (2.3), some amount of nonlinearity (say, *genuine nonlinearity*) in the flux f ensures that compactly supported (in space) solutions generate shock waves in finite time. Then the entropy (see the next section) dissipates accross shocks. Although being localized, this dissipation has a global effect in space as time tends to $+\infty$, because of two phenomena:

- On the one hand, some characteristics cross the shock and drive information in a linearly expanding cone. This information comes from one side of a shock and go to the other side, but is affected by the crossing of the discontinuity.
- On the other hand, and in the so-called *Lax shocks*, one family of characteristics fall into the shock and some information is lost forever.

The corresponding damping effect has been analyzed for scalar equations by Dafermos [10] and by T.-P. Liu [33, 34] for 2×2 with compactly supported initial data, and even for $n \times n$ systems with small data. Periodic data yields a faster decay, as shown by Glimm & Lax [15]. This looks reasonable since information needs only a finite time to travel from the shocks to every point in the domain, when the initial data is periodic.

For $n \times n$ systems and general data, either periodic or of compact support, the time asymptotics, at least at a formal level, is described by those solutions of (2.3) that never develop shocks[1]. For systems more general than 2×2, for instance with three unknowns or in dimension $d \geq 2$, such solutions are not understood so far.

Evidence of shock waves. Let us consider an inviscid system in one space dimension:

$$\partial_t u + \partial_x f(u) = 0, \tag{2.14}$$

together with its viscous approximation

$$\partial_t u + \partial_x f(u) = \epsilon \partial_x^2 u.$$

[1]Contact discontinuities, which are non-dissipative discontinuities, may occur.

Let U be a global solution of the ODE

$$\dot{U} = f(U) - sU - q,$$

for some constants s and q, and assume that the limits $U(\pm\infty) =: u_\pm$ exist. One says that U is a *heteroclinic connection* between u_- and u_+. Then

$$u^\epsilon(x,t) := U\left(\frac{x - st}{\epsilon}\right)$$

is an exact solution of the viscous system. If we believe that the inviscid system is the limit of the viscous one, then the limit of u^ϵ as $\epsilon \to 0+$ must be an admissible solution of (2.14). This limit is actually discontinuous, piecewise constant:

$$u(x,t) = \left\{ \begin{array}{ll} u_-, & x < st, \\ u_+, & x > st. \end{array} \right.$$

The discontinuity is located at $x = st$. It travels at velocity s, like the travelling waves u^ϵ.

We notice the important algebraic identity

$$q = f(u_-) - su_- = f(u_+) - su_+,$$

from which we derive the *Rankine–Hugoniot condition*

$$f(u_+) - f(u_-) = s(u_+ - u_-). \tag{2.15}$$

The Rankine–Hugoniot condition expresses the fact that u is a solution of (2.14) in the distributional sense.

2.7 Dissipation at the boundary

A similar phenomenon arises when (2.3) is posed in a bounded domain, and the boundary condition is not conservative. The possibility of influencing the solution in the whole domain by acting only at the boundary is widely used in control theory, where one wants to stabilize oscillations for instance. Let us give as an example the wave equation

$$\partial_t^2 u - c^2 \Delta_x u = 0$$

in a bounded domain, supplemented with the boundary condition

$$\frac{\partial u}{\partial \nu} = \gamma \frac{\partial u}{\partial t}$$

with $\gamma \leq 0$ and $\partial u / \partial \nu$ being the outer normal derivative of u. The total energy $E[u]$ is

$$E[u](t) := \frac{1}{2} \int_D \left((\partial_t u)^2 + c^2 |\nabla_x u|^2 \right) \, dx.$$

It would be conserved if $D = \mathbb{R}^d$, or if $\gamma = 0$. But if γ is negative, then

$$\frac{d}{dt}E[u] = \int_D \operatorname{div}(c^2(\partial_t u)\nabla u)\, dx = \int_{\partial D} c^2(\partial_t u)\partial_\nu u\, ds = \gamma c^2 \int_{\partial D} (\partial_t u)^2 ds$$

is negative, except if $\partial_t u$, hence $\partial_\nu u$, vanishes along the boundary.

At a formal level, it is not too complicated to understand the time asympotics. If we apply the Lasalle invariance principle, the limit behaviour is that of a solution of the wave equation with both $\partial_t u = 0$ and $\partial_\nu u = 0$ at the boundary. If such solutions exist and are non-trivial, then there must be eigenfunctions of Δ that satisfy both the Dirichlet and Neumann boundary conditions. Obviously, this is an overdetermined problem, with no solutions for generic domains in dimension $d \geq 2$. Thus for generic domains, we expect that the energy of $u(t)$ tends to zero as $t \to +\infty$. In particular $u(t)$ tends to a constant.

The analogue of the relaxation limit is here when $\gamma \to -\infty$. Formally, the limit problem is the wave equation with the boundary condition $\partial_t u = 0$. This is precisely the non-homegeneous Dirichlet boundary condition $u(x,t) = u_0(x)$, u_0 the initial data.

2.8 Finite difference schemes

A practical way to approximate the solution of the Cauchy problem for (2.3) is to employ a finite difference scheme. But because our system is formed of conservation laws, we prefer usually *conservative* schemes. In one space dimension, these write

$$\frac{u_j^{m+1} - u_j^m}{\Delta t} + \frac{f_{j+1/2}^m - f_{j-1/2}^m}{\Delta x} = 0, \qquad m \in \mathbb{N},\ j \in \mathbb{Z}. \tag{2.16}$$

Hereabove, u_j^n approximates the value of the unknown field u at the grid point $(j\Delta x, m\Delta t)$. Likewise, $f_{j+1/2}^n$ approximates the flux $f(u)$ at the intermediate point $((j+1/2)\Delta x, m\Delta t)$. It is determined in term of the unknowns u_k^m through the choice of a *numerical flux* F:

$$f_{j+1/2}^m = F(u_{j-p+1}^m, \dots, u_{j+q}^m). \tag{2.17}$$

In this formulation, the numerical flux depends upon $p+q$ grid points, and the scheme itself (2.16) determines u_j^{m+1} as a function of $p+q+1$ grid points at time $m\Delta t$.

The fact that the scheme (2.16, 2.17) is consistent with our system of conservation laws, in the sense that pointwise convergence of the approximate solutions ensures that the limit is a solution of (2.3), is ensured by the requirement

$$F(u, \dots, u) \equiv f(u), \qquad \forall u \in \mathcal{U}. \tag{2.18}$$

For the sake of simplicity, we shall consider *three-point* schemes, where $p = q = 1$:

$$f_{j+1/2}^m = F(u_j^m, u_{j+1}^m).$$

An obvious choice is the *centered scheme*,

$$F_c(u,v) = \frac{1}{2}(f(u) + f(v)),$$

but it is highly unstable as the grid parameters Δx and Δt tend to zero, and is unable to provide any kind of approximation of the exact solution of the Cauchy problem. This flaw is associated to a lack of dissipation.

Classical examples of three-point schemes are the Lax–Friedrichs, Godunov and Lax–Wendroff schemes. The simplest one is that of Lax–Friedrichs, where:

$$F_{LF}(u,v) = \frac{1}{2}(f(u)+f(v)) + \frac{\Delta x}{2\Delta t}(u-v).$$

It is stable under a limitation of the *aspect ratio* of the grid

$$\lambda := \frac{\Delta x}{\Delta t},$$

called a Courant–Friedrichs–Levy condition (CFL), here

$$\rho(\mathrm{d}f(u)) \leq \lambda.$$

This inequality has to be satisfied for every state $u \in \mathcal{U}$, or at least for those states relevant for the solution under consideration. The stability of the scheme is deeply related to a dissipation property.

The relevance of conservative finite difference schemes is given by the theorem of Lax & Wendroff given below. Before stating it, let us introduce a few notations. We define intervals of $\mathbb{R}$:

$$I_j := [(j-1/2)\Delta x, (j+1/2)\Delta x), \qquad I_{j+1/2} := [j\Delta x, (j+1)\Delta x).$$

Given the discrete unknowns u_j^m and the discrete fluxes $f_{j+1/2}^m$, and given $\Delta = (\Delta x, \Delta t)$, we define an approximate solution u^Δ and an approximate flux q^δ by

$$u^\Delta(x,t) :\equiv u_j^m, \qquad x \in I_j,\ t \in [m\Delta t, (m+1)\Delta t)$$

and

$$q^\Delta(x,t) :\equiv f_{j+1/2}^m, \qquad x \in I_{j+1/2},\ t \in [m\Delta t, (m+1)\Delta t).$$

Let us consider a sequence Δ that converges to zero. The first result is

Lemma 2.8.1 *Let us assume that the sequence u^Δ and q^Δ are uniformly bounded on every compact subset, and that they converge in the sense of distributions, towards u and q respectively. Then (u,q) satisfy*

$$\partial_t u + \mathrm{div} q = 0$$

in the weak sense, together with the initial condition $u(\cdot,0) = u^0$.

With the help of the dominated convergence theorem, we deduce

Theorem 2.8.1 (Lax & Wendroff) *Let us assume that $u^\Delta(x,t) \to u(x,t)$ almost everywhere, and that the scheme is consistent (one has (2.18)). Then u is a weak solution of the Cauchy problem (2.3, 2.5).*

As a matter of fact, the assumption ensures that

$$q^{\Delta}(x,t) \to F(u(x,t),\dots,u(x,t)) = f(u(x,t)),$$

whence $q \equiv f \circ u$. Thus Lemma 2.8.1 tells us that u is a weak solution. There remains to prove the lemma.

Proof

We define as well

$$u^{0\Delta}(x) :\equiv u_j^0, \qquad x \in I_j.$$

Then, given a test field $\phi \in \mathcal{D}(\mathbb{R}^2)^n$, we form

$$L_\Delta[\phi] := \int_0^\infty \int_{\mathbb{R}} (u^\Delta \cdot \partial_t \phi + q^\Delta \cdot \partial_x \phi)\, dxdt + \int_{\mathbb{R}} u^{0\Delta} \cdot \phi(\cdot, 0)\, dx.$$

We have

$$\begin{aligned} L_\Delta[\phi] &= \sum_{m\ge 0,j} u_j^m \cdot \int_{I_j \times (m\Delta t,(m+1)\Delta t)} \partial_t \phi \, dxdt + \sum_{m\ge 0,j} f_{j+1/2}^m \cdot \int_{I_j \times (m\Delta t,(m+1)\Delta t)} \partial_t \phi \, dxdt \\ &\quad + \sum_j u_j^0 \cdot \int_{I_j} \phi(x,0)\, dx \\ &= \sum_{m\ge 1,j} (u_j^{m-1} - u_j^m) \cdot \int_{I_j} \phi(x, m\Delta t)\, dx \\ &\quad + \sum_{m\ge 0,j} (f_{j-1/2}^m - f_{j+1/2}^m) \cdot \int_{m\Delta t}^{(m+1)\Delta t} \phi(j\Delta x, t)\, dt. \end{aligned}$$

With the help of (2.16), this gives

$$L_\Delta[\phi] = \sum_{m\ge 1,j} u_j^m \cdot a_j^m + \sum_j u_j^0 \cdot b_j,$$

with

$$b_j := \int_{I_j} \phi(x,0)\, dx - \frac{\Delta x}{\Delta t} \int_0^{\Delta t} \phi(j\Delta x, t)\, dt$$

and

$$a_j^m := \int_{I_j} (\phi(x,(m+1)\Delta t) - \phi(x, m\Delta t))\, dx - \frac{\Delta x}{\Delta t} \left(\int_{m\Delta t}^{(m+1)\Delta t} - \int_{(m-1)\Delta t}^{m\Delta t} \right) \phi(j\Delta x, t)\, dt$$

Taylor formula yields

$$b_j = O(\Delta x)\, O(\Delta x + \Delta t), \qquad a_j^m = O(\Delta x \Delta t)\, O(\Delta x + \Delta t).$$

Since ϕ has a compact support, the number of cells on which a_j^m is non-zero is bounded by a constant over $\Delta x \Delta t$. Likewise, the number of spatial cells I_j over which b_j is non-zero is counded by $c/\Delta x$. We deduce therefore

$$\begin{aligned} L_\Delta[\phi] &= O\left(\frac{1}{\Delta x \Delta t}\right) O(\Delta x \Delta t)\, O(\Delta x + \Delta t) + O\left(\frac{1}{\Delta x}\right) O(\Delta x)\, O(\Delta x + \Delta t) \\ &= O(\Delta x + \Delta t) \to 0. \end{aligned}$$

Passing to the limit, we therefore obtain

$$\int_0^{+\infty} \int_{\mathbb{R}} (u \cdot \partial_t \phi + q \cdot \partial_x \phi)\, dxdt + \int_{\mathbb{R}} u^0(x) \cdot \phi(x,0)\, dx = 0,$$

which means that (u, q) satisfy the conservation laws $\partial_t u + \partial_x q = 0$ together with the initial condition $u(\cdot, 0) = u^0$.

∎

2.9 Questions associated to dissipative models

1. The very first problem is: To which extent can we say that a given model is dissipative ? A mean to understand this concept is to ask for an explicit entropy dissipation. This is often a good notion, but it is not the only one. Physical systems usually have one non-trivial entropy, and dissipation could happen without this entropy being dissipated. For people familiar with the Initial Boundary Value Problem for hyperbolic systems, this is similar to the fact that an IBVP for a symmetric hyperbolic operator can be strongly well-posed (this is a kind of dissipation at the boundary) even in cases where the energy of the system (here the L^2-norm of u) is not dissipated. This happens especially in the study of the stability of multi-dimensional shock waves for gas dynamics. This makes the theory of the hyperbolic IBVP rather much complicated. See the recent book by S. Benzoni and the author [3].

2. Then comes the question of local or global well-posedness in spaces of classical solutions. In the linear case, Sobolev spaces are welcome. Without well-posedness, you cannot go ahead. A general principle is that entropy dissipativity implies well-posedness in the space associated to the entropy. This is almost clear in the linear case, but becomes false for nonlinear problems.

   Global well-posedness is of course an issue if we are going to study the time-asymptotics. Uniform well-posedness is required when one faces a singular limit. Both notions are often related.

3. All dissipative models involve small coefficients. When these parameters are set to zero, we are back to the quasilinear system of first-order PDEs. Does the solutions behave continuously as the diffusion coefficients tend to zero ?

4. As mentionned several times above, this singular limit is related to the time asymptotics at fixed diffusion parameter. However, this is not a complete equivalence and we have to study separately both limits.

5. To be physically reasonnable, a shock wave (in direction $\nu \in \mathbf{S}^1$, velocity c and end states $u^\pm$) of (2.3) must be associated to a travelling wave $\phi(x \cdot \nu - ct)$ (with $\phi(\pm\infty) = u^\pm$) of the dissipative model which we believe is representing faithfully the physics. Such a wave is called a *shock profile*. Thus we need to identify the heteroclinic travelling waves of the given model.

6. The admissibility of the shock is also subjected to the stability of its profile. This is a special instance of the time asymptotic problem.

7. It may happen that dissipative models also admit discontinuous solutions. We thus ask whether discontinuities can develop or if they are only propagated. If they may develop, does smallness and regularity of the initial data prevent from this shock breaking ? There must be a competition between dissipation and nonlinearity. Who wins ?

8. When the dissipative model admits discontinuities, what is the relation between its shock velocities and those of the non-dissipative model (2.3) ?

Remark. It may happen that several distinct dissipation processes are present simultaneously. This is the case for visco-elastic material with fading memory (viscosity + relaxation). This happens also when one wants to establish the global well-posedness of the Cauchy problem for a relaxation model or a coupled hyperbolic-elliptic system; the lack of regularization effect allows discontinuous solutions, for which the tools from functional analysis are useless. A way to attack this question is to add some artificial viscosity, and then to let it vanish.

Chapter 3

Classics about hyperbolic systems of conservation laws

Notation: Given a smooth function g on an open domain of $\mathbb{R}^n$, $\mathrm{d}g$ and $\mathrm{D}^2 g$ denote the differential and the Hessian matrix of a function $u \mapsto g(u)$. Thus $\mathrm{d}g$ is a field of linear forms, a differential form, and $\mathrm{D}^2 g$ is a field of bilinear symmetric forms.

The material of this chapter is presented in a sketchy way. For details and/or complete proofs, we refer to [43], [3] and to the chapters 5 to 8 below.

3.1 Linear, constant coefficient, first-order systems

Let us start with a linear first-order system of PDEs with constant coefficients:

$$\partial_t u + \sum_{\alpha=1}^{d} A^\alpha \partial_\alpha u = 0, \qquad (A^\alpha \in \mathbf{M}_n(\mathbb{R})). \tag{3.1}$$

We assume the space domain $\mathbb{R}^d$. This system is to be completed by an initial condition

$$u(x, t = 0) = u^0(x), \qquad x \in \mathbb{R}^d, \tag{3.2}$$

where u^0 is chosen in some functional space. Solving both (3.1, 3.2) is the *Cauchy problem.*

The classical theory of the Cauchy problem proceeds through Fourier analysis. Fourier transform in the x-variable yields the equivalent problem

$$\partial_t \hat{u}(\xi, t) + iA(\xi)\hat{u}(\xi, t) = 0, \qquad \left(A(\xi) := \sum_{\alpha=1}^{d} \xi_\alpha A^\alpha \right),$$
$$\hat{u}(\xi, 0) = \widehat{u^0}(\xi).$$

The field of matrices $\xi \mapsto A(\xi)$ is called the *symbol* of the system.

After Fourier transform, the system has become a linear ODE in the time variable, parametrized by the frequency variable ξ. The Cauchy problem in $\hat{u}$ has a unique, explicit solution:

$$\hat{u}(\xi, t) = \exp(-itA(\xi))\hat{u}^0(\xi).$$

There remains to invert the Fourier transform to get our solution $u(x, t)$. This can be done provided $\hat{u}(\cdot, t)$ is a reasonable function of ξ, say with a suitable decay at infinity. This requires that the map

$$\xi \mapsto \exp(-itA(\xi))$$

remains bounded over $\mathbb{R}^d$, for fixed $t \in \mathbb{R}^+$ (exercise: prove this claim under the assumption that for every initial data u^0 in $L^2(\mathbb{R}^d)^n$, the solution remains in the same space at every time t). This property is called *hyperbolicity.* Mind that this terminology refers to PDE theory and has nothing in common with the same terminology in dynamical systems theory. *Hyperbolic* is perhaps the most ubiquitous word in mathematics.

Consideration of eigenfields of the symbol yields immediately to a necessary condition of hyperbolicity:

> If a linear system is hyperbolic, then for every $\xi \in \mathbb{R}^d$, the eigenvalues of $A(\xi)$ are real numbers.

A slightly more carefull analysis yields a still necessary property:

> If a linear system is hyperbolic, then for every $\xi \in \mathbb{R}^d$, the eigenvalues of $A(\xi)$ are semi-simple. In other words, $A(\xi)$ is diagonalisable.

A necessary and sufficient, though not practical, condition has been provided by H.-O. Kreiss:

Theorem 3.1.1 (Kreiss) *A linear system is hyperbolic if, and only if, it is* uniformly *diagonalisable with real eigenvalues. This means that there exist fields of matrices $\xi \mapsto D(\xi)$ and $\xi \mapsto P(\xi)$, with $D(\xi)$ real and diagonal, $P(\xi)$ invertible, such that on the one hand $A(\xi) = P(\xi)^{-1}D(\xi)P(\xi)$ for every $\xi \in \mathbb{R}^d$, and on the other hand $\xi \mapsto \|P(\xi)\| \cdot \|P(\xi)^{-1}\|$ is bounded.*

We warn the reader that $\xi \mapsto P(\xi)$ does not need to be continous. However, it can always be chosen homogeneous, say of arbitrary degree ρ:

$$P(s\xi) = s^\rho P(\xi), \qquad \forall \xi \in \mathbb{R}^d, \, \forall s \in \mathbb{R}^+.$$

There are two interesting classes of hyperbolic systems:

Strictly hyperbolic systems. A system is *strictly* hyperbolic if $A(\xi)$ is diagonalisable with real and *simple* eigenvalues for every $\xi \neq 0$.. More generally, a system is *constantly* hyperbolic if the eiegenvalues of $A(\xi)$ are real and have multiplicities independent of $\xi \neq 0$.

As an exercise, show that strict hyperbolicity implies hyperbolicity (use the Implicit Function Theorem, plus the compactness of the unit sphere, then apply Kreiss' Theorem).

Symmetric hyperbolic systems. A system is *symmetric* hyperbolic if there exists a positive definite symmetric matrix S^0 such that $S(\xi) := S^0 A(\xi)$ is symmetric. In other words, $S^\alpha := S^0 A^\alpha$ is symmetric. As an exercise, show that symmetric hyperbolicity implies hyperbolicity (estimate directly the norm of the exponential ; use the square root of S^0).

3.2 Entropies and fluxes

We start with a quasilinear system (2.3) that we assume to be compatible with an *entropy identity.* The latter is an additional conservation law:

$$\partial_t \eta(u) + \mathrm{div}_x Q(u) = 0, \tag{3.3}$$

which is implied by (2.3), at least for smooth solutions. The function η is called an *entropy* of the system, and Q is its *flux.* Entropy-flux pairs are the solutions of the linear differential system

$$\mathrm{d}Q^\alpha = \mathrm{d}\eta \, \mathrm{d}f^\alpha, \qquad \alpha = 1, \dots, d. \tag{3.4}$$

In coordinates, this is

$$\frac{\partial Q^\alpha}{\partial u_i} = \sum_{j=1}^n \frac{\partial \eta}{\partial u_j} \frac{\partial f_j^\alpha}{\partial u_i}, \qquad \forall i = 1, \dots, n, \, \forall \alpha = 1, \dots, d.$$

This system has the trivial solution $\eta(u) = \ell(u) + \eta_0$, $Q^\alpha(u) = \ell(f^\alpha(u)) + q_0^\alpha$, where ℓ is a linear form and η_0, q_0^α are constant. Of course, we are interested in non-trivial, that is not affine entropies. But since the set of entropies is a vector space, we can use affine entropies to correct a non-trivial one, typical for fixing its value and its gradient at a given state $\bar{u}$.

We assume further that $\mathrm{D}^2\eta$ is positive definite at every point and we say that η is *strongly* convex. In particular, η is a strictly convex function of u. Given a state $\bar{u}$, we may replace η by the function

$$\hat{\eta}(u) := \eta(u) - \eta(\bar{u}) - \mathrm{d}\eta(\bar{u}) \cdot (u - \bar{u}),$$

and Q by

$$\hat{Q}(u) := Q(u) - Q(\bar{u}) - \mathrm{d}\eta(\bar{u}) \cdot (f(u) - f(\bar{u})).$$

The pair $(\hat{\eta}, \hat{Q})$ is still an entropy-flux pair. We have $\hat{\eta}(\bar{u}) = 0$ and $\mathrm{d}\hat{\eta}(\bar{u}) = 0$, whence locally $\hat{\eta}(u) \geq \gamma |u - \bar{u}|^2$ for some positive γ.

The primary role of an entropy is to provide an *a priori* estimate. If η is as above, and if a smooth solution tends to $\bar{u}$ at infinity sufficiently fast, then integrating (3.3) gives

$$\frac{d}{dt} \int_{\mathbb{R}^d} \eta(u(x,t)) \, dx = 0,$$

whence

$$\int_{\mathbb{R}^d} \eta(u(x,t)) \, dx = \int_{\mathbb{R}^d} \eta(u_0(x)) \, dx,$$

and therefore

$$\sup_{t \geq 0} \|u(\cdot, t) - \bar{u}\|_{L^2}^2 \leq \frac{1}{\gamma} \int_{\mathbb{R}^d} \eta(u_0(x)) \, dx.$$

Examples.

- Consider a linear system (3.1). It admits a strongly convex entropy if, and only if, it is symmetric hyperbolic. When this is the case, then an entropy is $\eta(u) = u^T S^0 u$, with flux $Q^\alpha(u) = u^T S^\alpha u$.

- Here is the so-called p-system, a model of isentropic gas dynamics in one space-dimension.

$$\partial_t v + \partial_x w = 0, \qquad \partial_t w + \partial_x p(v) = 0,$$

where $v \mapsto p(v)$ is a given function (an equation of state). We have $n = 2$, $u = (v, w)$ and the equations represent the conservation of mass and momentum, respectively, in terms of the so-called *Lagrangian coordinates*.

Let us look for an entropy of the form

$$\eta(u) = \frac{1}{2} w^2 + e(v).$$

One has

$$\partial_t \eta(u) + \partial_x (p(v) w) = e'(v) \partial_t v + p(v) \partial_x w,$$

which vanishes precisely when

$$\frac{de}{dv} = p.$$

This differential relation defines the function e, up to a constant, whence η. The convexity of η is equivalent to that of $v \mapsto e$, that is to

$$p' > 0.$$

We notice that this inequality is also the condition under which our system, written in the quasilinear form

$$\partial_t v + \partial_x w = 0, \qquad \partial_t w + p'(v) \partial_x v = 0,$$

is hyperbolic: its symbol is

$$\xi \begin{pmatrix} 0 & 1 \\ p'(v) & 0 \end{pmatrix},$$

whose eigenvalues are $\pm \xi \sqrt{p'(v)}$.

In this isentropic model, the mathematical entropy turns out to be the mechanical energy per unit mass. Thus the conservation of the total energy

$$\int \left(\frac{1}{2} w^2 + e(v) \right) dx$$

is here a consequence of the conservation of mass and momentum. We warn the reader that this is no longer the case in the full system of gas dynamics.

- Let us turn to the full system of gas dynamics, given in Eulerian coordinates by the unknown and flux (2.4). The conservation of mass writes

$$\partial_t\rho + \mathrm{div}(\rho v) = 0.$$

Elimination with the conservation of momentum yields

$$\partial_t v_j + v\cdot\nabla v_j + \frac{1}{\rho}\,\partial_j p = 0.$$

These two identities can be used to simplify the conservation of energy:

$$\partial_t e + v\cdot\nabla e + \frac{p}{\rho}\,\mathrm{div} v = 0.$$

Let us now choose a function $S(\rho, e)$, solution of the linear differential equation

$$\rho^2\frac{\partial S}{\partial e} + p\,\frac{\partial S}{\partial \rho} = 0.$$

This means that there exists a function $\theta(\rho, e)$, such that

$$\theta \mathrm{d}S = \mathrm{d}e + p\,\mathrm{d}\frac{1}{\rho}.$$

This is the well-known relation of thermodynamics, and S is called the physical entropy. The integrating factor $\theta > 0$ is the absolute temperature.

Using the evolution equations for ρ and e above, we obtain

$$\begin{aligned}(\partial_t + v\cdot\nabla)S &= S_\rho(\partial_t + v\cdot\nabla)\rho + S_e(\partial_t + v\cdot\nabla)\rho\\ &= -\rho S_\rho \mathrm{div} v - S_e\frac{p}{\rho}\,\mathrm{div} v \equiv 0.\end{aligned}$$

This expresses the constancy of the *physical entropy* S along the particle trajectories, since the particles have velocity v in this modelling. Recombining with the conservation of mass, we obtain an extra conservation law

$$\partial_t(\rho S) + \mathrm{div}(\rho S v) = 0.$$

Whence a mathematical entropy

$$\eta(u) = -\rho S, \qquad Q(u) = -\rho S v,$$

where the minus sign has been chosen in such a way that η is convex when the system of Euler equations is hyperbolic.

As an exercise, show that a system (2.3) admits a strongly convex entropy if, and only if its *linearization*

$$\partial_t v + \sum_{\alpha=1}^{d} \mathrm{d}f^\alpha(\bar u)\partial_\alpha u = 0 \tag{3.5}$$

is symmetric hyperbolic, for every $\bar u \in \mathcal{U}$.

3.3 Symmetric hyperbolic linear systems

A linear system of first-order PDEs, with variable coefficients, is called *symmetric*, in the sense of Friedrichs, if it can be put in the form

$$S_0(x,t)\partial_t u + \sum_{\alpha=1}^{d} S^\alpha(x,t)\partial_\alpha u + M(x,t)u = f(x,t), \tag{3.6}$$

where u is the unknown, and the matrices S_0, S^α are symmetric and depend smoothly and boundedly upon (x,t), and S_0 is uniformly positive definite. Finding such a form may require that one multiplies the system by some matrix, and/or one makes a linear change of unknown. Remark that M is only smooth and bounded, but needs not be symmetric.

The symmetric form yields an *a priori* estimate. Multiply (3.6) by u^T, one obtains

$$\partial_t\langle S_0 u, u\rangle + \sum_{\alpha=1}^{d} \partial_\alpha\langle S_\alpha u, u\rangle = \langle Nu, u\rangle + 2\langle f, u\rangle,$$

with

$$N = \partial_t S_0 + \sum_{\alpha=1}^{d} \partial_\alpha S^\alpha - 2M.$$

Let us integrate over a large ball of $\mathbb{R}^d$. If u decays sufficiently fast to zero at infinity, we can let the radius going to infinity. At the end, we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^d} \langle S_0 u, u\rangle \, dx = \int_{\mathbb{R}^d} (\langle Nu, u\rangle + 2\langle f, u\rangle)\, dx.$$

Since $S_0(x,t) \geq \omega I_n$ with some positive and constant ω, there exists constants c verifying

$$\langle Nv, v\rangle \leq c\langle S_0 v, v\rangle, \qquad |v|^2 \leq c\langle S_0 v, v\rangle, \qquad \forall v \in \mathbb{R}^n, \forall (x,t).$$

We therefore deduce that

$$Y(t) := \int_{\mathbb{R}^d} \langle S_0 u, u\rangle \, dx$$

satisfies an inequality

$$Y' \leq c\,(Y + \|f(t)\|_{L^2})\,,$$

whence an explicit bound for $Y(t)$, by means of a Gronwall estimate.

This estimate shows that $f \equiv 0$ and $u(t=0) \equiv 0$ imply $u \equiv 0$. This is the uniqueness property. As it is often the case, the existence in this linear situation is a consequence of the Hahn–Banach and Riesz theorems. This requires establishing an *a priori* estimate for an adjoint problem, but this is an obvious matter once one realizes that this adjoint problem is of the same kind as our Cauchy problem. See [3], Chapter 2 for the details. We shall not state here a precise result, but let us just say that if the coefficients are reasonnably smooth, and if they are constant outside of a compact set, then the Cauchy problem for (3.6) is well-posed in the Sobolev space $H^s(\mathbb{R}^d)^n$ for every $s \in \mathbb{R}$.

3.4 Symmetric hyperbolic quasi-linear systems

The only difference between linear and quasi-linear systems is the dependence of the coefficients upon the unknown. For the sake of simplicity, say that the system writes:

$$S_0(u)\partial_t u + \sum_{\alpha=1}^{d} S^\alpha(u)\partial_\alpha u = f(x,t), \tag{3.7}$$

with S_0 and S^α symmetric, smooth, bounded, and S_0 uniformly positive definite.

The first important fact is

Theorem 3.4.1 (Godunov, Lax & Friedrichs) *Let us assume that a system of conservation laws (2.3) admits a strongly convex entropy η. Then it can be put in the symmetric form (3.7).*

Proof

Let us consider the change of variable $z := \mathrm{d}\eta(u)$, that is

$$z_j = \frac{\partial \eta}{\partial u_j}.$$

Then (3.4) writes

$$\mathrm{d}Q^\alpha = \sum_{j=1}^{n} z_j \mathrm{d}f_j^\alpha = \mathrm{d}\left(\sum_{j=1}^{n} z_j f_j^\alpha\right) - \sum_{j=1}^{n} f_j^\alpha \mathrm{d}z_j.$$

This tells us that

$$f_j^\alpha = \frac{\partial M^\alpha}{\partial z_j}, \qquad M^\alpha := \sum_{j=1}^{n} z_j f_j^\alpha - Q^\alpha.$$

Since on an other hand, one has

$$u_j = \frac{\partial \eta^*}{\partial z_j},$$

where η^* is the Legendre transform of η, we see that (2.3) rewrites

$$\partial_t \frac{\partial \eta^*}{\partial z_j} + \sum_{\alpha=1}^{d} \partial_\alpha \frac{\partial M^\alpha}{\partial z_j} = 0, \qquad \forall j = 1, \dots, n.$$

Applying the chain rule, this amounts to writing

$$S_0(z)\partial_t z + \sum_{\alpha=1}^{d} S^\alpha(z)\partial_\alpha z = 0,$$

where

$$S_0 = \mathrm{D}^2\eta^* = \left(\mathrm{D}^2\eta\right)^{-1} > 0n, \qquad S^\alpha = \mathrm{D}^2 M^\alpha.$$

■

Solving the Cauchy problem. The main idea is to construct a sequence of approximate solutions. Say that the initial data u_0 has compact support. Then we state $u^0(x,t) \equiv u_0(x)$. We then define by induction u^m, the solution of the linear Cauchy problem

$$\begin{aligned} S_0(u^{m-1})\partial_t u^m + \sum_{\alpha=1}^{d} S^\alpha(u^{m-1})\partial_\alpha u^m &= 0, \\ u^m(x,0) &= u_0(x). \end{aligned}$$

It is possible, and desirable, to replace u_0 by a sequence of smooth initial data u_0^m, converging fast enough to u_0 in $H^s(\mathbb{R}^d)$.

According to Theorem 3.4.1 and to the previous paragraph, the sequence u^m is well-defined. Each term u^m has an *a priori* estimate in $H^s(\mathbb{R}^d)$. However, it seems that this estimate does depend on m. Fortunately, it happens that one can find a uniform (thus m-independent) estimate provided

- on the one hand, $s > 1 + \frac{d}{2}$,
- on the other hand, one restricts to a finite time-interval $(0,T)$, with $T > 0$ small enough. Of course this T depends on the initial data.

With such an estimate at hand, one can show that the sequence u^m is Cauchy, thus converges. One verifies easily that its limit is a solution of the Cauchy problem. Actually, one can establish an *a priori* estimate of the difference of two solutions[1], which yields uniqueness. Finally we obtain the so-called *local-in-time* well-posedness:

Theorem 3.4.2 *Let the system (8.1) have smooth fluxes f^α. Assume that there exists a strongly convex ($\mathrm{D}^2\eta > 0$) entropy η. Then, given an initial data $u_0 \in H^s(\mathbb{R}^d)$ with $s > 1+d/2$, there exists a time $T > 0$ and a unique solution u in the class*

$$\mathcal{C}(0,T;H^s) \cap \mathcal{C}^1(0,T;H^{s-1})$$

of the Cauchy problem.

We notice that since $s > 1 + d/2$, one has the Sobolev embeddings

$$H^s(\mathbb{R}^d) \subset \mathcal{C}_b^1(\mathbb{R}^d), \qquad H^{s-1}(\mathbb{R}^d) \subset \mathcal{C}_b^0(\mathbb{R}^d).$$

There follows that the solution given by Theorem 3.4.2 belongs to $\mathcal{C}^1([0,T) \times \mathbb{R}^d)$: it is a *classical* solution.

[1] It is used along the proof above, especially to prove that one has a Cauchy sequence.

3.5 The breakdown of smooth solutions

Smooth solutions do not always exist globally in time, despite the local existence result stated in Section 3.4. What happens is that the first-order derivatives may blow-up in finite time, and do so in general. For general systems, it is difficult to give a complete proof of that, and only partial results are known. But the situation is very clear for scalar conservation laws ($n = 1$). We shall illustrate it in a one-dimensional setting.

Thus let $f : \mathbb{R} \to \mathbb{R}$ be a smooth flux. We only assume that f is not affine. Whenever a solution u of the Cauchy problem is smooth, we have

$$(\partial_t + f'(u)\partial_x)u = 0,$$

which means that u is constant along the so-called *characteristic curves* $t \mapsto (X(t), t)$, defined by the ODE

$$\frac{dX}{dt} = f'(u(X, t)).$$

Since

$$u(X(t), t) \equiv \mathrm{cst} = u(x_0, t), \qquad (x_0 := X(0)),$$

we have $\dot{X} \equiv \mathrm{cst}$ and therefore

$$X(t) = x_0 + f'(u_0(x_0))t.$$

Consequently, u obeys the following implicit relation

$$u\left(x_0 + tf'(u_0(x_0)), t\right) = u_0(x_0), \qquad \forall x_0 \in \mathbb{R}, \forall t > 0. \tag{3.8}$$

When the map $y \mapsto f'(u_0(y))$ is non-decreasing, this formula can be used to calculate the solution u for all x and $t > 0$. When the opposite holds true, there exists two values $x_0 < x_1$ such that

$$f'(u_0(x_1)) < f'(u_0(x_0)).$$

Then there exists a positive time t such that

$$x_0 + tf'(u_0(x_0)) = x_1 + tf'(u_0(x_1)).$$

Applying formula (3.8), we deduce that $u_0(x_0) = u_0(x_1)$, which is a contradiction. In conclusion, we have

Theorem 3.5.1 *For a scalar equation in one space-dimension, the smooth solution of the Cauchy problem is globally defined for $t > 0$ if, and only if, the initial data u_0 is such that $f' \circ u_0$ is monotonous non-decreasing.*

Remark that the smooth solution is global for negative times if, and only if, $f' \circ u_0$ is non-increasing. In conclusion, the solution is global over $t \in \mathbb{R}$ only when $f' \circ u_0$ is constant !!

3.6 Discontinuous solutions

Another evidence of breakdown. First of all, let us show that derivatives are the problem. Again, we make a calculation for a scalar equation in dimension one. Let $v := f' \circ u$. By the chain rule, we have

$$\partial_t v + v\partial_x v = 0.$$

Let us denote $w := \partial_x v$. Differentiating above, we obtain

$$(\partial_t + v\partial_x)w + w^2 = 0.$$

This identity can be recast as an ODE along characteristic lines:

$$\frac{d}{dt}w(X(t), t) + w^2 = 0.$$

This is a Ricatti equation, whose solution blows up in finite time whenever the initial data $w(x_0, 0) = \partial_x(f' \circ u_0)(x_0)$ is negative. This confirms the fact that blow-up does occur, except when $f' \circ u_0$ is non-decreasing.

Weak solutions. The blow-up result is not the end of the history. The implicit formula (3.8) shows that the values taken by u are values taken by u_0, and therefore

$$\|u(\cdot, t)\|_\infty \leq \|u_0\|_\infty.$$

This inequality turns out to be valid for a scalar equation in any dimensions. The situation becomes much more complicated for systems, even in one space-dimension, but one expects that the solution remains bounded in L^∞ in most cases. At least, the *a priori* estimate of Paragraph 3.2 tells us that it remains bounded in some Lebesgue or Orlicz space. Therefore, blow-up concerns only the derivatives of u in general. This suggests that one could extend the solution beyond the breakdown points, within the class of bounded measurable functions. This makes sense because then a system of conservation laws can be interpreted in the distributional sense. We speak then of *weak solutions*:

Definition 3.6.1 *We say that a bounded measurable field $u : \Omega \mapsto \mathcal{U}$ is a* weak solution *of the system (2.3) if, for every test function $\phi \in \mathcal{D}(\Omega)^n$, one has*

$$\iint_\Omega \left(u \cdot \partial_t \phi + \sum_{\alpha=1}^d f^\alpha(u) \cdot \partial_\alpha \phi \right) dx\, dt = 0. \tag{3.9}$$

In practice, we shall be concerned more often with the Cauchy problem, which is treated with the

Definition 3.6.2 *Let $u_0 : \mathbb{R}^d \mapsto \mathcal{U}$ be a given bounded measurable function.*

We say that a bounded measurable field $u : \mathbb{R}^d \times (0, T) \mapsto \mathcal{U}$ is a weak solution *of the Cauchy problem (2.3, 3.2) if, for every test function $\phi \in \mathcal{D}(\mathbb{R}^d \times (-\infty, T))^n$, one has*

$$\int_0^T \int_{\mathbb{R}^d} \left(u \cdot \partial_t \phi + \sum_{\alpha=1}^d f^\alpha(u) \cdot \partial_\alpha \phi \right) dx\, dt + \int_{\mathbb{R}^d} u_0(x) \cdot \phi(x, 0)\, dx = 0. \tag{3.10}$$

Thanks to the Green formula, one easily verifies that a classical solution of the system (2.3) does satisfy (3.9), and that a classical solution of the Cauchy problem satisfies (3.10). We therefore have

$$\text{Classical} \implies \text{Weak.}$$

The converse is false in general. However, the Green formula gives the easy result that

$$(\text{Weak} \ + \ \text{cont. differentiable}) \implies \text{Classical.}$$

These two implications are exactly those expected for a generalization of the theory of classical solutions.

The Rankine–Hugoniot relation. Typical weak solutions are piecewise smooth, with discontinuities along smooth hypersurfaces Σ of $\mathbb{R}^{d+1}$. Since each conservation law

$$\partial_t u_j + \mathrm{div} f_j(u) = 0$$

can be recast as

$$\mathrm{div}_{t,x}(u_j, f_j(u)) = 0,$$

the field u is a distributional solution if, and only if

- on the one hand, it is a classical solution away from the discontinuity surfaces,
- on the one hand, the normal flux of the vector field $(t,x) \mapsto (u_j, f_j(u))$ is continuous accross each discontinuity surface Σ. Denoting by $\mathbf{n} \in \mathbf{S}^d$ a unit normal vector to Σ, this writes

$$n_0[u_j] + \sum_{\alpha=1}^{d} n_\alpha [f_j^\alpha(u)] = 0, \tag{3.11}$$

where $[h](\bar{z}) := h_+(\bar{z}) - h_-(\bar{z})$ is the difference between the limits of a piecewise smooth quantity h on each sides of Σ (called the *jump* of h). Say that

$$h_\pm(\bar{z}) = \lim_{\epsilon \to 0+} h(\bar{z} \pm \epsilon \mathbf{n}(\bar{z})).$$

Because of (3.11), and since f is Lipschitz, we see that $(n_1, \dots, n_d)$ cannot be zero. This shows that the trace of Σ at each time t is a hypersurface Σ_t of $\mathbb{R}^d$. It has a unit normal vector

$$\nu := \frac{1}{\sqrt{1 - n_0^2}} \begin{pmatrix} n_1 \\ \vdots \\ n_d \end{pmatrix}.$$

As time varies the manifold Σ_t travels with a normal speed[2] equal to

$$s := -\frac{n_0}{\sqrt{1 - n_0^2}}\,.$$

[2]For a one-parameter family of hypersurfaces, the only meaningful velocity is the normal velocity.

Then the limits can be defined equivalently by

$$h_\pm(\bar{x},t) = \lim_{\epsilon\to 0+} h(\bar{x} \pm \epsilon\nu(\bar{x},t),t).$$

Finally, the jump relation writes

$$[f_j(u)]\cdot\nu = s[u_j], \qquad \forall j = 1,\dots,n. \tag{3.12}$$

This is the so-called *Rankine–Hugoniot condition.* It can be rewritten in a compact form

$$[f(u;\nu)] = s[u],$$

where the denote

$$f(u;\xi) := \sum_{\alpha=1}^{d} \xi_\alpha f^\alpha(u).$$

Remark that in the linear case, we just have $f(u;\xi) = A(\xi)u$.

The one-dimensional situation. The case of piecewise smooth weak solutions ressembles very much, at least in the neighbourhood of a point of discontinuities, the one-dimensional case

$$\partial_t u + \partial_x f(u) = 0.$$

Then Σ is a curve, parametrized by the time $t \mapsto X(t)$. The Rankine–Hugoniot condition reduce to

$$[f(u)]] = s[u] = \frac{dX}{dt}[u]. \tag{3.13}$$

It is remarquable that when the amplitude $[u]$ of the discontinuity is small, this equation implies both that

- the direction jump $[u]$ is approximately an eigen-direction of $\mathrm{d}f(u)$,
- the velocity s is close to the corresponding eigenvalue.

In particular, in the scalar case, the Taylor formula tells us that

$$s = \frac{f(u_+) - f(u_-)}{u_+ - u_-} = f'(u_m)$$

for some u_m between $u_\pm$. This shows that the curve Σ is approximately a characteristic curve.

3.7 The uniqueness issue

Contrary to (local) classical solutions, weak ones are far from unique in general. Let us illustrate this assertion with the analysis of the following Cauchy problem:

$$\partial_t u + \partial_x \frac{u^2}{2} = 0, \qquad \text{(Burgers equation)}$$

with the trivial initial condition $u_0 \equiv 0$. The obvious and natural solution is $u \equiv 0$. However for every $a > 0$, the following function is a weak solution (Exercise: verify both conditions in the paragraph above. Draw a figure.):

$$u^a(x,t) = \begin{cases} 0, & x < -at, \\ -2a, & -at < x < 0, \\ 2a, & 0 < x < at, \\ 0, & x > at. \end{cases}$$

Since hyperbolic systems of conservation laws do represent physical processes of the real world, this non-uniqueness feature is irrelevant and must be ruled out by some physical principle which has a mathematical formulation. Such criteria wear various clothes and are named by the generic words *entropy condition.*

The entropy inequality. A popular criterion arises when we assume that the solution is actually the limit, as $\epsilon \to 0+$ of the solution u^ϵ of some dissipative model. This shall be discussed in more details in the next chapter, but for the moment, let us take the one-dimensional viscous model

$$\partial_t u + \partial_x f(u) = \epsilon \partial_x^2 u.$$

Assuming that there exists a convex entropy η, multiplication by $\mathrm{d}\eta(u)$ yields

$$\partial_t \eta(u) + \partial_x Q(u) = \epsilon \partial_x (\mathrm{d}\eta(u) \partial_x u) - \mathrm{D}^2 \eta(u) : (\partial_x u, \partial_x u).$$

Since $\mathrm{D}^2\eta(u)$ is a positive definite quadratic form, the latter term is non-positive, and there remains

$$\partial_t \eta(u) + \partial_x Q(u) \leq \epsilon \partial_x (\mathrm{d}\eta(u) \partial_x u).$$

In this inequality, u stands actually for u^ϵ. We wish to pass to the limit. We shall see in Section 4.1 that the right-hand side tends to zero in the sense of distribution. If moreover u^ϵ converges boundedly almost everywhere, the dominated convergence theorem allows us to pass to the limit in the left-hand side, and we obtain the so-called *entropy inequality*, due to P. Lax:

$$\partial_t \eta(u) + \partial_x Q(u) \leq 0, \tag{3.14}$$

which must be read in the distributional sense, that is

for every **non-negative** test function ϕ, one has

$$\int_0^T \int_{\mathbb{R}} (\eta(u) \cdot \partial_t \phi + Q(u) \cdot \partial_x \phi)\, dx\, dt + \int_{\mathbb{R}} \eta(u_0(x)) \cdot \phi(x,0)\, dx \geq 0.$$

In the regions where u is a classical solution, (3.14) is a straighforward consequence of (2.3), with actually an equality instead of an inequality. Thus the only information brought by (3.14) concerns the jumps accross discontinuity curves. One obtains the equivalent inequality

$$[Q(u)] \leq s[\eta(u)]. \tag{3.15}$$

More generally, in several space dimensions, we have the restriction

$$[Q(u;\nu)] \leq s[\eta(u)],$$

with the now standard notation

$$Q(u;\xi) := \sum_{\alpha=1}^{d} \xi_\alpha Q^\alpha(u).$$

Remark that inequality (3.15) tells us that in general one cannot exchange the roles of u_- and u_+. More precisely, if u is an admissible solution[3], then

$$\tilde{u}(x,t) := u(-x, T-t)$$

is not so, despite the fact that the system (2.3) was formally space-time reversible. Inequation (3.14) does not enjoy this reversibility. The role of an entropy condition is thus to put a small amount of irreversibility in the formally reversible system (2.3). The entropy condition is therefore reminiscent to the *second principle of thermodynamics*.

Example. Let us go back to the Burgers equation. The Rankine–Hugoniot condition tells us that

$$s = \frac{u_- + u_+}{2}.$$

Let us choose the entropy-flux pair

$$\eta(u) = \frac{u^2}{2}, \qquad Q(u) = \frac{u^3}{3}.$$

Then

$$[Q(u)] - s[\eta(u)] = \frac{1}{12}[u]^3.$$

Thus (3.15) tells us that admissible discontinuities satisfy

$$u^+ \leq u^-.$$

This calculation, though the simplest among entropy calculations, gives rise to a typical result: for a general system, under a generic assumption, the *rate* $[Q(u;\nu)] - s[u]$ of the entropy dissipation across a discontinuity is of cubic order, with respect to $[u]$, when this jump is small.

[3]By *admissible*, we mean satisfying an entropy condition.

Other entropy conditions. For scalar conservation laws with non-convex fluxes f, the situation is more complicated than in the case of the Burgers equation. Writing (3.15) for a single entropy-flux pair is not sufficient to recover uniqueness. A cure is to write (3.15) for *all* the entropy-flux pairs at our disposal. This yields a satisfactory theory in the case of conservation laws (see Kružkhov [29]), because every function is an entropy. The corresponding admissibility condition for discontinuities is the so-called *Oleĭnik condition*.

Unfortunately, realistic[4] systems admit only one or two non-trivial entropy-flux pairs, either because $n \geq 3$ or because[5] of $n = 2$ while $d \geq 2$. Therefore, the entropy inequalities are often too weak to select the physically relevant discontinuities, and thus to characterize the admissible solutions. There is a need for a refined criterion. An interesting one is the existence of a *shock profile*. This notion depends upon the choice of a dissipative model associated to (2.3). A refinement, which has a sound justification is the existence of a *stable* shock profile.

It is interesting to notice that, in the cases where a nice uniqueness result is known, then we observe that all the approaches yield the same characterization of admissible discontinuities. For instance, this is the case in scalar conservation laws.

3.8 The Riemann problem

A fundamental property of systems (2.3) is that they are invariant under the scaling $(x,t) \mapsto (\mu x, \mu t)$. The main consequence is that if the initial data is scale invariant,

$$u^0(\mu x) = u(x), \qquad \forall x \in \mathbb{R}^n,\ \mu > 0,$$

and if the Cauchy problem is uniquely sovable, then the solution $u(x,t)$ must satisfy

$$u(\mu x, \mu t) \equiv u(x,t),$$

which means that u depends only upon x/t (choose $\mu = 1/t$ above). Such a solution is called self-similar. The search of a self-similar solution is called the *Riemann problem*.

The Riemann problem is mostly used in one space dimension, where a system writes

$$\partial_t u + \partial_x f(u) = 0.$$

In this one-dimensional setting, the RP can often be solved in a closed form, by means of algebraic calculations and integration of ODEs. The initial data has the simple form

$$u^0(x) = \begin{cases} u_\ell, & x < 0, \\ u_r, & x > 0. \end{cases}$$

The first step is to write $u(x,t) = U(x/t)$, where $U(y) := u(y,1)$. Then U solves the differential equation

$$\frac{d}{dy} f(U) = y \frac{dU}{dy}. \tag{3.16}$$

[4]A system of conservation laws taken at random usually does not admit a single non-trivial entropy-flux pair. By *realistic*, we mean systems arising for instance in continuum mechanics, often associated with some variational principle.

[5]Remark that the case $n = 2$, $d = 1$, called the 2×2-case, is intermediate in difficulty.

The initial data becomes

$$\lim_{y\to-\infty} U(y) = u_\ell, \qquad \lim_{y\to+\infty} U(y) = u_r. \tag{3.17}$$

When a solution exists (most likely for systems endowed with a strongly convex entropy), it is piecewise Lipschitz, except at finitely many points where it is discontinous.

Zones of Lipschitz regularity. When U is Lipschitz, the chain rule yields the equivalent ODE

$$(\mathrm{d}f(U) - yI_n)\frac{dU}{dy} = 0.$$

This tells that either $U' = 0$, or it is an eigenvector of the Jacobian matrix $\mathrm{d}f(U)$ and y is the corresponding eigenvalue. There are thus two regimes:

- zones of constancy, where $y \mapsto U(y)$ does not vary. An instance of such a regime is for large values of y, where $U(y) \equiv u_\ell$ or u_r, depending on the sign of y.
- so-called "rarefaction waves", where

  $$y = \lambda_k(U(y)) \tag{3.18}$$

  for some index $k = 1, \ldots, n$, while

  $$U'(y) \parallel r_k(U(y)). \tag{3.19}$$

  In such zones, $y \mapsto U'(y)$ is a parametrization of an arc of an integral curve of the eigenfield r_k. The parametrization is determined by the following identity, obtained by differentiating (3.18):

  $$\mathrm{d}\lambda_k(U(y)) \cdot U'(y) = 1. \tag{3.20}$$

  Notice that this requires

  $$\mathrm{d}\lambda_k(u) \cdot r_k(u) \neq 0, \tag{3.21}$$

  a property called *genuine nonlinearity.*

Discontinuities. When U experiences a discontinuity at y_0, we rewrite (3.16) as

$$\frac{d}{dy}(f(U) - yU) = -U,$$

which shows that $y \mapsto f(U(y) - yU(y)$ must be Lipschitz continuous. This implies that the left an right limits $u_\pm$ of U at $y_0 \pm 0$ satisfy

$$f(u_-) - y_0 u_- = f(u_+) - y_0 u_+.$$

This is precisely the Rankine–Hugoniot relation for a discontinuity of velocity y_0:

$$f(u_+) - f(u_-) = y_0(u_+ - u_-).$$

To summarize, the Riemann problem is solved by gluing elementary solutions on different intervals of $\mathbb{R}$:

- rarefaction waves,
- admissible discontinuities,
- constant states.

An elementary case is that of a linear system, which is detailed in the list of exercises.

We shall not discuss here the solvability of the Riemann problem. We content ourselves to mention, without proof, the

Theorem 3.8.1 (Lax) *Let $u^* \in \mathcal{U}$ be given. Let us assume that there exists a neighbourhood $\mathcal{W}$ of u^* in which the system (2.3) is strictly hyperbolic with characteristic fields satisfying either $\mathrm{d}\lambda_k \cdot r_k \neq 0$ (genuine nonlinearity) or $\mathrm{d}\lambda_k \cdot r_k \equiv 0$ (linear degeneracy).*

Then for every neighbourhood $\mathcal{V} \subset \mathcal{W}$, there exists a neighbourhood $\mathcal{V}_0 \subset \mathcal{V}$ with the property that, given any states $u_\ell, u_r \in \mathcal{V}_0$, there exists a unique admissible solution of the Riemann problem with values in $\mathcal{V}$.

The proof of this result is given in every book on hyperbolic systems of conservation laws, for instance in [10, 43].

Chapter 4

Dissipation and entropy

At the beginning of this chapter, there is a first-order system of conservation laws (2.3), admitting a strongly convex entropy η of flux Q. This is what we call a *physically relevant* system.

Let $\bar{u} \in \mathcal{U}$ be given. Without loss of generality, we assume that η reaches its minimum at $\bar{u}$.

4.1 The viscous case

Let us begin with a viscous extension of (2.3), which has the same unknown u:

$$\partial_t u + \mathrm{div} f(u) = \mathrm{div}(B(u)\nabla u) = \sum_{\alpha,\beta} \partial_\alpha (B^{\alpha\beta}(u)\partial_\beta u). \tag{4.1}$$

Adapting the calculation of Paragraph 3.7, we have

$$\begin{aligned} \partial_t \eta(u) + \mathrm{div} Q(u) &= \mathrm{d}\eta(u)\mathrm{div}(B(u)\nabla u) \\ &= \mathrm{div}(\mathrm{d}\eta(u)B(u)\nabla u) - \sum_{\alpha\beta}(\mathrm{D}^2\eta(u)\partial_\alpha u, B^{\alpha\beta}(u)\partial_\beta u) \end{aligned}$$

The last term is quadratic in the Jacobian matrix ∇u. We recall that if $d = n = 1$, this term is simply $-b(u)\eta''(u)(\partial_x u)^2$ and is non-positive because η is convex and $b > 0$ (for the Cauchy problem to be well-posed for (4.1).)

We thus say that the viscous model is *weakly entropy-dissipative* if (4.1) implies the inequality

$$\partial_t \eta(u) + \mathrm{div} Q(u) \leq \mathrm{div}(\mathrm{d}\eta(u)B(u)\nabla u).$$

This definition is a bit too weak for our concern. For instance, when $d = 1$ and $B^{11} =: B$ is the matrix of a rotation of ninety degree (a Schrödinger-like second-order term), we have entropy conservation (an equality instead of an inequality) if the entropy has the form $\eta = |u|^2$. Of course, nobody should call such a system a dissipative one. This is why we prefer the following stronger definition:

We say that the viscous model is *entropy-dissipative* if (4.1) implies an inequality

$$\partial_t\eta(u)+\mathrm{div}Q(u)+\omega\sum_\alpha\left|\sum_\beta B^{\alpha\beta}(u)\partial_\beta u\right|^2\leq\mathrm{div}(\mathrm{d}\eta(u)B(u)\nabla u),\tag{4.2}$$

where ω is some strictly positive constant. This amounts to saying that

$$\sum_{\alpha,\beta}(\mathrm{D}^2\eta(u)X_\alpha|B^{\alpha\beta}(u)X_\beta)\ \geq\ \omega\sum_\alpha\left|\sum_\beta B^{\alpha\beta}(u)X_\beta\right|^2\tag{4.3}$$

$$=:\omega|B(u)\mathbf{X}|^2,\qquad\forall u,\ \forall X_1,\dots,X_d\in\mathbb{R}^n.\tag{4.4}$$

Note that the term $|B(u)\mathbf{X}|^2$ (the reader is warned that the notation (4.4) is a little bit confusing) is the strongest quadratic form that we may expect to control, since if it vanishes, then the left-hand side of (4.2) vanishes too. We shall see below that we can relax a little bit the assumption (4.3).

4.1.1 The vanishing viscosity limit.

If we replace B by ϵB ($\epsilon>0$ a viscosity parameter) in (4.1), we have the same factor in the corresponding terms of (4.2):

$$\partial_t\eta(u)+\mathrm{div}Q(u)+\epsilon\omega|B(u)\nabla u|^2\leq\epsilon\mathrm{div}(\mathrm{d}\eta(u)B(u)\nabla u).$$

To see how our definition is practical, let us consider, at a formal level, the singular limit as $\epsilon\to0+$. We assume that the initial data (say, independent from ϵ) is asymptotically constant (that is $u(|x|=\infty,t=0)=\bar u$). Recall that $\eta(\bar u)=0$ and $\mathrm{d}\eta(\bar u)=0$, and as well $Q(\bar u)=0$, $\mathrm{d}Q(\bar u)=0$. Say that we have globally (and not only locally) $\eta(u)\geq c_0|u-\bar u|^2$ with c_0 a positive constant. We integrate our dissipation inequality over $\mathbb{R}^d$ and obtain, if $u(\cdot,t)$ decays fast enough to $\bar u$ at infinity,

$$\frac{d}{dt}\int_{\mathbb{R}^d}\eta(u^\epsilon)dx+\epsilon\omega\int_{\mathbb{R}^d}|B(u^\epsilon)\nabla u^\epsilon|^2dx\leq0.$$

Integrating again with respect to time, we find

$$\sup_{t\geq0}\int\eta(u^\epsilon(x,t))dx\leq\int\eta(u(x,0))dx,\qquad\epsilon\omega\int_0^T dt\int|B(u^\epsilon)\nabla u^\epsilon|^2dx\leq\int\eta(u(x,0))dx.$$

From these estimates, it follows that $u^\epsilon-\bar u$ is bounded in $L^\infty_t(L^2_x)$, while $\epsilon^{1/2}B(u^\epsilon)\nabla u^\epsilon$ is bounded in $L^2_{x,t}$. The latter point is extremely useful, since it implies that $\epsilon B(u^\epsilon)\nabla u^\epsilon$ tends to zero in $L^2_{x,t}$, and therefore the viscous term

$$\epsilon\mathrm{div}(B(u^\epsilon)\nabla u^\epsilon)$$

tends to zero in the distributional sense. Thus if in addition we are able to pass to the limit in the nonlinear flux $f(u^\epsilon)$ (for instance if u^ϵ converges almost everywhere[1]), then we find that the limit of u^ϵ is the solution of (2.3).

TODO : l'inégalité d'entropie à la limite.

4.1.2 Range and kernels.

About the algebraic condition (4.3), it is worth noticing that the viscosity is in general *incomplete*, in the sense that there exists at least one non-trivial linear form ℓ such that $\ell B^{\alpha\beta} = 0$ for every pair of indices α, β. This means that the dissipative system contains at least one non-viscous conservation law:

$$\partial_t \ell(u) + \mathrm{div}(\ell \circ f(u)) = 0.$$

A typical illustration is of course the conservation of mass, which in the Navier-Stokes system remains the same as in the Euler system:

$$\partial_t \rho + \mathrm{div}(\rho v) = \text{ no term here; just zero.}$$

Let us form the symbols

$$B^\alpha(\xi; u) := \sum_\beta \xi_\beta B^{\alpha\beta}(u), \qquad B(\xi; u) := \sum_\alpha \xi_\alpha B^\alpha(\xi; u) = \sum_{\alpha,\beta} \xi_\alpha \xi_\beta B^{\alpha\beta}(u).$$

Using a single vector X and applying (4.3) to $X_\alpha := \xi_\alpha X$, we obtain the necessary condition

$$(\mathrm{D}^2\eta(u)X | B(\xi; u)X) \geq \omega \sum_\alpha |B^\alpha(\xi; u)X|^2, \qquad \forall X \in \mathbb{R}^n. \tag{4.5}$$

Remark that we have been strongly demanding when we asked for a *pointwise* dissipation inequality, where we could have been content with an *integrated* inequality. Passing from the first (strong) to the second (weaker) assumption amounts to replacing a convexity assumption, for the quadratic form

$$F \mapsto \sum_{\alpha\beta} (\mathrm{D}^2\eta(u) F_{\cdot\alpha}, B^{\alpha\beta}(u) F_{\cdot\beta})$$

over $\mathbf{M}_{n\times d}(\mathbb{R})$, by *rank-one convexity*. The latter is precisely (4.5). Recall that a quadratic form $F \mapsto Q(F)$ is rank-one convex if, and only if, the functional

$$v \mapsto \int_{\mathbb{R}^d} Q(\nabla v)\, dx$$

[1]The convergence almost everywhere is of course a very difficult issue, largely open. It is the question of *stability*, in some appropriate topology. What we have discussed so far is merely the *consistency* of the vanishing viscosity method. Consistency is the fact that if the approximate solution converges to something, then the limit is a solution of the limit problem.

is convex over $H^1(\mathbb{R}^d)^n$. Thus (4.5) means that for a constant $\bar{u}$ and arbitrary v, smooth with compact support,

$$\int_{\mathbb{R}^d} \sum_{\alpha\beta} (\mathrm{D}^2\eta(\bar{u})\partial_\alpha v, B(\bar{u})^{\alpha\beta}\partial_\beta v)\, dx \geq \omega \int_{\mathbb{R}^d} \sum_\alpha \left| \sum_\beta B(\bar{u})^{\alpha\beta}\partial_\beta v \right|^2 dx.$$

In many multi-dimensional problems, for instance in models of visco-elasticity, assumption (4.5) is much more realistic than (4.3).

Proposition 4.1.1 *Let the viscous system be entropy-dissipative (in the weaker sense (4.5)). Then*

1. *one has*

$$\ker B(\xi; u) = \bigcap_\alpha \ker B^\alpha(\xi; u), \qquad \forall \xi \in \mathbb{R}^d, \tag{4.6}$$

2. *the spectrum of the symbol $B(\xi; u)$ is contained in the union of the right half-plane $\{z; \mathrm{Re} z > 0\}$ and of the origin $z = 0$,*
3. *the kernel of $B(\xi; u)$ and its range are orthogonal with respect to the scalar product defined by $\mathrm{D}^2\eta(u)$,*
4. *the zero eigenvalue is semi-simple (that is, its multiplicity equals the dimension of the kernel).*

Proof

- The first point follows immediately from (4.5) and the definition of the symbols.
- Decomposing a vector into its real and imaginary parts, we see that we also have

$$\mathrm{Re}(\mathrm{D}^2\eta(u)\bar{X}|B(\xi; u)X) \geq \omega \sum_\alpha |B^\alpha(\xi; u)X|^2, \qquad \forall X \in \mathbb{C}^n.$$

  When X is an eigenvector of $B(\xi; u)$, with λ the eigenvalue, there comes

$$(\mathrm{Re}\lambda)(\mathrm{D}^2\eta(u)\bar{X}|X) \geq \omega \sum_\alpha |B^\alpha(\xi; u)X|^2 \geq 0.$$

  Since $\mathrm{D}^2\eta > 0_n$, the factor $(\mathrm{D}^2\eta(u)\bar{X}|X)$ is positive. There follows $\mathrm{Re}\lambda \geq 0$, with equality only if $B^\alpha(\xi; u)X = 0$ for every α, which implies $B(\xi; u)X = 0$. In this latter case, we have $\lambda = 0$.

- Let Y belong to the kernel, and X be an arbitrary vector. Apply the dissipation (4.5) to $X + sY$. Since we also have $B^\alpha(\xi; u)Y = 0$ (see the previous point), there remains

$$(\mathrm{D}^2\eta(u)(sY + X)|B(\xi; u)X) \geq \omega \sum_\alpha |B^\alpha(\xi; u)X|^2.$$

  Since $s \in \mathbb{R}$ is arbitrary, this implies

$$(\mathrm{D}^2\eta(u)Y|B(\xi; u)X) = 0.$$

  This is the orthogonality of the kernel and the range of $B(\xi; u)$, with respect to the scalar product induced by $\mathrm{D}^2\eta(u)$.

- The orthogonality, plus the fact that $\mathrm{D}^2\eta(u)$ is positive definite, imply that the intersection of the kernel and the range is trivial. Because of dimensionality, this means

$$\mathbb{R}^n = \ker B(\xi, u) \oplus R(B(\xi; u)).$$

  This exactly tells us that 0 is semi-simple.

∎

In particular, the knowledge of either the kernel or the range determines completely the other; if one of both is independent of $\xi \neq 0$, the other one is so, too. Since in practice, the range of $B(\xi; u)$ does not depend on ξ (see the discussion above), we deduce that the kernel does not as well. This was one of Kawashima's assumption in his thesis [23]. This fact is illustrated by the Navier-Stokes system, where the kernel has dimension one (it is the tangent space to the line $\{v = \text{cst}, \theta = \text{cst}\}$) and the null form ℓ is obviously constant, corresponding to the conservation of mass. In general, the kernel does depend on u and ξ, unlike the range (again, see gas dynamics above).

4.2 Viscous models: the nature of dissipated quantities

Physically relevant viscous models not only have a dissipative structure. They also contain a few first-order conservation laws. This means that there exist linear forms (i.e. coordinates) ℓ, such that $\ell B(\xi; u) \equiv 0$ for every state u and frequency ξ. With a change of coordinates, we may always assume that the p first rows of $B(\xi; u)$ are null, so that the system contains the conservation laws

$$\partial_t u_j + \sum_\alpha \partial_\alpha f_j^\alpha(u) = 0, \qquad j = 1, \dots p. \tag{4.7}$$

Of course, system (4.7) is not closed, since the fluxes also involve the components $u_{p+1}, \dots, u_n$. As mentionned above, a typical example is the conservation of mass, in absence of chemical or nuclear process. Another is the conservation of momentum in gas dynamics, if we disregard the Newtonian viscosity. In the sequel, we make the rather natural assumption that the range

$R(B(\xi;u))$ is not only contained in, but also equal to $\{0\}\times\mathbb{R}^{n-p}$, for every non-zero frequency ξ. This will be called *Assumption (L)* in the sequel. It amounts to saying that $B(\xi;u)$ has a constant range for $\xi\neq 0$.

Recall Proposition 4.1.1: $\ker B(\xi;u)$ is $\mathrm{D}^2\eta(u)$-orthogonal to $R(B(\xi;u))$. In other words, when $\xi\neq 0$, $\ker B(\xi;u)$ is defined by the following linear equations

$$\ker B(\xi;u)=\left\{z\in\mathbb{R}^n\,;\,\left(\mathrm{d}\frac{\partial\eta}{\partial u_j}\right)z=0,\,\forall j\geq p+1\right\}.$$

Because of (4.6), we deduce that the kernel of $B^\alpha(\xi;u)$ contains

$$\bigcap_{j=p+1}^{n}\ker\mathrm{d}\frac{\partial\eta}{\partial u_j}\,,.$$

Let $\{\mathbf{v}_1,\dots,\mathbf{v}_n\}$ be the dual basis of

$$\left\{\mathrm{d}\frac{\partial\eta}{\partial u_1}\,,\dots,\mathrm{d}\frac{\partial\eta}{\partial u_n}\right\}.$$

It is formed of the columns of the matrix $\mathrm{D}^2\eta^*$, with η^* the Legendre transform of η. In particular, it depends smoothly on u. Denoting

$$Y_j^\alpha(\xi;u):=B^\alpha(\xi;u)\mathbf{v}_j,$$

we can write

$$B^\alpha(\xi;u)=\sum_{j=1}^{n}Y_j^\alpha\mathrm{d}\frac{\partial\eta}{\partial u_j}=\sum_{j=p+1}^{n}Y_j^\alpha\mathrm{d}\frac{\partial\eta}{\partial u_j}\,.$$

Remarking that

$$Y_j^\alpha(\xi;u)=\sum_{\beta=1}^{d}\xi_\beta Y_j^{\alpha\beta}(u),\qquad Y_j^{\alpha\beta}:=B^{\alpha\beta}\mathbf{v}_j,$$

we find

$$B^{\alpha\beta}(u)=\sum_{j=p+1}^{n}Y_j^{\alpha\beta}\mathrm{d}\frac{\partial\eta}{\partial u_j}\,.$$

Our first conclusion is that the fluxes in the second order terms involve only the first order derivatives of

$$\frac{\partial\eta}{\partial u_{p+1}}\,,\dots,\frac{\partial\eta}{\partial u_n}\,,$$

namely:

$$B^{\alpha\beta}\partial_\beta u=\sum_{j=p+1}^{n}Y_j^{\alpha\beta}\partial_\beta\frac{\partial\eta}{\partial u_j}\,.$$

To summarize, we have

Theorem 4.2.1 *Assume that in an entropy-dissipative system (4.1), the p first rows are first-order conservation laws,*

$$\partial_t u_j + \mathrm{div} f_j(u) = 0, \qquad j = 1, \dots, p,$$

while the symbol $B(\xi; u)$ has rank $n - p$ for every non-zero $\xi \in \mathbb{R}^d$.

Then the second order part can be rewritten in the form

$$\sum_{\alpha,\beta} \sum_{j=p+1}^{n} \partial_\alpha \left(Y_j^{\alpha\beta} \partial_\beta \frac{\partial \eta}{\partial u_j} \right).$$

Comment. This result tells us that the class of systems of the form

$$\begin{aligned}
&(4.8) \qquad & \partial_t v + \mathrm{div} g(v, w) &= 0, \\
&(4.9) \qquad & \partial_t w + \mathrm{div} h(v, w) &= \sum_{\alpha,\beta} \partial_\alpha (C^{\alpha\beta} \partial_\beta w),
\end{aligned}$$

is not relevant from a physical point of view, unless the entropy η splits as the sum of a v-part and a w-part ; that is, unless $\mathrm{d}_v \mathrm{d}_w \eta \equiv 0$. For an isentropic flow, the total energy plays the role of the entropy, and the property above applies in Lagrangian variables but not in Eulerian ones, a rather uncomfortable fact ! For a non-isentropic flow, the property applies only for an ideal gas (the temperature is a function of the internal specific energy e only), and only in Lagrangian variables ; real gases or Eulerian variables cannot be handled within the class of systems (4.8, 4.9).

Consequently, articles which focus on systems of the form (4.8, 4.9) miss the full applicative range that they target. In addition, one may anticipate that their authors encounter mathematical difficulties that would not arise if working about entropy-dissipative systems in the sense that we have considered here.

Ellipticity. We now establish a property of the vectors $Y_j^{\alpha\beta}(u)$. To begin with, they belong to $R(B^{\alpha\beta}(u))$ and therefore their coordinates $Y_{jk}^{\alpha\beta}(u)$ vanish for $k \le p$. Let us next rewrite (4.5) in terms of the tensor Y. We shall employ as usual the symbols $Y^\alpha(\xi; u)$ and $Y(\xi; u)$, which are $n \times n$ matrices. Their columns are Y_j^α and Y_j, respectively, while their entries are Y_{ij}^α and Y_{ij}. We point out that, since $Y_j^{\alpha\beta}$ belongs to the range of $B^{\alpha\beta}$, these entries vanish for $i = 1, \dots, p$: we may write

$$Y_j^{\alpha\beta} = \begin{pmatrix} 0 \\ Z_j^{\alpha\beta} \end{pmatrix}.$$

Since the rank of $B(\xi)$ equals $n - p$, the rank of $\{Z_{p+1}(\xi), \dots, Z_n(\xi)\}$ equals $n - p$ when $\xi \neq 0$.

Making the linear transformation

$$X \mapsto V := \mathrm{D}^2 \eta(u) X,$$

we have

$$B^\alpha(\xi;u)X = \sum_{j\geq p+1} v_j Y_j^\alpha(\xi), \qquad B(\xi;u)X = \sum_{j\geq p+1} v_j Y_j(\xi).$$

The dissipative inequality (4.5) thus writes

$$V\cdot \sum_{j\geq p+1} v_j Y_j(\xi) \geq \omega(u)\sum_\alpha \left|\sum_{j\geq p+1} v_j Y_j^\alpha(\xi)\right|^2, \qquad \forall V\in\mathbb{R}^n,$$

Which amounts to

$$\sum_{i,j\geq p+1} v_i v_j Y_{ij}(\xi) \geq \omega(u)\sum_\alpha \left|\sum_{j\geq p+1} v_j Z_j^\alpha(\xi)\right|^2, \qquad \forall V\in\mathbb{R}^n. \tag{4.10}$$

When $\xi\neq 0$, the right-hand side of (4.10) is a norm for the vector $(v_{p+1},\dots,v_n)$, for if this quantity vanishes, then so does the linear combination

$$\sum_{j\geq p+1} v_j Z_j(\xi).$$

Since $\{Z_{p+1}(\xi),\dots,Z_n(\xi)\}$ is a free family, this implies $v_{p+1}=\cdots=v_n=0$. This means that the right-hand side in (4.10) is a norm for the vector $(v_{p+1},\dots,v_n)^T$. Whence a positive number $c(\xi;u)$ such that

$$\sum_{i,j\geq p+1} v_i v_j Y_{ij}(\xi) \geq c(\xi;u)\sum_{j\geq p+1} v_j^2.$$

By homogeneity of the left-hand side, we deduce that there exists a positive $c_0(u)$ such that

$$\sum_{i,j\geq p+1} v_i v_j Y_{ij}(\xi) \geq c_0(u)|\xi|^2\sum_{j\geq p+1} v_j^2.$$

This is nothing but the *Legendre–Hadamard* condition for the four-indices tensor $Z_{ij}^{\alpha\beta}$. In conclusion, we have

Proposition 4.2.1 *Under the assumptions of Theorem 4.2.1, the tensor Z satisfies the Legendre–Hadamard condition. In other words, the operator*

$$\mathbf{z} \stackrel{L}{\longmapsto} \sum_{\alpha,\beta}\sum_{j=p+1}^{n} \partial_\alpha\left(Z_j^{\alpha\beta}(u)\partial_\beta\frac{\partial\eta}{\partial u_j}\right)$$

is elliptic.

If we make the stronger dissipative assumption (4.2), we obtain that L is *strongly* elliptic, which means that

$$\sum_{\alpha,\beta=1}^{d}\sum_{i,j\geq p+1} F_{\alpha i}F_{\beta j}Z_{ij}^{\alpha\beta}(u) \geq c_0(u)\|F\|^2, \qquad \forall F\in\mathbf{M}_{d\times(n-p)}(\mathbb{R}).$$

The 'equilibrium' case. Let us consider fields for which the dissipation rate

$$\int_{\mathbb{R}^d} \sum_\alpha \left| \sum_\beta B(u)^{\alpha\beta} \partial_\beta v \right|^2 dx$$

vanishes. Such fields $x \mapsto u(x)$ are called *equilibrium states*. With $\mathbf{z}$ as above, our assumptions means that $L\mathbf{z} = 0$. Since L is elliptic, we expect that every bounded solution of $L\mathbf{z} = 0$ is constant, or that the only solution in $L^2(\mathbb{R}^d)^{n-p}$ is the trivial one. Let us give two evidences of this claim.

First of all, in one space dimension, this system is nothing but

$$Z(u)\partial_x \mathbf{z} = 0,$$

where $Z(u)$ is non-singular. Therefore we have $\partial_x \mathbf{z} = 0$ and $\mathbf{z}$ is constant.

Secondly, let us consider the constant coefficient version of $L\mathbf{z} = 0$:

$$\sum_\beta Z^{\alpha\beta}(\bar{u})\partial_\beta \mathbf{z} = 0, \qquad \forall \alpha = 1, \dots, d.$$

Let us assume that $\mathbf{z}$ is a tempered distribution. Then, applying the Fourier transform, we obtain

$$\sum_\beta \xi_\beta Z^{\alpha\beta}(\bar{u})\hat{\mathbf{z}}(\xi) = 0, \qquad \forall \alpha = 1, \dots, d,$$

which implies

$$Z(\xi; \bar{u})\hat{\mathbf{z}}(\xi) = 0.$$

Since $Z(\xi; \bar{u})$ is non-singular for $\xi \neq 0$, we deduce that the support of $\hat{\mathbf{z}}$ is contained in $\{0\}$. In other words, $\mathbf{z}$ must be a polynomial. If $\mathbf{z}$ is bounded, it is therefore a constant, while if it is square-integrable, then it vanishes identically.

To summarize these arguments, we

Claim 4.2.1 *For a state at equilibrium, the expressions*

$$\frac{\partial \eta}{\partial u_j}, \qquad \forall j = p+1, \dots, n$$

do not depend upon the x-variable.

We warn the reader that this conclusion is not the end of the story. The claim does not say anything about the components $u_1, \dots, u_p$, and it does not speak about the dependency upon the time variable. To go forward, we shall need the so-called *Kawashima–Shizuta condition.*

4.2.1 The reduced hyperbolic system

We now introduce a notion that is related to the formal limit of the system

$$\partial_t u + \sum_\alpha \partial_\alpha f^\alpha(u) = \kappa \sum_{\alpha,\beta} \partial_\alpha (B^{\alpha\beta}(u)\partial_\beta u)$$

when $\kappa \to +\infty$. This limit is related to the time asymptotics $t \to +\infty$.

The system being dissipative, we have an estimate

$$\int_{\mathbb{R}} |B(u^\kappa)\nabla_x u^\kappa|^2 dx \leq \frac{1}{\omega\kappa} \int_{\mathbb{R}} \eta(u_0^\kappa)\, dx,$$

where the right-hand side tends to zero if the initial data remains bounded. Thus it is likely, at least formally, that the limit u is non-dissipated, thus takes values in a level set $\{u \,|\, \mathbf{z} = \mu\}$ as described[2] by Claim 4.2.1.

Furthermore, the p first rows of the system do not depend on κ. Thus we expect that the limit satisfies them. Finally, our hope is that $v := (u_1, \dots, u_p)^T$ satisfies the closed system of first-order conservation laws

$$\partial_t v + \sum_\alpha \partial_\alpha F^\alpha(v; \mu) = 0, \tag{4.11}$$

where F is defined by $F(v, \mathbf{z}) := f(u)$ when

$$v = (u_1, \dots, u_p)^T, \qquad \mathbf{z} = \left(\frac{\partial \eta}{\partial u_{p+1}}, \dots, \frac{\partial \eta}{\partial u_n} \right)^T. \tag{4.12}$$

This definition turns out to be meaningful. As a matter of fact, because of the strong convexity of η, the transformation

$$u \mapsto \left(u_1, \dots, u_p, \frac{\partial \eta}{\partial u_{p+1}}, \dots, \frac{\partial \eta}{\partial u_n} \right)^T$$

is a change of variable. Notice that if $u' \neq u$, then

$$\langle \mathrm{d}\eta(u') - \mathrm{d}\eta(u), u' - u \rangle > 0$$

by strict convexity, and the left-hand side is linear in $(v, \mathbf{z})$. Therefore this strict inequality does not permit $(v, \mathbf{z}) = (v', \mathbf{z}')$. In other words, the map $u \mapsto (v, \mathbf{z})$ is one-to-one.

The above analysis suggests the following

Definition 4.2.1 *System (4.11) is called the* reduced hyperbolic system *of the viscous one (4.1).*

This terminology anticipates on the following result, which is due to Boillat & Ruggeri [4].

[2] Here, we go a little beyond the claim, by saying that in the limit, the variable $\mathbf{z}$ is also time-independent.

Theorem 4.2.2 *Let the viscous system (4.1) be as above: strong entropy-dissipation, $B(u)$ having a constant range.*

Then the reduced hyperbolic system admits a strongly convex entropy. In particular, it is Friedrichs symmetrizable, hence hyperbolic.

Proof

Let us denote

$$\lambda_j := \frac{\partial \eta}{\partial u_j}$$

the dual variables. In particular $\lambda_j = z_j$ for $j \geq p+1$. We shall write θ for $(\lambda_1, \dots, \lambda_p)^T$. The Legendre–Fenchel transform η^* of η is a strongly convex function of λ, with

$$\mathrm{D}^2_\lambda \eta^* = (\mathrm{D}^2_u \eta)^{-1}.$$

Recall that we have $u = \mathrm{d}\eta^*(\lambda)$ and that there exists smooth functions $M^\alpha(\lambda)$ such that $f^\alpha(u) = \mathrm{d}M^\alpha(\lambda)$ (actually, $M^\alpha(\lambda) := f^\alpha(u) \cdot \lambda - Q^\alpha(u)$, with Q the entropy-flux). Since $\lambda = (\theta, \mathbf{z})$, the reduced hyperbolic system writes

$$\partial_t(\mathrm{d}\hat{\eta^*}(\theta)) + \sum_\alpha \partial_\alpha(\mathrm{d}\hat{M}^\alpha(\theta)) = 0, \tag{4.13}$$

where $\hat{g}(\theta) := g(\theta, \mu)$. Since the restriction of η^* to the subspace $\mathbf{z} \equiv \mu$ is a strongly convex function, we have our Friedrichs symmetrization:

$$\mathrm{D}^2_\theta \hat{\eta^*} \partial_t \theta + \sum_\alpha \mathrm{D}^2_\theta \hat{M} \partial_\alpha \theta = 0. \tag{4.14}$$

From (4.13), we also deduce an additional conservation law

$$\partial_t(\mathrm{d}\hat{\eta^*} \cdot \theta - \hat{\eta^*}) + \sum_\alpha \partial_\alpha(\mathrm{d}\hat{M}^\alpha \cdot \theta - \hat{M}^\alpha) = 0.$$

With $v = \mathrm{d}\hat{\eta^*}$, the expression $E := \mathrm{d}\hat{\eta^*} \cdot \theta - \hat{\eta^*}$ is nothing but the Legendre–Fenchel transform of $\hat{\eta^*}$, thus is a strongly convex function of v. This is the entropy of the reduced system.

∎

Practical issues.

- From the proof above, we have

  $$E = \mathrm{d}_\theta \eta^* \cdot \theta - \eta^* = \mathrm{d}_v \eta \cdot v - \eta^*,$$

  which yields the explicit formula

  $$E = \eta - \mathrm{d}_w \eta \cdot w, \tag{4.15}$$

  where $w = (u_{p+1}, \dots, u_n)$.

- Let us write blockwise the Hessian matrix of η at $u = (v, w)^T$:

$$\mathrm{D}^2\eta(u) = \begin{pmatrix} s & r^T \\ r & \sigma \end{pmatrix} > 0_n,$$

  with $s \in \mathbf{SPD}_p$. Then the Hessian of the entropy of the reduced system at v, when $\mathrm{d}_w\eta = \mu$ is fixed, is given by the Schur complement $s - r^T\sigma^{-1}r$ of σ. It is a classical fact that since $\mathrm{D}^2\eta$ is positive definite, $s - r^T\sigma^{-1}r$ is so.

The sub-characteristic property. In a hyperbolic first-order system $z_t + A(z)z_x = 0$, infinitesimal disturbances, for instance singularities of derivatives, travel at finite speeds, the characteristic speeds. These are the eigenvalues $a_- := a_1 \leq \cdots \leq a_n =: a_+$ of $A(z)$ and thus are functions of the state z. More generally, for a system written in the form $A_0(z)z_t + A(z)z_x = 0$, the velocities are the generalized eigenvalues of the pair (A_0, A), given by the equation

$$\det(A(z) - a_j(z)A_0(z)) = 0.$$

Weyl's infsup formulæ. If the system is in symmetric form ($A_0(z)$ and $A(z)$ symmetric, with $A_0(z)$ positive definite), the characteristic speeds may be written as Weyl's infsup formulæ of Rayleigh ratio. In particular

$$a_-(z) = \inf_{X\neq 0} \frac{(A(z)X, X)}{(A_0(z)X, X)}, \qquad a_+(z) = \sup_{X\neq 0} \frac{(A(z)X, X)}{(A_0(z)X, X)}. \tag{4.16}$$

Let us apply this principle to the hyperbolic system (2.3)

$$\partial_t u + \sum_\alpha \partial_\alpha f^\alpha(u) = 0,$$

which has the symmetric form

$$\mathrm{D}^2_\lambda \eta^* \partial_t \lambda + \sum_\alpha \mathrm{D}^2_\lambda M^\alpha \partial_x \lambda = 0,$$

and to the reduced system (4.11), in symmetric form (4.14). One passes from the former to the latter by replacing a symmetric pair $(A_0(u), A(\xi; u))$ by the pair $(S_0(u), S(\xi; u))$ of upper-left $p \times p$ blocs. In the Rayleigh ratio, this amounts to restrict to vectors X of the form $(Y, 0)^T$ with $Y \in \mathbb{R}^p$. Making this restriction increases the infimum and lowers the supremum. We therefore obtain the following result, called *sub-characteristic property.*

Proposition 4.2.2 *With the same assumptions as above, the characteritic velocities $a_j(\xi; u)$ of (2.3) and those $b_j(\xi; u)$ of the reduced system (4.11) satisfy the inequalities*

$$a_j \leq b_j \leq a_{j+n-p}, \qquad j = 1, \ldots, p.$$

Conclusion. A strongly entropy-dissipative viscous extension of a given hyperbolic system of conservation laws is far from being arbitrary, since the dissipation can concern only some very special quantities. As an example, we take again the gas dynamics. The entropy, in our mathematical sense, is $\eta = -\rho S$ (S the physical entropy), while the conserved variables are $u = (\rho, m = \rho v, \varepsilon := \frac{1}{2}\rho v^2 + \rho e)$. The above analysis tells that $\partial\eta/\partial\varepsilon$ must be a function of θ only (since the Fourier–Euler system is strongly dissipative), and that similarly $\partial\eta/\partial m$ must be a function of (v, θ) only (since the Fourier–Navier-Stokes system is strongly dissipative). We leave the reader establishing the identity

$$\theta \mathrm{d}\eta = \left(e + \frac{p}{\rho} - \theta S - \frac{1}{2}v^2\right)\mathrm{d}\rho + v\mathrm{d}m - \mathrm{d}\varepsilon,$$

which confirms these assertions. Hint: start from $\theta \mathrm{d}S = \mathrm{d}e + p\mathrm{d}\frac{1}{\rho}$.

If we start from the Fourier–Euler system ($n = 2 + d$ and $p = 1 + d$), the reduced hyperbolic system is the isothermal Euler system. We now calculate the reduced entropy E, constructed in the proof of Theorem 4.2.2. To do so, we employ Formula (4.15),

$$\begin{aligned} E &= \eta - \varepsilon \frac{\partial\eta}{\partial\varepsilon} = -\rho s + \frac{\varepsilon}{\theta} \\ &= \frac{1}{\theta}\left(\rho e - \rho\theta s + \frac{1}{2}\rho|v|^2\right). \end{aligned}$$

We deduce that E is nothing but the mechanical energy, renormalized by the temperature. The internal energy per unit mass is now

$$e_0(\rho; \theta) := e - \theta s,$$

also known as the *free energy*.

4.3 Relaxation

At first glance, a relaxation model writes

$$\partial_t u + \mathrm{div} g(u, v) = 0, \tag{4.17}$$

$$\partial_t v + \mathrm{div} h(u, v) = -\frac{1}{\tau} r(u, v), \tag{4.18}$$

where r is a reaction term that has a damping effect, and $\tau > 0$ is a scaling parameter, a *relaxation time*. Thus dissipation is more or less damping here. Remark that we immediately incorporated the fact that the damped system still contains p conservation laws, those in (4.17).

To understand what damping means, let us consider an initial data that is constant. Then the solution is a function $(u_0, v(t))$ of the time only, with u constant. It is governed by the ODE $\tau\dot{v} = -r(u_0, v)$. We ask that the latter has, for every state u_0, a unique equilibrium $V(u_0)$ (that is $r(u_0, V(u_0)) = 0$), and that this equilibrium is strongly stable:

$$\mathrm{Sp}\left(\mathrm{d}_v r(u_0, V(u_0))\right) > 0. \tag{4.19}$$

The graph Γ_{eq} of $u \mapsto V(u)$ is the *equilibrium manifold*. At last, we define the equilibrium flux by $f_{eq}(u) := g(u, V(u))$. The equilibrium system that we obtain formally as $\tau \to 0+$, is

$$\partial_t u + \mathrm{div} f_{eq}(u) = 0. \tag{4.20}$$

This is the analogue, at the level of relaxation models, of the reduced system which we encountered in viscous models.

We say that (4.17, 4.18) is *entropy-dissipative* if there exists an entropy-flux pair $(E(u,v), Q(u,v))$, with $\mathrm{D}^2 E > 0$, and a positive number $\omega > 0$ such that this system implies formally the inequality

$$\partial_t E(u,v) + \mathrm{div} Q(u,v) + \frac{\omega}{\tau}|r(u,v)|^2 \leq 0. \tag{4.21}$$

Of course, we may replace, at least locally, the dissipative term $|r(u,v)|^2$ by $|v - V(u)|^2$. As in the viscous case, we cannot expect a stronger dissipation.

Dissipativity can be described in algebraic terms. On the one hand, (E, Q) must be an entropy-flux pair of the homogeneous system (that with $r \equiv 0$), which requires differential relations between E, Q, g and h. On the other hand, one must have

$$\mathrm{d}_v E(u,v) r(u,v) \geq \omega |r(u,v)|^2. \tag{4.22}$$

Making a Taylor expansion in v about $V(u)$, we obtain a first- and a second-order conditions at equilibrium. First of all, since $\mathrm{d}_v r(u, V(u))$ is invertible, we must have

$$\mathrm{d}_v E(u, V(u)) = 0. \tag{4.23}$$

Secondly, we have

$$\mathrm{D}_v^2 E_{(u,V(u))}\left(X, \mathrm{d}_v r(u, V(u)) X\right) \geq \omega |\mathrm{d}_v r(u, V(u)) X|^2, \qquad \forall X \in \mathbb{R}^n. \tag{4.24}$$

Notice that (4.24) implies (4.19), *via* an argument similar to the viscous case, but here with the additional fact that zero is not an eigenvalue of $\mathrm{d}_v r$, by assumption.

Once again, Inequality (4.21) is usefull both for well-posedness and the limit $\tau \to 0+$. For let us integrate over $\mathbb{R}^d$, assuming that everything vanishes at infinity. Then

$$\frac{d}{dt}\int_{\mathbb{R}^d} E(u^\tau, v^\tau)\, dx + \frac{\omega}{\tau}\int_{\mathbb{R}^d} |r(u^\tau, v^\tau)|^2 dx \leq 0.$$

Integrating again in time, we obtain that u^τ, v^τ are bounded in $L_t^\infty(L_x^2)$ and that $r(u^\tau, v^\tau)$ is an $O(\tau^{1/2})$ in $L_{x,t}^2$. This amounts to saying that $v^\tau - V(u^\tau)$ is an $O(\tau^{1/2})$ in $L_{x,t}^2$. If moreover u^τ converges almost everywhere, we may pass to the limit in (4.17) and its limit satisfies the equilibrium system (4.20). This is the same kind of consistency that we proved in the viscous case.

TODO : l'inégalité d'entropie à la limite.

Remark. The existence of an equilibrium $V(u)$ can be deduced from the dissipative assumption (4.22). If E, a strictly convex function, is coercive, then the section $v \mapsto E(u,v)$ admits a unique minimum $V(u)$, where $\mathrm{d}_v E(u, V(u))$ vanishes. From Cauchy–Schwarz, we have $\omega|r(u,v)| \le |\mathrm{d}_v E(u,v)|$ and therefore $r(u, V(u)) = 0$. Remark that we proved above the converse: if $r = 0$, then $\mathrm{d}_v E = 0$. Thus, $v \mapsto r(u,v)$ vanishes only once.

4.3.1 The structure of the equilibrium system

Remarkably, the structure of the equilibrium system (4.20) is very much similar to that of the reduced system that we discussed in the viscous case. This is the consequence of a similar set of assumptions. Namely, our relaxation system is entropy-dissipative and it contains a set of p conservation laws (4.17), while the dissipative term r has 'rank' $n-p$.

The first fact is a rephrasing of (4.23). We emphasize this identity in that it is analogous to Claim 4.2.1.

Proposition 4.3.1 *Assume that the system (4.17, 4.18) is entropy-dissipative (E the entropy), with $\mathrm{d}_v r(m)$ of full rank for every m on the equilibrium manifold Γ_{eq}.*

Then Γ_{eq} is a level set of the map $u \mapsto (\partial E/\partial u_{p+1}, \dots, \partial E/\partial u_n)$. More precisely, it is the zero-level set.

The rest is of course completely parallel to the analysis made in Paragraph 4.2. The main result is

Theorem 4.3.1 *Assume that the system (4.17, 4.18) is entropy-dissipative (E the entropy), with $\mathrm{d}_v r(m)$ of full rank at every m on the equilibrium manifold Γ_{eq}.*

Then the equilibrium system (4.20) admits a convex entropy. In particular, it is Friedrichs symmetrizable. Finally, its characteristic velocities span an interval smaller than or equal to the interval spanned by the velocities of the original system (4.17, 4.18).

4.3.2 Kinetic models

We shall not develop much the analysis of kinetic models. Let us mention in passing that kinetic formulations (abstract devices) of scalar equations

$$\partial_t u + \mathrm{div}_x A(u) = 0$$

were designed by Lions, Perthame and Tadmor in such a way that they dissipate every Kruzkhov entropy. The same holds for their BGK counterpart

$$(\partial_t + \mathbf{a}(\xi)\cdot\nabla_x) f = \frac{1}{\tau}(\chi_f(\xi; u(x,t)) - f), \qquad \mathbf{a} := \frac{dA}{du}, \quad u(x,t) := \int_{\mathbb{R}} f(x,t;\xi)\, d\xi,$$

where χ_f is defined by

$$\chi_f(\xi, u) := \begin{cases} +1 & \text{for } 0 < \xi < u, \\ -1 & \text{for } u < \xi < 0. \end{cases}$$

The most important example is however the Boltzman equation, where the dissipation is given by the H-Theorem. The entropy is the opposite of the Kulback functional:

$$\eta(f) := -\int_{\mathbb{R}^3} f \log f \, dv.$$

This theorem is nothing but the identity

$$\partial_t \eta(f) + \operatorname{div}_x \left(\int_{\mathbb{R}^3} f \log f \, v \, dv \right) + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (\cdots) \, dv \, dv' = 0. \tag{4.25}$$

In the last integral above, v, v' stand for the incident velocities of colliding particles, and the quantity in brackets vanishes if, and only if

$$f(x,t,v) f(x,t,v') = f(x,t,v^*) f(x,t,v^{*\prime}) \tag{4.26}$$

for every collision, where the velocities $v*$ and $v^{*\prime}$ must be compatible with the conservation of momentum and kinetic energy in an elastic collision. It is an interesting exercise to show that (4.26) implies that $f(x,t,\cdot)$ is a Gaussian function of v. In the context of the Boltzman equation, we speak of a *Maxwellian*:

$$f(x,t,v) = a(x,t) \exp \frac{|v - \bar{v}(x,t)|^2}{b(x,t)}. \tag{4.27}$$

Hereabove, $\bar{v}$ is the mean velocity and a and b are related to the mass density and the temperature of the fluid.

4.3.3 The time asymptotics

In presence of a strictly dissipative system, we may wish to apply the Lasalle invariance principle that we used for dissipative ODEs. In general, it is hard to justify rigorously this procedure ; in particular, the compactness of trajectories may be a difficult issue. At least, since we need to use an *a priori* estimate, we assume that the initial data is asymptotically constant as $|x| \to +\infty$. Thus we shall remain in this paragraph at a formal level.

The time asymptotics should be described by the solutions of the system, for which there is no dissipation at all. Because the dissipation controls, by assumption, the term responsible for it in the PDEs, it amounts to saying that the solution also solves the system of conservation laws. Let us take a few examples.

A formal argument. Let us consider a system of conservation laws (balance laws in the case of relaxation) with some entropy dissipation. If the solution tends to $\bar{u}$ as $|x| \to +\infty$ fast enough, and the entropy is normalized by $\eta(\bar{u}) = 0$, $\mathrm{d}\eta(\bar{u}) = 0$, then we have by integration

$$\frac{d}{dt} \int_{\mathbb{R}^d} \eta(u(x,t)) \, dx + \int_{\mathbb{R}^d} D[u](x,t) \, dx = 0,$$

where $D[u]$ is the dissipation rate, a non-negative quantity.

A formal application of the Lasalle's invariance principle (which is not justified as long as we lack of a compactness argument) suggests that the (expected) omega-limit set Ω is invariant by the flow of the system, and contained in the zero-set of D. Since the vanishing of D means the vanishing of the dissipative terms (because of (4.22), or (4.3)), we conclude that taking an element $\hat{u}_0 \in \Omega$, the solution $\hat{u}$ of the Cauchy problem with initial data $\hat{u}_0$ satisfies the homogeneous first-order system (2.3), together with contraints. The latter are of the form

$$\mathrm{d}_v \eta \equiv 0,$$

in the relaxation case, or

$$\nabla_x \frac{\partial \eta}{\partial u_j} = 0, \qquad j = p+1, \dots, n,$$

in the viscous case.

In every situation, the resulting system (conservation laws plus constraints) is highly overdetermined and is likely to imply that $\hat{u} \equiv \bar{u}$. This shows that Ω is a singleton, and therfore we can conclude that $u(\cdot, t)$ converges towards the constant state $\bar{u}$. When this is the case, we speak of *convergence to equilibrium*. Of course, even when we can prove the relevance of the Lasalle's principle, it does not tell us at which rate does $u(\cdot, t)$ converge.

We give below a few examples, without justification of the Lasalle's principle.

Jin–Xin relaxation. The conservation laws and the constraint are

$$\partial_t u + \mathrm{div}_x v = 0, \qquad \partial_t v + a^2 \nabla_x u = 0, \qquad v = f(u).$$

Let us assume for the sake of simplicity that we are in one-space dimension. Then the PDEs tell that $v + au = w(x - at)$ and $v - au = z(x + at)$. There remains to understand which functions w and z of one variable may satisfy the functional equation

$$\frac{1}{2}(w(\xi) + z(\eta)) = f\left(\frac{1}{2a}(w(\xi) - z(\eta))\right), \qquad \forall \xi, \eta \in \mathbb{R}.$$

Under the so-called sub-characteristic condition, which tells that $|f'| < a$, this implies that w and z are constants. Therefore the dissipation makes the solutions tend to a constant state, of course an equilibrium.

Broadwell system. The Broadwell system is a kinetic model with a discrete set of velocities. In two space-dimensions, the unknown consists in four densities u_N, u_W, u_E, u_S, which evolve according the differential equations

$$(\partial_t + \partial_x) u_W = \frac{1}{\tau}(u_N u_S - u_E u_W), \qquad (\partial_t - \partial_x) u_E = \frac{1}{\tau}(u_N u_S - u_E u_W),$$
$$(\partial_t + \partial_y) u_S = \frac{1}{\tau}(u_E u_W - u_N u_S), \qquad (\partial_t - \partial_y) u_N = \frac{1}{\tau}(u_E u_W - u_N u_S),$$

with $\tau > 0$ a relaxation time. This is an entropy-dissipative system of PDEs, with the entropy

$$\eta(u) := -(u_N \log u_N + u_S \log u_S + u_E \log u_E + u_W \log u_W).$$

The dissipation balance writes

$$\partial_t \eta(u) + \mathrm{div}\vec{q}(u) = -\frac{1}{\tau}(\log(u_N u_S) - \log(u_E u_W))(u_N u_S - u_E u_W),$$

in which the right-hand side is non-positive and vanishes only when $u_N u_S - u_E u_S$, that is at equilibrium.

The time asymptotics is thus described by those solutions of

$$(\partial_t + \partial_x) u_W = 0, \quad (\partial_t - \partial_x) u_E = 0, \quad (\partial_t + \partial_y) u_S = 0, \quad (\partial_t - \partial_y) u_N = 0$$

which are at equilibrium. We thus have $u_W = f_W(x-t, y)$, ... ,$u_N = f_N(x, y+t)$ with

$$f_W(x-t, y) f_E(x+t, y) = f_S(x, y-t) f_N(x, y+t) \qquad \forall x, y \in \mathbb{R}.$$

This functional equation yields the parametrization

$$u_W(x, y, t) = \exp(a(x+y-t) + b(x-y-t) + c(y)), \ldots$$

where $a, b, c \ldots$ are suitable functions of one variable. We observe that the flow can be quite complicated as $t \to +\infty$, since there remain a lot of freedom degrees.

Viscous systems. We must have, in the asympotic limit, $B(u)\nabla u = 0$. This usually tells that $\nabla_x u$ takes values in an affine space parallel to $\ker B(u; \xi)$ (recall that the latter does not depend on ξ. This, together with the system (2.1), makes an overdetermined system, from which we may extract some simplification.

For instance, let us consider the Navier-Stokes equations with heat diffusion. We find that the velocity v and the temperature θ must be functions of the time variable only. The Euler equations then reduce to

$$\rho_t + v \cdot \nabla \rho = 0, \qquad \rho \partial_t v + \left. \frac{\partial p}{\partial \rho} \right|_\theta \nabla \rho = 0,$$

where the derivative is taken at constant temperature. From this it follows the overdetermined system of PDEs

$$\nabla \left(\rho^{-1} \left. \frac{\partial p}{\partial \rho} \right|_\theta \nabla \rho \right) = 0,$$

and we deduce that ρ is function of t alone, provided it is constant at infinity. Finally, $\rho_t = 0$ and $\partial_t v = 0$, so that the flow become uniform as $t \to +\infty$.

If we drop the Newtonian viscosity instead, though keeping the Fourier law, the dissipation only tells that $\theta = \theta(t)$ asymptotically. From $S_t + v \cdot \nabla S = 0$ (S the physical entropy) and the conservation of mass, we derive both

$$\rho_t + v \cdot \nabla \rho = 0 \qquad \text{and} \qquad \rho \, \mathrm{div} v = 0.$$

Then the conservation of momentum gives

$$\partial_t v + (v \cdot \nabla) v + \rho^{-1} \left.\frac{\partial p}{\partial \rho}\right|_\theta \nabla \rho = 0.$$

We thus have $d+2$ PDEs for only $d+1$ unknowns. Amazingly enough, taking the curl of the last equation yields the identity

$$\partial_t \Omega + (v \cdot \nabla)\Omega = (\Omega \cdot \nabla) v, \qquad \Omega := \operatorname{curl} v.$$

In other words, v is asymptotically the velocity field of an inviscid, incompressible flow ! The overdetermination mentionned above reflects in the fact that the pressure π associated to this flow, which happens as a Lagrange multiplier of the incompressibility constraint, is in fact convected by the flow itself:

$$(\partial_t + v \cdot \nabla)\pi = 0.$$

The Boltzman equation. As discussed above, the non-dissipative solutions must be local Maxwellians, of the form (4.27). Inserting this formula into the remaining part of the system $\partial_t f + v \cdot \nabla_x f = 0$ yields the conclusion that f is a *uniform* Maxwellian, meaning that $a, \bar{v}$ and b are constants. We leave this as an exercise to the reader. Thus the solution of the Boltzman equation is expected to behave as a uniform Maxwellian when $t \to +\infty$, say if the initial data was uniform at infinity. The justification of this picture is still an open problem.

4.4 Dissipation in finite difference schemes

Although the theory has a much wider range, we restrict to the one-dimensional situation for the sake of simplicity. We recall the general form of a conservative finite difference scheme (2.16):

$$\frac{u_j^{m+1} - u_j^m}{\Delta t} + \frac{f_{j+1/2}^m - f_{j-1/2}^m}{\Delta x} = 0, \qquad m \in \mathbb{N},\ j \in \mathbb{Z}.$$

The consistency with (2.3) is ensured by (2.18):

$$f_{j+1/2}^m := F(u_{j+1-p}^m, \dots, u_{j+q}^m), \qquad F(u, \dots, u) \equiv f(u), \qquad \forall u \in \mathcal{U}.$$

We recall that consistency does not preclude of the stability of the scheme. For instance, the centered scheme

$$F(a,b) = \frac{1}{2}(f(a) + f(b))$$

is consistent but highly unstable.

Let us now assume that (2.3) is endowed with a strongly convex entropy η and entropy flux Q.

Definition 4.4.1 *The scheme (2.16) is* entropy-dissipative *if there exists a state function $\psi : \mathcal{U}^{p+q} \to \mathbb{R}$, such that (2.16) implies*

$$\frac{\eta(u_j^{m+1}) - \eta(u_j^m)}{\Delta t} + \frac{\psi_{j+1/2}^m - \psi_{j-1/2}^m}{\Delta x} \le 0, \qquad m \in \mathbb{N},\ j \in \mathbb{Z}, \tag{4.28}$$

with as usual

$$\psi_{j+1/2}^m := \psi(u_{j-p+1}^m, \cdots, u_{j+q}^m).$$

The function ψ is called the numerical entropy flux.

Again for the sake of simplicity, we shall restrict to three point schemes, that is to

$$f_{j+1/2}^m = F(u_j^m, u_{j+1}^m).$$

In this case, we have also

$$\psi_{j+1/2}^m = \psi(u_j^m, u_{j+1}^m).$$

The inequality (4.28) then reduces to

$$\eta(b) + \frac{1}{\lambda}(\psi(a,b) - \psi(b,c)) - \eta\left(b + \frac{1}{\lambda}(F(a,b) - F(b,c))\right) \ge 0, \qquad \forall a,b,c \in \mathcal{U}, \tag{4.29}$$

where we recall $\lambda = \Delta x/\Delta t$. The left-hand side of (4.29) is called the *entropy dissipation rate* of the scheme, and denoted $D(a,b,c)$. The inequality is trivially satisfied as $a = b = c$. If F and ψ are differentiable, a Taylor expansion gives then

$$\mathrm{d}_\ell \psi(b,b) = \mathrm{d}\eta(b)\mathrm{d}_\ell F(b,b), \qquad \ell = 1,2,$$

where d_ℓ denotes the differential of F or ψ with respect to the ℓ-th argument. Defining $\Psi(u) := \psi(u,u)$, we deduce

$$\mathrm{d}\Psi(b) = (\mathrm{d}_1 + \mathrm{d}_2)\psi(b,b) = \mathrm{d}\eta(b)(\mathrm{d}_1 + \mathrm{d}_2)F(b,b) = \mathrm{d}\eta(b)\mathrm{d}f(b) = \mathrm{d}Q(b).$$

Whence Ψ equals Q up to an additive constant. The latter may be fixed at zero. We therefore have the analogue of consistency, but at the level of the entropy flux:

$$\psi(u,u) \equiv Q(u), \qquad \forall u \in \mathcal{U}. \tag{4.30}$$

4.4.1 Dissipation in the Lax–Friedrichs scheme

Recall the Lax–Friedrichs scheme

$$F_{LF}(u,v) := \frac{1}{2}(f(u) + f(v)) + \frac{\lambda}{2}(u - v).$$

It is natural to take

$$\psi_{LF}(u,v) := \frac{1}{2}(Q(u) + Q(v)) + \frac{\lambda}{2}(\eta(u) - \eta(v)).$$

In this case, the dissipation rate depends only on the arguments a and c, but not upon b:

$$D(a,c) = \frac{1}{2}(\eta(a)+\eta(c)) - \eta\left(\frac{a+c}{2}+\frac{1}{2\lambda}(f(a)-f(c))\right)+\frac{1}{2\lambda}(Q(a)-Q(c)).$$

Although it has an explicit algebraic form, it can be justified at a PDE level as follows. Say that u_j^m is known for all $j\in 2\mathbb{Z}$ at a given step m. Then $u(\cdot,m\Delta t)$ is approximated by the step function u^m that takes the constant value u_j^m over the interval $[(j-1)\Delta x,(j+1)\Delta x)$. The Cauchy problem with initial data u^m is then solved during a time laps Δt, the solution being denoted $z^m(x,t)$. This gives an approximation $z(\cdot,\Delta t)$ of u at time $(m+1)\Delta t$, but this approximation is not any more a step function. To recover a step function, we average

$$u_k^{m+1} := \frac{1}{2\Delta x}\int_{(k-1)\Delta x}^{(k+1)\Delta x} z^m(x,\Delta t)\,dx, \qquad \forall k\in 1+2\mathbb{Z}.$$

Remark that the new grid points are shifted by a half-mesh length (here the mesh length is $2\Delta x$, instead of Δx).

The reason why this procedure gives rise to an explicit formula is the following. Applying the Green formula to the divergence-free vector field $(z^m,f(z^m))$, we have

$$u_k^{m+1} = \frac{1}{2}(u_{k-1}^m+u_{k+1}^m)+\frac{1}{2\Delta x}\int_0^{\Delta t}(f(z^m((k-1)\Delta x,t)-f(z^m((k+1)\Delta x,t)))\,dt.$$

However, because we deal with hyperbolic PDEs, it is an experimental fact that the support of a solution propagates at a finite speed, say the maximal wave speed V. In the procedure above, this can be verified whenever we have a *Riemann solver*, that is when we are able to solve the so-called *Riemann problem*. This is the Cauchy problem where the initial data is constant on each side of the origin. One first considers the solution of the Riemann problem between two consecutive states u_{k-1}^m and u_{k+1}^m. It is constant at left and at right of the wedge defined by $|x-k\Delta x|<Vt$. Thanks to this property, the solutions of various Riemann problems as $k\in 1+2\mathbb{Z}$ match along the vertical lines $x=j\Delta x$ for $j\in 2\mathbb{Z}$ as long as $0<t<\Delta t$, and can be glued in order to form the field z^m, provided $\Delta x\le V\Delta t$. As a matter of fact, the function

$$t\mapsto z^m(j\Delta x,t), \qquad j\in 2\mathbb{Z}$$

is constant over $(0,\Delta t)$, equal to u_j^m. Whence the formula

$$u_k^{m+1} = \frac{1}{2}(u_{k-1}^m+u_{k+1}^m)+\frac{1}{2\lambda}(f(u_{k-1}^m)-f(u_{k+1}^m)), \qquad k\in 1+2\mathbb{Z}.$$

Riemann solver yields dissipation. Let us prove that dissipation does occur when a Riemann solver is at our disposal: We may use the function z^m as above. Since η is convex, the Jensen inequality gives

$$\eta(u_k^{m+1}) \le \frac{1}{2\Delta x}\int_{(k-1)\Delta x}^{(k+1)\Delta x}\eta(z^m(x,\Delta t))\,dx, \qquad \forall k\in 1+2\mathbb{Z}.$$

The right-hand side is treated by applying the Green formula to the vector field $(\eta(z^m), Q(z^m))$, whose divergence is non-positive:

$$\begin{aligned}\eta(u_k^{m+1}) &\leq \frac{1}{2\Delta x}\left(\int_0^{\Delta t}\int_{(k-1)\Delta x}^{(k+1)\Delta x}(\partial_t\eta(z^m)+\partial_x Q(z^m))\,dxdt+\int_{(k-1)\Delta x}^{(k+1)\Delta x}\eta(z^m(x,0))\,dx\right.\\ &\quad\left.+\int_0^{\Delta t}(Q\circ z^m((k-1)\Delta x,t)-Q\circ z^m((k+1)\Delta x,t))\,dt\right)\\ &\leq 0+\frac{1}{2}(\eta(u_{k-1}^m)+\eta(u_{k+1}^m))+\frac{\Delta t}{2\Delta x}(Q(u_{k-1}^m)-Q(u_{k+1}^m)).\end{aligned}$$

This is exactly saying that the dissipation rate D is non-negative. Let us point out that it will be positive whenever $u_{k-1}^m \neq u_{k+1}^m$ since then $z^m(\cdot,\Delta t)$ is not constant and the Jensen inequality is strict.

An unexpected inequality. Let us apply the dissipation to the case where the flux f is the gradient of a potential $H : \mathcal{U} \to \mathbb{R}$:

$$f_j(u) = \frac{\partial H}{\partial u_j}.$$

The system (2.3) is hyperbolic, since $\mathrm{d}f = \mathrm{D}^2 H$ is symmetric. In particular, one can choose the entropy-flux pair

$$\eta(u) = \frac{1}{2}|u|^2, \qquad Q(u) = u\cdot f(u) - H(u).$$

When a Riemann solver is at our disposal and the Courant–Friedrichs–Levy condition

$$-\lambda I_n \leq \mathrm{D}^2 H \leq \lambda I_n \tag{4.31}$$

is satisfied, then the dissipation tells us that

$$|f(a)-f(c)|^2 \leq \lambda^2|a-c|^2 + 2\lambda(a-c)\cdot(f(a)+f(c)) + 4\lambda(H(c)-H(a)). \tag{4.32}$$

We do not know how to construct a Riemann solver for general functions satisfying condition (4.31). However, we **conjecture** that this (4.31) still implies the inequality (4.32) for every a and c, with strict inequality if $a \neq c$.

Remarks.

- The implication is trivial if H is a quadratic form (case of a linear system). If the CFL condition is saturated, that is if $\lambda = \rho(S)$, then D vanishes for some non-trivial pairs, namely when $c - a$ is an eigenvector associated to the eigenvalue $\pm\lambda$. This suggests that only the strict CFL condition $\rho(\mathrm{d}f) < \lambda$ be considered in order to have a strict dissipation $D(a,c) > 0$ whenever $c \neq a$.

- The assumption tells that $u \mapsto H(u) \pm \lambda|u|^2/2$ are convex and concave, respectively. Writing that their graphs lie above (resp. below) their tangents yields the inequality

$$H(c) - H(a) \geq \max\left\{(c-a)\cdot f(a) - \frac{1}{2}|c-a|^2, (c-a)\cdot f(c) - \frac{1}{2}|c-a|^2\right\}.$$

  This is however weaker than (4.32).

- Differentiating with respect to c, one obtains

$$\mathrm{d}_c D(a,c) = 2(\lambda I_n - \mathrm{D}^2 H(c))(f(c) - f(a) + \lambda(c-a)).$$

  An analogous formula holds true for $\mathrm{d}_a D(a,c)$. Under the strict CFL condition, the vector $f(c) - f(a) + \lambda(c-a)$ does not vanish, unless $a = c$, because of the strict convexity of $H(u) + \lambda|u|^2/2$. Likewise, the matrix $\lambda I_n - \mathrm{D}^2 H(c)$ is non-singular. Therefore $\mathrm{d}_c D(a,c) \neq 0$ for $c \neq a$. Likewise, $\mathrm{d}_a D(a,c) \neq 0$.

  On an other hand, D is positive in a neighbourhood of $c = a$, because $\lambda^{-1} f$ is a strict contraction and the remainder

$$(a-c)\cdot(f(a) + f(c)) + 2(H(c) - H(a))$$

  is an $O(|c-a|^3)$. Therefore the inequality (4.32) would follow if one had a control (a kind of Palais–Smale condition) of $D(a,c)$ and its gradient when a or c tends to the boundary of the domain $\mathcal{U}$, for instance if a or c tends to infinity. This is however an indirect way to prove (4.32). One could prefer a direct calculation in the style of those in convex analysis.

4.4.2 The Godunov scheme and its dissipation

The Godunov scheme ressembles very much the Lax–Friedrichs one. At each step $m\Delta t$, the solution is approximated by a step function, then the system (2.3) is solved exactly, by gluing solutions of Riemann problems, yielding a field $z^m(x,t)$. After a time laps Δt, this field is projected onto step functions, by averaging on meshes. The only difference is that the grid at time $(m+1)\Delta t$ is the same as the one at time $m\Delta t$ (for the Lax–Friedrichs scheme, it was shifted by a half-mesh length[3]).

Since we do not deal any more with half-mesh length, we may go back to our original notation, where Δx is the mesh-length and u_j^m is defined for every $m \in \mathbb{N}$ and $j \in \mathbb{Z}$ (instead of $m + j \in 2\mathbb{Z}$ for the Lax–Friedrichs scheme). Thus z^m is the solution of the Cauchy problem with initial data piecewise constant, equal to u_j^m over $I_j = ((j-1/2)\Delta x, (j+1/2)\Delta x)$. We build z^m by gluing the solutions of the Riemann problems between consecutive states u_j^m and u_{j+1}^m as j runs over $\mathbb{Z}$. The gluing yields an exact solution of (2.3) whenever the solutions of the Riemann problems do not interact along the mid-lines $x = j\Delta x$. Since waves travel at speeds that are roughly eigenvalues of the Jacobian matrices $\mathrm{d}f$, this requires the CFL condition

$$\Delta t \leq \frac{\rho(\mathrm{d}f)}{2}\,\Delta x.$$

[3]In French: “pour le schéma de Lax-Friedrichs, deux grilles successives sont *en quinconce*”.

This condition is however too restrictive ; it has long been observed that an interaction of neighbouring Riemann problem is harmless, provided it does not affect the constancy of z^m along the lines $x = (j+1/2)\Delta x$. This only requires the CFL condition

$$\Delta t \leq \rho(\mathrm{d}f)\,\Delta x.$$

The updated value u_j^{m+1} is calculated following the same ideas as in Paragraph 4.4.1: one writes the Green formula associated to the conservation laws satisfied by z^m, in the domain $I_j \times (0, \Delta t)$. We use the fact that z^m is constant along the lines $x = (j+1/2)\Delta x$; we denote the corresponding values $u_{j+1/2}^m$. This yields the formula

$$u_j^{m+1} = u_j^m + \lambda(f(u_{j-1/2}^m) - f(u_{j+1/2}^m)). \tag{4.33}$$

The state $u_{j+1/2}^m$ is the value $U(0)$ of the solution $U((x-(j+1/2)\Delta x)/t)$ of the Riemann problem between u_j^m and u_{j+1}^m. For this reason, one often denotes

$$u_{j+1/2}^m = R(u_j^m, u_{j+1}^m; 0).$$

Remark that it may happen that $U(y)$ be discontinuous at $y = 0$. If this is the case, it seems that $u_{j+1/2}^m$ is not well-defined: should we take the limit at left u_-, or the limit at right u_+ ? This ambiguity is actually harmless: thanks to the Rankine-Hugoniot condition, we have $f(u_-) = f(u_+)$. This shows that the value of $f(u_{j+1/2}^m)$ involved in (4.33) does not depend on the choice we make between u_- and u_+.

We notice that the Godunov scheme requires more work than the Lax–Friedrichs scheme, since one has to solve Riemann problems. For this reason, we can apply the scheme only to those systems for which the Riemann solver is known in closed form. The extra work is rewarded by the fact that the Godunov scheme is much less dissipative than that of Lax–Friedrichs, which is a important feature for numerical simulations.

Dissipation of the Godunov scheme. The same calculation as in Paragraph 4.4.1 works out. Thanks to the Jensen inequality, the Godunov scheme is dissipative. However, since we need to solve Riemann problems, the analysis of the dissipation rate is not so easy. At first glance, we may say that D does depend on the three arguments a, b and c.

Chapter 5

The linear Cauchy problem with constant coefficients

Before going into more details about the nonlinear models, we need to understand basic questions such as well-posedness of the Cauchy problem and time asymptotics, at the linear level with constant coefficients. We shall use often the Fourier transform in the space variable, which we denote by $\mathcal{F}$. We also write $\hat{g}$ for $\mathcal{F}g$.

5.1 First-order systems and relaxation

5.1.1 Ordinary well-posedness

Let us consider a system of the form

$$\partial_t u + \sum_{\alpha=1}^{d} A^\alpha \partial_\alpha u + Bu = 0, \tag{5.1}$$

where A^α, B are $n \times n$ matrices with real entries. We consider the Cauchy problem where the initial data is

$$u(x,0) = a(x), \qquad \forall x \in \mathbb{R}^d.$$

We say that the Cauchy problem is *strongly well-posed* in $H^s(\mathbb{R}^d)$ if, for every initial data in this space, there exists a unique solution in the class $\mathcal{C}^0(0,T;H^s) \cap \mathcal{C}^1(0,T;H^{s-1})$. By using the isomorphism $\Lambda^s : H^s \to L^2$, defined by $\mathcal{F}(\Lambda^s g) = (1+|\xi|^2)^{s/2}\hat{g}$, we easily see that the wel-posedness in H^s is equivalent to that in L^2, so that we shall concentrate on the latter. In case of well-posedness, we have a bounded operator $S_t : a \mapsto u(\cdot,t)$. The set $(S_t)_{t\in\mathbb{R}}$ is a one-parameter group, with a property of pointwise (in L^2) continuity: for every $a \in L^2$, the map $t \mapsto S_t u$ is continuous from $\mathbb{R}$ with values in L^2. Note however that the map $t \mapsto S_t$ is *not* continuous in the operator norm, since such a property happens only for ODEs, and never in PDEs.

A useful remark is that if the Cauchy problem is well-posed, then we can solve as well the problem

$$\partial_t u + \sum_{\alpha=1}^{d} A^\alpha u + Bu = f(x,t), \qquad u(x,0) \equiv 0$$

by means of the Duhamel formula:

$$u(t) = \int_0^t S_{t-s} f(s)\, ds, \tag{5.2}$$

where we now denote $g(s)$ for $g(\cdot, s)$. The integral is convergent as soon as $f \in L^1(0,T;L^2(\mathbb{R}^d))$.

An important consequence is that the well-posedness of the Cauchy problem does not depend at all on the matrix B. The reason is a classical argument from ODEs: if it is well-posed for the matrix B, then one may solve (5.1) with B_1 instead, by applying a fixed point argument to the integral formula, deduced from the Duhamel principle:

$$u(t) = S_t a + \int_0^t S_{t-s}(B_1 - B)u(s)\, ds.$$

The right-hand side is a linear contraction in $\mathcal{C}(0,T;L^2)$ whenever T is small enough. Thus there exists a unique local solution. Since the smallness of T does not depend on the size of the data, we have an other solution on $(T, 2T)$, etc... Finally, the solution is global.

We are therefore led to the study of the Cauchy problem when $B = 0_n$. To this end, we employ the Fourier transform, which convert (5.1) into an ODE in time, parametrized by the frequency ξ:

$$\partial_t \hat{u}(\xi,t) + iA(\xi)\hat{u} = 0.$$

Hereabove, the symbol is defined by

$$A(\xi) := \sum_\alpha \xi_\alpha A^\alpha.$$

The solution of the ODE, with initial data $\hat{a}(\xi)$, is obviously

$$\hat{u}(\xi,t) = \exp(-itA(\xi))\hat{a}(\xi).$$

Since $\mathcal{F}$ is an automorphism of L^2, saying that $a \mapsto u(t)$ is a bounded operator over L^2 amounts to saying that

$$b \mapsto [\xi \to \exp(itA(\xi))b(\xi)]$$

is such an operator. Since this is just a pointwise multiplication (by a matrix-valued function), this is equivalent to saying that this function is bounded. Therefore well-posedness is equivalent[1] to the property that for every $t > 0$, one has

$$\sup_{\xi\in\mathbb{R}^d} \| \exp(-itA(\xi))\| < \infty.$$

This actually is independent of $t \neq 0$ because of the linearity $tA(\xi) = A(t\xi)$. Whence the statement

[1] To be complete, we have to prove the continuity in time with values in L^2. This is a consequence of the dominated convergence Theorem.

Theorem 5.1.1 *The Cauchy problem for (5.1) is well-posed (in one or in each of the H^s) if and only if there holds*

$$\sup_{\xi \in \mathbb{R}^d} \| \exp(iA(\xi)) \| < \infty. \tag{5.3}$$

Under this condition, it is actually well-posed forward and *backward in time.*

The property (5.3) is called *strong hyperbolicity.* We shall just say *hyperbolicity.*

It is not always easy to check whether a given first-order system is hyperbolic or not. A well-known necessary condition is the fact that for every $\xi \in \mathbb{R}^d$, the matrix $A(\xi)$ is diagonalizable with real eigenvalue. Unfortunately, although this is also sufficient in one space dimension ($d = 1$), it is not in higher dimension. Kreiss gave a characterization, which is not extremely practical however:

Theorem 5.1.2 *A first-order operator $\partial_t + A(\nabla_x)$ is hyperbolic if, and only if its symbol $A(\xi)$ is diagonalisable ($A(\xi) = P(\xi)^{-1}D(\xi)P(\xi)$, $P(\xi)$ the matrix of eigenvectors), with real eigenvalues, and with the property that $\xi \mapsto \|P(\xi)^{-1}\| \, \|P(\xi)\|$ is bounded over $\mathbb{R}^d$ (this is called* uniform diagonalization*).*

For applied mathematicians, the life is a bit easier. This because there are two important classes of first-order operators for which the hyperbolicity is automatic. The first one is that of Friedrichs-symmetric systems, for which the matrices A^α are symmetric. Then $P(\xi)$ may be choose orthogonal, and $\xi \mapsto \|P(\xi)^{-1}\| \, \|P(\xi)\| \equiv 1$. More generally, there are the *symmetrizable* systems, for which there exists a symmetric positive definite S_0 such that every product $S_0 A^\alpha =: S^\alpha$ is symmetric. This amounts to saying that the corresponding system admits a quadratic entropy:

$$\partial_t(S_0 u, u) + \sum_\alpha \partial_\alpha(S^\alpha u, u) = 0.$$

In this case, the *a priori* estimate follows immediately and there is no need of the Fourier transform.

The second important class is that of *strictly hyperbolic* operators, defined as those for which the eigenvalues of $A(\xi)$ are real and simple for every $\xi \neq 0$. Then we may choose $P(\xi)$ continuous and homogeneous of degree zero, thus bounded as well as its inverse. In physical examples, strict hyperbolicity is rare[2], but the argument applies to the more general class that we commonly encounter, of *constantly hyperbolic* operators. In this class, we require that $A(\xi)$ is diagonalizable with real eigenvalues of $A(\xi)$, and that their multiplicities form a set of integers $(m_1, \dots, m_s)$ that does not depend on $\xi \neq 0$.

We remark finally that when $B = 0_n$, the operators S_t are uniformly bounded as t varies. Actually, the invariance under space-time dilations implies that for $s, t \neq 0$, S_t and S_s are conjugated by an isometry, and therefore $\|S_t\|_{\mathcal{L}(L^2)}$ does depend on $t \neq 0$. When $B \neq 0_n$, things are different and we only may say that there exist two numbers c and ω such that

$$\|S_t\|_{\mathcal{L}(L^2)} \leq c e^{\omega |t|}.$$

This is a standard fact in semi-group theory.

[2]Lax showed that strictly hyperbolic operators do not exist for some pairs (n, d).

5.1.2 Relaxation models: uniform well-posedness

A linear relaxation model is nothing but a special kind of first-order system (5.1), where B is a singular matrix. The eigenforms associated to the null eigenvalue define *conserved quantities* $\ell(u)$. Relaxation means that in addition the matrix B has a damping effect, which is observed when time increases. Separating the conserved variables (here denoted as v) from the "damped" ones (denoted as w), and introducing a relaxation time scale τ, we rewrite the system in the form

$$\partial_t v + C(\nabla_x)v + D(\nabla_x)w = 0, \tag{5.4}$$

$$\partial_t w + E(\nabla_x)v + F(\nabla_x)w = -\frac{1}{\tau}Kw. \tag{5.5}$$

We have blockwise

$$A(\xi) = \begin{pmatrix} C(\xi) & D(\xi) \\ E(\xi) & F(\xi) \end{pmatrix}, \qquad B = \begin{pmatrix} 0_p & 0 \\ 0 & K \end{pmatrix},$$

where K is non-singular. So far, we only have assumed that 0 is a semi-simple eigenvalue of B. We shall soon give a justification of that assumption.

As far as well-posedness at fixed $\tau > 0$ is concerned, Theorem 5.1.1 gives the answer: it happens if and only if the operator $\partial_t + A(\nabla_x)$ is hyperbolic. Thus we shall always assume hyperbolicity in the sequel.

In relaxation theory, we actually ask more. At least, we want the L^2-stability of the forward Cauchy problem as $\tau \to 0+$. This amounts to saying that for fixed time $t > 0$

$$\sup_{0<\tau<1} \|S_t^\tau\|_{\mathcal{L}(L^2)} < \infty.$$

Hereabove, the superscript τ recalls that the semi-group is associated with the parameter τ that appears in the system. Remark that this *uniform well-posedness* will hold only forward, but not backward, contrary to the unparametrized well-posedness.

Thanks to the Fourier transform, the uniform well-posedness is equivalent to

$$\sup_{0<\tau<1} \sup_{\xi\in\mathbb{R}^d} \| \exp(-t\left(\tau^{-1}B + iA(\xi)\right))\| < \infty,$$

for every positive t. Posing $\eta := t\xi$ and $s := t/\tau$, we see that this is equivalent to the uniform-in time stability at given parameter, say $\tau = 1$:

$$\sup_{s>1} \sup_{\eta\in\mathbb{R}^d} \| \exp(-s\,(B + iA(\eta)))\| < \infty \tag{5.6}$$

However, since hyperbolicity ensures well-posedness, which is always uniform on bounded time intervals, the first supremum in (5.6) may be taken over all of $s > 0$.

We have seen in the previous section a case of uniform well-posedness, when the system above admits a weakly dissipative entropy, here a quadratic one. We leave to the reader the derivation of stability from the *a priori* estimate given by such an entropy. But since a general system might have no entropies, or at least no dissipative one, it is useful to find either necessary or sufficient criteria for the uniform well-posedness. We begin with a necessary condition.

Theorem 5.1.3 *Assume that the system (5.4, 5.5) is uniformly well-posed in L^2 (or in H^s, this is equivalent). Then the* reduced operator $\partial_t + C(\nabla_x)$ *is hyperbolic.*

Proof

Since $\mathbb{C}^n = \mathbb{C}^p \oplus \mathbb{C}^{n-p}$ splits into the direct sum of invariant subspaces of B associated to distinct parts of the spectrum, there is a neighbourhood of B in $\mathbf{M}_n(\mathbb{C})$ in which every matrix B' has invariant subspaces $N(B')$ and $G(B')$. These spaces depend analytically on B', which means that the corresponding eigenprojectors $P(B')$ and $I_n - P(B')$ are analytic in B'. Besides, $N(B) = \mathbb{C}^p \oplus \{0\}$ and $G(B) = R(B) = \{0\} \oplus \mathbb{C}^{n-p}$. Because of $[P(B'), B'] = 0_n$ and of $P(B')^2 = P(B')$, one has $P(B')B'^k P(B') = (P(B')B'P(B'))^k$ for every $k \in \mathbb{N}$, and more generally $P(B')f(B')P(B') = f(P(B')B'P(B'))$ for every holomorphic function f. In particular,

$$P(B')(\exp B')P(B') = \exp(P(B')B'P(B')). \tag{5.7}$$

Let M denote the left-hand side in (5.6). For every positive time s, frequency ξ, we have

$$\|\exp(-s\,(B + iA(\xi)))\| \le M.$$

For ξ small enough, $B' := B + iA(\xi)$ is as above, and we denote $N(\xi)$ instead of $N(B + iA(\xi))$, and so on. If X is in $N(\xi)$, then (5.7) and the above inequality yield

$$\|\exp(-sP(\xi)\,(B + iA(\xi))\,P(\xi))\| \le M\|P(\xi)\|^2. \tag{5.8}$$

We now make the rescaling $\eta := s\xi$. We keep η fixed and let $s \to +\infty$, meaning that ξ tends to 0. The right-hand side of (5.8) tends to M. On the one hand,

$$sP(\xi)iA(\xi)P(\xi) = iP(\xi)A(\eta)P(\xi) \longrightarrow iP(0)A(\eta)P(0) = \begin{pmatrix} iC(\eta) & 0 \\ 0 & 0_{n-p} \end{pmatrix}.$$

On the other hand, we now that $P(0)B = BP(0) = 0_n$, and therefore both $P(\xi)B$ and $BP(\xi)$ are $O(s^{-1})$. Since we can find a matrix R such that $B = BRB$, we have

$$sP(\xi)BP(\xi) = s(P(\xi)B)R(BP(\xi)) = O(s^{-1}) \longrightarrow 0_n.$$

Passing to the limit in (5.8), we obtain therefore

$$\|\exp(-iC(\eta))\| \le M, \qquad \forall \eta \in \mathbb{R}^d,$$

and this is the hyperbolicity of $\partial_t + C(\nabla)$.

∎

Remark that the above argument can be carried out for every invariant subspace $\mathcal{B}$ of B associated to some part of the spectrum. However the conclusion will be different: the restriction (defined through an eigenprojector and a natural embedding of $\mathcal{B}$) of (5.4, 5.5) must be uniformly well-posed. When $\mathcal{B}$ corresponds to a pair of pure imaginary eigenvalues[3], this is quite a complicated condition.

[3] This situation occurs for instance in the equations of a rotating fluid, where Bu is the effect of the Coriolis force.

Of course the reader must be unsatisfied at this stage, since we have only treated uniform stability, which is not as a strong property as what we should expect in relaxation. For instance, a Coriolis force passes our test, although it conserves the total energy and thus is not exactly relaxing.

5.1.3 The sub-characteristic property

We thus have a pair of hyperbolic operators, of which H is somehow a restriction of L. We want to compare their wave velocities.

Let us begin with the system $Lu = 0$. If the initial data u_0 depends only upon one coordinate $y := x \cdot \xi$ (ξ a unit vector), then it is natural to look for a solution of the form $u(x, t) = U(y, t)$. Then we have to solve the one-dimensional Cauchy problem

$$(\partial_t + A(\xi)\partial_y)U = 0, \qquad U(y, 0) = U^0(y).$$

Let $a_1, \dots, a_n$ be the eigenvalues and $\{r_1, \dots, r_n\}$ the corresponding eigenbasis of $A(\xi)$. We decompose both the data and the solution over this basis:

$$U^0(y) = \sum_j \phi_j(y) r_j, \qquad U(y, t) = \sum_j \alpha_j(y, t) r_j.$$

Then we must solve

$$(\partial_t + a_j \partial_y)\alpha_j = 0, \qquad \alpha_j(y, 0) = \phi_j(y).$$

This is done explicitly with the formula $\alpha_j(y, t) = \phi_j(y - a_j t)$, whence

$$u(x, t) = \sum_j \phi_j(x \cdot \xi - a_j t) r_j.$$

This shows that the eigenvalues a_j are wave velocities for the operator L. In particular, we have a quantitative estimate of the propagation of the support:

$$\operatorname{Supp} U(t) \subset \operatorname{Supp} U^0 + t[a_-, a_+], \tag{5.9}$$

where $a_\pm$ denote the smallest and the largest eigenvalue of $A(\xi)$.

Likewise, the wave velocities of the operator H in the direction ξ are the eigenvalues $c_1, \dots, c_p$ of $C(\xi)$. We denote $c_\pm$ the extreme ones. Our goal below is to compare $c_\pm$ with $a_\pm$. We shall say that the relaxed system obeys the *sub-characteristic property* if both L and H are hyperbolic, and $[c_-, c_+] \subset [a_-, a_+]$.

We point out that the estimates (5.9) is accurate, in the sense that one cannot replace $[a_-, a_+]$ by a smaller interval: it is enough to choose U_0 so that $\phi_\pm$ are non-zero.

Symmetric case. Recall that a linear entropy-dissipative model (the entropy being a positive definite quadratic form) has the property of uniform well-posedness (this follows immediately from the *a piori* estimate). When one of the models is entropy-dissipative, for the quadratic entropy $\eta(u) := u^T S u$, the change of unknown $z := S^{1/2} u$ yields an equivalent system in which the new symbol $\tilde{A}(\xi) = S^{1/2}A(\xi)S^{-1/2}$ is a symmetric matrix. Thus we may suppose that $A(\xi)$ is symmetric. Since $C(\xi)$ is a principal sub-matrix, it is symmetric too. The symmetry ensures trivially that both L and H are hyperbolic, but it gives more. Here, sub-characteristicity follows from the characterization of symmetric eigenvalues by the Rayleigh ratio: the eigenvalues of $C(\xi)$ are those of $A(\xi)$ satisfy an even more precise set of inequalities:

$$a_j \le c_j \le a_{j+n-p}.$$

Let us illustrate (5.11) with gas dynamics, in which we take in account the heat diffusion but not the Newtonian viscosity. The linear system is obtained by linearization about a constant flow. It inherits the entropy-dissipativity from the nonlinear system. The operator L is that of the linearized Euler equations, with velocities $v \cdot \xi, v \cdot \xi \pm c_{ad}|\xi|$ in direction ξ, with v the material velocity and c_{ad} the adiabatic sound speed. The operator H is that of the Euler system for isothermal flows. Its velocities are $v \cdot \xi, v \cdot \xi \pm c_{iso}|\xi|$ with now c_{iso} the isothermal sound speed. The subcharacteristic property tells that the isothermal sound speed is always smaller than the adiabatic sound speed

$$c_{ad}(\rho, e) \le c_{iso}(\rho, e). \tag{5.10}$$

Actually, the inequality is strict, this because $A(\xi)$ is not block diagonal (in the sense that $D(\xi)$ is non-zero). In the case of an ideal fluid, where the equation of state is $p(\rho, e) = (\gamma - 1)\rho e$, $\gamma > 1$ the adiabatic constant, we have

$$c_{iso} = \sqrt{(\gamma - 1)e}, \qquad c_{ad} = \sqrt{\gamma(\gamma - 1)e}.$$

Observe that a gas with γ close to one, typically a gas of which the molecules are complex with many possible deformations, behaves almost as an isothermal one, since $c_{iso} \sim c_{ad}$.

General case. We drop the assumption of entropy-dissipativity and retain only that of uniform well-posedness.

Proposition 5.1.1 *Assume that the Cauchy problem for the relaxation system (5.4,5.5) is uniformly well-posed. Then the extreme eigenvalues of the symbols $A(\xi)$ and $C(\xi)$ satisfy the sub-characteristic property*

$$a_- \le c_- \le c_+ \le a_+. \tag{5.11}$$

This statement follows from the convergence of the solution as the relaxation time goes to zero (singular limit):

Theorem 5.1.4 *Assume that the Cauchy problem for the relaxation system (5.4,5.5) is uniformly well-posed. Given an initial data $u_0 = (v_0, w_0)$ in $L^2(\mathbb{R}^d)$, denote by u^τ the solution associated to the relaxation time $\tau > 0$. As $\tau \to 0^+$, u^τ converges weakly in L^2 towards $(v, 0)$, where v is the solution of the Cauchy problem for the reduced system*

$$Hv = 0, \qquad v(\cdot, 0) = v_0.$$

Proof

By uniform well-posedness, the sequence u^τ is bounded in $\mathcal{C}(0,T;L^2)$. Thus it is relatively compact for the weak-star topology of $L^\infty(0,T;L^2)$. We have

$$w^\tau = -\tau R^{-1}(\partial_t w^\tau + \cdots),$$

in which the parenthesis is bounded as a distribution. Thus w^τ converges weakly to zero. Restricting to a converging subsequence with limit $(v,0)$, we may pass to the limit in the equation (5.4) and obtain $Hv = 0$. We prove now that v satisfies the expected initial condition. For every $\phi \in \mathcal{D}(\mathbb{R}^{d+1})$, we have

$$\int_0^{+\infty}\int_{\mathbb{R}^d}\left(v^\tau(\partial_t\phi + C^T(\nabla_x)\phi) + w^\tau D^T(\nabla_x)\phi\right)\,dxdt + \int_{\mathbb{R}^d}\phi(x,0)v_0(x)\,dx = 0.$$

We easily pass to the limit in each terms since u^τ converges weakly in L^2. Therefore we obtain

$$\int_0^{+\infty}\int_{\mathbb{R}^d} v\left(\partial_t\phi + C^T(\nabla_x)\phi\right)\,dxdt + \int_{\mathbb{R}^d}\phi(x,0)v_0(x)\,dx = 0,$$

which is the rigorous formulation for the initial condition $v(\cdot,0) = v_0$. Thus v is an $L^\infty_t(L^2)$-solution of the reduced Cauchy problem. The Fourier transformation can be applied in this class and we obtain that v is the unique solution of class $\mathcal{C}(0,T;L^2)$. Because of this uniqueness, the convergence holds for the whole sequence u^τ, and not only a subsequence.

■

To deduce the Proposition 5.1.1, it is enough to consider the one-dimensional system for the planar waves in direction ξ. Our assumption, written at the level of the symbols and their exponentials, contains the fact that the Cauchy problem for planar waves is uniformly well-posed too, in $L^2(\mathbb{R})$. In this context, we have $L = \partial_t + A\partial_x$ and $H = \partial_t + C\partial_x$. Theorem 5.1.4 applies in this framework. Given v_0 in $L^2(\mathbb{R})$, we set $w_0 \equiv 0$ and $u_0 := (v_0, 0)$.

Lemma 5.1.1 *Assume that v_0 has a compact support, in an interval I. Then the solution u^τ has a compact support at every time $t > 0$, included in $I + t[a_-, a_+]$.*

The limit u of u^τ inherits this property. Thus the support of the solution of the Cauchy problem $Hv = 0$, $v(0) = v_0$ is contained in $I + t[a_-, a_+]$. Since this is true for every data v_0 with support in I, and since (5.9) is accurate, we deduce that $[a_-, a_+]$ contains $[c_-, c_+]$ (remark that v may be computed explicitly by integration along characteristics).

There remains to prove Lemma 5.1.1.

Proof

Let $(T_t)_{t\geq 0}$ be the semi-group generated by L. It satisfies (5.9). We may fix $\tau = 1$ for instance. Recall that u solves the fixed point problem

$$u(t) = T_t u_0 - \int_0^t T_{t-s}Bu(s)\,ds,$$

and that it can be obtained by a Picard iteration: u is the limit of u^m, where $u^0 \equiv 0$ and

$$u^{m+1}(t) = T_t u_0 - \int_0^t T_{t-s} B u^m(s)\, ds,$$

Because of (5.9), the property that

$$\operatorname{Supp} u^m(t) \subset \operatorname{Supp} u_0 + t[a_-, a_+]$$

implies the same at the order $m+1$. Therefore it is true for every $m \in \mathbb{N}$. Passing to the limit, it is also satisfied by u.

■

5.2 Viscous systems

5.2.1 Ordinary well-posedness

In this paragraph, we consider systems that are partially of second-order:

$$\partial_t v + C(\nabla_x)v + D(\nabla_x)w = 0, \tag{5.12}$$

$$\partial_t w + E(\nabla_x)v + F(\nabla_x)w = \sum_{\alpha,\beta} b^{\alpha\beta}\partial_\alpha\partial_\beta w, \tag{5.13}$$

in which the symbol $b(\xi)$ is non-singular for every $\xi \neq 0$. In this setting, we have *a priori* assumed that both the kernel and the range of $B(\xi)$ are independent of $\xi \neq 0$ and that zero is a semi-simple eigenvalue, hypotheses that we have proved reasonnable in presence of a dissipative entropy. Then, thanks to linearity, we have selected coordinates (v, w) in such a way that the symbol $B(\xi)$ be in block-diagonal form. We recall that when the system has a dissipative entropy (here a quadratic one), this is equivalent to require only that the range is independent of ξ. Thus we do not violate the likely physics that such a system could govern.

As usual, our main concern here is the forward L^2-well posedness. It is however a non-trival question here, since the operator in the right-hand side is not bounded. Applying again the Fourier transform, we find that well-posedness amounts to having

$$\sup_{\xi \in \mathbb{R}^d} \| \exp(-t(iA(\xi) + B(\xi)))\| < \infty, \qquad \forall t > 0. \tag{5.14}$$

More precisely, the left-hand side must be bounded by some $ce^{\omega t}$.

As in the previous paragraph, the well-posedness followed from the existence of a dissipative entropy ; once again, we leave the reader establish the *a priori* estimates. The latter property however is not necessary, as shown by the following fundamental result.

Theorem 5.2.1 *Assume that the Cauchy problem for (5.12, 5.13) is well-posed in L^2. Then the operator $H := \partial_t + C(\nabla_x)$ is hyperbolic.*

Conversely, assume that H is hyperbolic and that the operator $P := \partial_t - b(\nabla_x)$ is parabolic (this is characterized by the property that there exists a $\theta > 0$ such that the eigenvalues of the symbol $b(\xi)$ have a real part larger than or equal to $\theta|\xi|^2$). Then the Cauchy problem for (5.12, 5.13) is well-posed in L^2 (actually in every H^s).

Proof
There are two parts, a necessary and a sufficient conditions.

Necessity is that for the well-posedness to happen, H must be hyperbolic. The proof follows the ideas developped in that of Theorem 5.1.3. A hint is given in [3], page 35. To begin with, we examine the spectrum of the symbol $iA(\xi) + B(\xi)$ when the real vector ξ has a large norm. With $\xi = s\eta$ and $s := |\xi|$, the symbol equals $isA(\eta) + s^2B(\eta)$, where $s >> 1$ and η is a unit vector. Since $b(\eta)$ is uniformly non-singular, we know that the spectrum splits into two subsets $\Lambda(\xi)$ and $\sigma(\xi)$, the sets of large and of small eigenvalues. The former is approximately $s^2 \cdot \mathrm{Sp}(b(\eta))$ while the latter is approximately $is \cdot \mathrm{Sp}(C(\eta))$. To see this, observe that

$$isA(\eta) + s^2B(\eta) \sim s^2 \begin{pmatrix} 0_p & 0 \\ 0 & b(\eta) \end{pmatrix}.$$

Let us make the transformation (an inversion) $\xi \mapsto \theta := |\xi|^{-2}\xi$. At the origin, the map $\theta \mapsto s^{-2}(isA(\eta) + s^2B(\eta))$ is Lipschitz continuous, and it is smooth along rays. To each of $\Lambda(\xi)$ and $\sigma(\xi)$, there corresponds an invariant subspace under the symbol. It depends Lipschitz continuously on θ. Denoting the latter by $N(\theta)$, it satisfies $N(\theta) = \mathbb{C}^p \times \{0_{n-p}\}$. Therefore there exists a (compact) neighborhood $\mathcal{V}(0)$ and a Lipschitz continuous matrix-valued map $\theta \mapsto K(\theta)$ such that $N(\theta)$ has an equation $w = K(\theta)v$ when $\theta \in \mathcal{V}(0)$. We have $K(0) = 0_{(n-p)\times p}$.

By assumption, for every $u_0 \in \mathbb{C}^n$, the solution $u(t)$ of $\dot{u} = -(iA(\xi) + B(\xi))u$ satisfies $|u(t)| \leq c_0|u_0|$ for every positive time t. Let us choose u_0 in $N(\theta)$, that is $w_0 = K(\theta)v_0$. Then we have

$$|v(t)| \leq |u(t)| \leq c_0|u_0| \leq c_0(1 + \|K(\theta)\|)|v_0| \leq c_1|v_0|,$$

for every positive t and θ in $\mathcal{V}(0)$. Remark that since $w(t) \equiv K(\theta)v(t)$, the evolution of v is driven by

$$\dot{v} = -i(C(\xi) + D(\xi)K(\theta))v.$$

We thus have proved that

$$\sup_{t>0} \sup_{\theta\in\mathcal{V}(0)} \| \exp -it(C(\xi) + D(\xi)K(\theta))\| \leq c_1.$$

Let now fix a vector $\nu \in \mathbb{R}^d$. For $\tau > 0$ small enough, $\theta := \tau\nu$ belongs to $\mathcal{V}(0)$. Choosing $t = \tau|\nu|^2$ above, we deduce

$$\| \exp -i(C(\nu) + D(\nu)K(\tau\nu))\| \leq c_1.$$

We finally let $\tau \to 0^+$, to obtain

$$\| \exp(-iC(\nu)))\| \leq c_1, \qquad \forall \nu \in \mathbb{R}^d$$

which is hyperbolicity of H.

The sufficiency part is that if P is parabolic and H is hyperbolic, then the Cauchy problem is well-posed. Since we do not know a reference for this result, we give a rather complete sketch of the proof. Recall that, since our operator is linear, with coefficients independent of time, it suffices to establish a local existence result, because the time of existence T will not depend on the size of the data, and therefore we can extend it to intervals $(T/2, 3T/2)$, $(T, 2T)$, ...

For the sake of clarity, we begin with the case where $C(\xi)$ and $b(\xi)$ are **symmetric** ; the latter is then positive definite. The correct procedure, which we develop afterwards, is to proceed with a fixed point argument à la Picard, but for the moment, we content ourselves with an *a priori* estimate. Thus let (v, w) be a solution of the Cauchy problem, smooth enough and decaying at infinity. Multiplying the equations by v^T and w^T respectively, and integrating by parts, we obtain

$$\frac{d}{dt}|v|^2 \leq 2\|D\|\,|v|\,\|w\|$$

and

$$\frac{d}{dt}|w|^2 + 2\gamma\|w\|^2 \leq 2\|w\|(\|E\|\,|v| + \|F\|\,|w|),$$

where $|w|$ denotes the L^2-norm and $\|w\|$ the H^1-norm. Summing up, and using the Young inequality, we obtain

$$\frac{d}{dt}(|v|^2 + |w|^2) \leq c_0(|v|^2 + |w|^2).$$

Then Gronwall Lemma gives[4]

$$|v(t)|^2 + |w(t)|^2 \leq e^{ct}\left(|v(0)|^2 + |w(0)|^2\right).$$

General case. We know drop the symmetry assumption for the symbols $C(\xi)$ and $b(\xi)$. Let us denote by $(R_t)_{t\geq 0}$ and $(S_t)_{t\geq 0}$ the semi-groups associated to P and H. They are uniformly bounded on L^2:

$$\sup_{t\geq 0}\|R_t\|_{\mathcal{L}(L^2)} =: \rho < \infty, \qquad \sup_{t\geq 0}\|S_t\|_{\mathcal{L}(L^2)} =: \sigma < \infty.$$

Actually, we can prove a regularizing property for the parabolic equation $Pz = f$, analogous to that of the symmetric case. This requires the following result of linear algebra:

Lemma 5.2.1 (Liapunov) *The spectrum of a matrix $M \in \mathbf{M}_n(\mathbb{R})$ has a strictly positive real part if, and only if, there exists a positive definite symmetric matrix Σ such that the symmetric part* $\mathrm{Re}(\Sigma M) := \frac{1}{2}(\Sigma M + M^T\Sigma)$ *of ΣM is positive definite.*

[4]We warn the reader that the constant c is positive, and that this is not an artefact of the proof. The semi-group is well-defined on L^2, but it can grow as time increases. Of course, in an entropy dissipative system, it does not. More generally, a system for which the semi-group does grow is not called dissipative.

Moreover, if M depends smoothly on some parameter, then Σ can be chosen smooth as well. This allows us to symmetrize the operator P. We thus have the standard estimate

$$|z(t)|^2 + \int_0^t \|z(s)\|^2 ds \le c_1 \int_0^t \|f(s)\|_{H^{-1}}^2 ds.$$

Likewise, we have

$$\|\nabla R_t\|_{\mathcal{L}(L^2)} \le \frac{\rho}{\sqrt{t}}.$$

We now rewrite our Cauchy problem as a fixed point for a map $\mathcal{T}$. The Cauchy problem writes

$$Hv = -D(\nabla)w, \qquad Pw = -E(\nabla)v - F(\nabla)w,$$

Using the Duhamel Formula, this amounts to writing $u = \mathcal{T}u$, where $\mathcal{T}$ is defined as follows:

$$\mathcal{T}\begin{pmatrix} v \\ w \end{pmatrix}(t) := \begin{pmatrix} v_1(t) \\ w_1(t) \end{pmatrix}, \qquad Pw_1 = -E(\nabla)v - F(\nabla)w, \quad Hv_1 = -D(\nabla)w_1, \tag{5.15}$$

and of course $v_1(0) = v_0$, $w_1(0) = w_0$ are the prescribed initial data. We warn the reader that we define w_1 first, and then v_1 in terms of w_1. To proceed, we have to show that $\mathcal{T}$ is contractive on $\mathcal{C}(0, T : L^2)$ for some $T > 0$. For that purpose, we may take $u_0 = 0$. We then have

$$v_1(t) = -\int_0^t S_{t-s} D(\nabla) w_1(s)\, ds, \qquad w_1(t) = -\int_0^t R_{t-s}(E(\nabla)v(s) + F(\nabla)w(s))\, ds.$$

Let N_T be the norm of u in $\mathcal{C}(0, T; L^2)$. On the one hand, we have

$$|w_1(t)| \le \rho N_T \int_0^t (\|E\| + \|F\|) \frac{ds}{\sqrt{t-s}} = c_2 \sqrt{t} N_T.$$

On the other hand

$$\int_0^t \|w_1(s)\|^2 ds \le c_1 \rho^2 \int_0^t (\|E\|\, |v(s)| + \|F\|\, |w(s)|)^2 ds \le c_3 t N_T^2. \tag{5.16}$$

From this it follows

$$|v_1(t)| \le \sigma c_3 \|D\| t N_T.$$

In conclusion, the norm of $\mathcal{T}u$ in $\mathcal{C}(0, T; L^2)$ is bounded by $c_4(T + \sqrt{T})N_T$. Thus $\mathcal{T}$ is contractive provided that $c_4(T + \sqrt{T}) < 1$.

∎

5.2.2 Hyperbolicity in hyperbolic–parabolic systems

Theorem 5.2.1 suggests that the hyperbolic operator H plays an important role in the evolution. Actually, H is *the* hyperbolic part of the full system, the only one. The equation $Hv = 0$ is that which remains when the viscous part κB has a large coefficient κ that we let tend to $+\infty$. Then the component w is frozen because of the elliptic equation $B(\nabla_x)w = 0$ in the limit. Another point of view consists in looking at small scales, by means of a change of space-time variables $(x, t) \mapsto (x/\kappa, t/\kappa)$, with the same effect. Thus at small scales, w looks to be constant (because it is smoothed out by the parabolic part), and v is, at leading order, governed by the operator $\partial_t + C(\nabla_x)$.

The operator H plays an important role even at finite κ, say $\kappa = 1$. This can be illustrated by the analysis of the propagation of singularities of the solutions u of (5.12, 5.13). A fully parabolic operator regularizes immediately the solution, which becomes $\mathcal{C}^\infty$ whenever $t > 0$. This is not true any more for operators as above, where the matrix B is singular. We only expect that the parabolic component w will be smoothed out a little bit. At least, for L^2 data, $w(t)$ is H^1 in space and time for $t > 0$ (this is clear from the proof above). In particular, it cannot be discontinuous across a hypersurface. Thus discontinuities concern v and ∇w, but not w. The evolution of the discontinuities can be understood by writing the Rankine–Hugoniot condition for (5.12). Here, ν is the normal to the hypersurface and s is its normal velocity

$$(C(\nu) - s)[v] = 0. \tag{5.17}$$

Thus discontinuities for the full system obey the same polarization law as those of the reduced hyperbolic system $Hv = 0$. Of course the coupling of the pair (v, w) still occurs across the hypersurface. The gradient of w does jump, and obeys the following Rankine–Hugoniot conditions. On the one hand, the compatibility relations

$$\partial_\alpha \partial_\beta w = \partial_\beta \partial_\alpha w$$

implies

$$\nu_\alpha[\partial_\beta w] = \nu_\beta[\partial_\alpha w],$$

whence the existence of a vector field λ, defined on the hypersurface, such that

$$[\partial_\alpha w] = \nu_\alpha \lambda.$$

On the other hand, (5.13) gives

$$\sum_\alpha b^\alpha(\nu)[\partial_\alpha w] = E(\nu)[v].$$

In other words, we have

$$b(\nu)\lambda = E(\nu)[v]. \tag{5.18}$$

This equation determines the jump of ∇w, once we now that of v.

This coupling has a striking consequence on the evolution of the amplitude of the discontinuities. It is however not possible to give a detailed description here for multi-dimensional problem. That would require pseudo-differential theory and Egorov's Theorem. Thus we content to examine the situation in one space-dimension, where the equations are

$$v_t + Cv_x + Dw_x = 0, \qquad w_t + Ev_x + Fw_x = bw_{xx}.$$

Because of (5.17), the discontinuities of v follow straight lines of equations $x = x_0 + st$, s an eigenvalue of C. In addition, $[v]$ writes $\alpha(t)r$ where r is an associated eigenvector, because of (5.17). Moreover, we know from (5.18) that $[w_x] = b^{-1}E[v]$. Let ℓ be the associated eigenform[5], normalized by $\ell r = 1$. Multiplying by ℓ the differential equation for v, and using the identity $[v]' = [v_t + sv_x]$, we see that the amplitude satisfies the ODE

$$\alpha' = -(\ell Db^{-1}Er)\alpha. \tag{5.19}$$

This shows that, for the full system to be said *dissipative*, the scalar $\beta := \ell Db^{-1}Er$ must be positive. Under this assumption, the discontinuities decay exponentially[6].

In the analysis above, the matrix C is diagonalisable: $C = P\Lambda P^{-1}$, where P is the matrix whose columns are the eigenvectors and P^{-1} is that of eigenforms. Thus, if the system is dissipative, then the diagonal of $P^{-1}Db^{-1}EP$ is positive. Remark that if the whole system is symmetric, then $\beta = \langle b^{-1}Er, Er\rangle$ is obviously non-negative, and is actually positive if $Er \neq 0$, which means that the vector $(r, 0)^T \in \ker B$ is not an eigenvector of A ; we shall see later on that this is a crucial condition in the study of the time asymptotics.

5.2.3 Uniform well-posedness

So far, the first-order part of (5.12,5.13) did not need to be hyperbolic. Only H had to be so. Additionally, the sole assumption of L^2-well-posedness did not tell anything about the large time behaviour of the solutions, essentially because it did not give any global bound of trajectories. We now go one step beyond, asking the system to be *uniformly* (in time) *well-posed*. This assumption tells us that the semi-group associated to (5.12,5.13) is globally bounded for $t > 0$. Using the Fourier transform in space, this amounts to saying that

$$\sup_{t>0} \sup_{\xi\in\mathbb{R}^d} \|\exp(-t(B(\xi) + iA(\xi)))\| := M < +\infty. \tag{5.20}$$

Theorem 5.2.2 *Assume that the Cauchy problem for (5.12,5.13) be uniformly well-posed in L^2. Then*

- *The first-order part $\partial_t + A(\nabla_x)$ of the system is hyperbolic.*
- *Given a simple eigenvalue a_j of $A(\xi)$, together with eigenvector R_j and eigenform L_j, the number*

$$\gamma_j := \frac{L_j B(\xi) R_j}{L_j R_j}$$

is non-negative.

---

[5] Here we assume for the sake of simplicity that s is a simple eigenvalue.

[6] Even though the solution itself does not decay that fast, but only at an algebraic rate.

- *The coefficients*
$$\beta_j(\xi) := \frac{\ell_j(\xi)D(\xi)b(\xi)^{-1}E(\xi)r_j(\xi)}{\ell_j(\xi)r_j(\xi)}$$
are non-negative, where $\ell_j(\xi)$ and $r_j(\xi)$ are the eigen-form and -vector of $C(\xi)$ associated to some simple eigenvalue.

Remarks – Observe that we cannot improve the result into $\beta_j > 0$, since a Schrödinger-like system ($A(\xi)$ symmetric, $B(\xi)$ skew-adjoint) satisfies the assumption but then $\beta_j = 0$. – The function $\xi \mapsto \beta_j$ is homogeneous of degree zero. It is a damping rate for the discontinuities polarized along the mode $(r_j(\xi), b(\xi)^{-1}E(\xi)r_j(\xi))$, in the direction ξ. – As we shall see below, the γ_j's are decay rates for non-linear diffusion waves, as $t \to +\infty$.

Proof

Large wave-length analysis. Given $\eta \in \mathbb{R}^d$ and $s, \tau \in \mathbb{R}^+$, let us apply (5.20) to $\xi = s\eta$ and $t = s^{-1}\tau$. We have
$$\| \exp(-\tau(sB(\eta) + iA(\eta)))\| := M.$$
Letting $s \to 0^+$, we obtain
$$\| \exp(-i\tau A(\eta)))\| := M,$$
which is the hyperbolicity of $\partial_t + A(\nabla_x)$.

Besides, the eigenvalues of $sB(\eta) + iA(\eta)$ have to be of non-negative real part. Standard perturbation theory tells us that they obey the asymptotic expansions ($j = 1, \dots, n$)
$$ia_j + s\gamma_j(\eta) + O(s^2),$$
whenever a_j is a simple eigenvalue of $A(\eta)$. Whence the second point of the Theorem.

High frequency analysis. Given $\eta \in \mathbb{S}^{d-1}$, the matrix $B(\eta)$ splits $\mathbb{C}^n$ into two invariant subspaces $\mathbb{C}^p \times \{0\}$ and $\{0\} \times \mathbb{C}^{n-p}$. When $s \in \mathbb{C}$ is small, these spaces are smoothly deformed as invariant subspaces of $B(\eta) + isA(\eta)$. The first one[7], denoted by $N(s;\eta)$, has an equation
$$N(s;\xi) = \{x = (y,z)^T \mid z = K(s;\eta)y\},$$
with $K(s;\eta) \in \mathbf{M}_{(n-p)\times p}(\mathbb{C})$ and $K(0;\eta) = 0$. Writing the invariance, we see that K is a solution (the small one) of the Ricatti equation

$$KD(\xi)K + KC(\xi) - F(\xi)K + ib(\xi)K = E(\xi), \qquad \xi = \frac{1}{s}\eta. \tag{5.21}$$

[7] $N(s;\eta)$ corresponds to modes that are not damped directly by the viscosity, but could only be through the coupling between the v- and the w-equations.

The space $N(s;\eta)$ is actually invariant under $B(\xi)+iA(\xi)$ with ξ as above. The restriction of $B(\xi)+iA(\xi)$ to this space coincides with $i(C(\xi)+D(\xi)K(s;\eta))$ acting on the coordinate y. For s and η as above, our assumption thus tells us

$$\sup_{t>0}\sup_{\xi\in\mathbb{R}^d}\|\exp(-it(C(\xi)+D(\xi)K(s;\eta)))\| := M_1 < +\infty. \tag{5.22}$$

This implies that the spectrum of $i(C(\xi)+D(\xi)K(s;\eta))$ has a non-negative real part. When $s\to 0$, one has $K(s;\eta)\sim -ib(\xi)^{-1}E(\xi)=O(s^{-1})$, whence $iD(\xi)K(s;\eta)\to D(\eta)b(\eta)^{-1}E(\eta)$. Finally,

$$i(C(\xi)+D(\xi)K(s;\eta))=\frac{i}{s}\,C(\eta)+D(\eta)b(\eta)^{-1}E(\eta)+O(s).$$

Standard perturbation theory tells us that the eigenvalues of $i(C(\xi)+D(\xi)K(s;\eta))$ obey the asymptotic expansions ($j=1,\dots,p$)

$$\frac{i}{s}\,c_j+\beta_j(\eta)+O(s),$$

whenever c_j is a simple eigenvalue of $C(\eta)$. Since the real part of these eigenvalues have to be non-negative, we obtain the third part of the Theorem.

∎

Time asymptotics. The coefficients γ_j computed above also appear in the time asymptotics. If the initial data is smooth with good decay at infinity, one can show (see for instance Liu & Zeng [35]) that the solution behaves like a sum of diffusion waves. Let us assume for the sake of simplicity that $d=1$ and A has simple eigenvalues $a_1,\dots,a_n$. Then

$$u(x,t)\sim\sum_{j=1}^{n}m_jK^t\left(\frac{x-a_jt}{\sqrt{\gamma_j}}\right)R_j\qquad t\to+\infty,$$

where K^t s the heat kernel. The j-th wave travels with velocity a_j. Its crest decays as $t^{-1/2}$ ($t^{-d/2}$ in dimension d) while its mass spreads on a slowly expanding domain, of width $\sqrt{t}$. It does not decay in the L^1-topology. Actually the mass of the j-th wave is asymptotically $m_j\sqrt{\gamma_j}$.

We emphasize that the numbers β_j and γ_j govern completely different phenomena. The former are associated to the well-posedness, which should be viewed as a local well-posedness since it is the property that will be involved in the local well-posedness of non-linear hyperbolic-parabolic systems. They appear in the local analysis of the propagation of discontinuities. When β_j is positive (a fact that is implied by the uniform well-posedness), the corresponding shocks are damped exponentially fast. In particular, they disappear as $t\to+\infty$: this is an asymptotic regularization property. On the contrary, the γ_j's have no meaning at a local level. Their reality follows from the uniform well-posedness and then they are non-negative. When they are positive, they describe at which rate the waves approach diffusion waves as $t\to+\infty$.

Relaxation model	kind of well-posedness	Viscous model
$\partial_t + A(\nabla_x)$	ordinary	$\partial_t + C(\nabla_x)$
$\partial_t + C(\nabla_x)$	uniform	$\partial_t + A(\nabla_x)$

Figure 5.1: Hyperbolicity needed for ordinary and uniform well-posedness.

Viscosity *vs* relaxation. In both the relaxation and the viscosity models, the well-posedness of the Cauchy problem implies that some first order operator is hyperbolic: $L = \partial_t + A(\nabla_x)$ in the former case, $H = \partial_t + C(\nabla_x)$ in the latter. Then uniform well-posedness implies that the other operator (H in the former case, L in the latter) is hyperbolic too. However L and H are flipped between the cases. This is emphasized in the tableau 5.1.

5.2.4 Sub-characteristic property

The situation is less clear than for the relaxation systems, because the support of the solution u^ϵ ($\epsilon > 0$ the viscosity parameter) propagates at infinite speed. Therefore the argument used in the relaxation case does not apply. We conjecture that the subcharacteristic property still hold. We prove this when $p = n - 1$ (this extends the exemple of an isothermal gas). We even prove a little bit more in this context:

Proposition 5.2.1 *Assume that Cauchy problem for the viscous system (5.12,5.13) is uniformly well-posed. Assume also that $p = n - 1$. Then the extreme eigenvalues of the symbols $A(\xi)$ and $C(\xi)$ satisfy the sub-characteristic property (5.11). Actually, the eigenvalues of $C(\xi)$ separate those of $A(\xi)$:*

$$a_1(\xi) \leq c_1(\xi) \leq a_2(\xi) \leq \cdots \leq c_{n-1}(\xi) \leq a_n(\xi).$$

Proof

Once again, we may assume that $d = 1$, so that $C(\xi) = \xi C$ and $b(\xi) = \xi^2 b$. Here b is a scalar because $d = 1$, a positive one because of the spectral assumption upon b. One may assume that $b = 1$.

Let R be the matrix whose columns are the eigenvectors r_j of C. Then the rows ℓ_j of $L := R^{-1}$ are eigenforms. Conjugating A by $\mathrm{diag}\{R, 1\}$, we obtain

$$A \sim \mathcal{A} = \begin{pmatrix} \Lambda & LD \\ ER & LFR \end{pmatrix} =: \begin{pmatrix} \Lambda & Y \\ Z^T & f \end{pmatrix},$$

where Λ is the diagonal matrix similar to C, X and Y belong to $\mathbb{R}^{n-1}$ and f is a scalar. Each $\beta_j = \ell_j D b^{-1} E r_j$ is known to be non-negative ; in the present situation, this means $z_j y_j \geq 0$.

We now compute the characteristic polynomial of A, the same as for $\mathcal{A}$:

$$P_A(X) = (X - f)\prod_1^p (X - c_j) - \sum_1^p z_j y_j \prod_{i \neq j}(X - c_i).$$

Let us evaluate P_A at c_p:

$$P_A(c_p) = -z_p y_p \prod_1^{p-1}(c_p - c_i) \leq 0.$$

Since $P_A(+\infty) = +\infty$, we deduce that $a_n \geq c_p$. A similar argument yields $a_1 \leq c_1$.

$\blacksquare$

For a general $p \in [1, \ldots, n-2]$ we certainly cannot conclude with only the information that $\beta_j \geq 0$ (hereabove, we actually use only $\beta_1 \geq 0$ and $\beta_p \geq 0$). Thus we should have to fully exploit the uniform well-posedness, of which the sign of the β_j's was only a partial consequence.

Chapter 6

The Kawashima–Shizuta condition

6.1 Lyapunov functions for dissipative systems

The Kawashima condition can be stated for general strongly entropy-dissipative system, to ensure an ultimate form of dissipativity. This powerful tool for the time asymptotics was first developed in the context of parabolic-hyperbolic systems, by Kawashima in his thesis [23]. It was later extended to the hyperbolic-elliptic coupling in [27], and then to relaxation models by many authors, see [16, 40, 50, 55]. The basic idea is that since the time asymptotics is not governed by the sole principal part of a dissipative system, the decay must follow from the existence of a Lyapunov function that is *not* homogeneous in the derivatives of the unknown. At the linear level, it is often enough to look for a function of the form

$$\mathcal{Y}[u] = \int_{\mathbb{R}^d} E(u, \nabla u)\, dx$$

with E a positive semi-definite quadratic form. Why is such a Lyapunov function useful ? After all, we already have the entropy

$$\mathcal{E}[u] := \int_{\mathbb{R}^d} \eta(u)\, dx,$$

where η is our convex entropy, a positive definite quadratic form. How can the decay of $\mathcal{Y}$ help when the decay of $\mathcal{E}$ is helpless ? Actually, it cannot in general, since decay does not always mean a decay to zero. What is important is that this decay is associated to some dissipation:

$$\frac{d}{dt}\mathcal{E}[u] + \mathcal{Q}_0[u] \le 0, \quad \frac{d}{dt}\mathcal{Y}[u] + \mathcal{Q}_1[u] \le 0,$$

where $\mathcal{Q}_j$ are non-negative quadratic functionals. We thus have the following informations, if $\mathcal{Y}$ is convex. On the one hand, the functions of time $t \mapsto \mathcal{E}[u(t)]$ and $t \mapsto \mathcal{Y}[u(t)]$ are non-increasing. On the other hand, the functions $t \mapsto \mathcal{Q}_j[u(t)]$ are integrable over $\mathbb{R}^+$, for an initial data u_0 either in L^2 or in H^1.

Kawashima's idea is that if $\mathcal{Q}_1$ dominates the L^2-norm, then for $u_0 \in H^1(\mathbb{R})$ the function $t \mapsto \mathcal{E}[u(t)]$ is simultaneously non-increasing and integrable. Therefore it has a bound

$$\mathcal{E}[u(t)] \le \frac{m}{t}, \qquad m := \int_0^{+\infty} \mathcal{E}[u(s)]\, ds. \tag{6.1}$$

This implies the decay

$$\|u(t)\|_{L^2} = O\left(t^{-1/2}\right), \qquad \forall u_0 \in H^1(\mathbb{R}). \tag{6.2}$$

We warn the reader that this decay rate might not hold for a data in L^2. We can however deduce a decay without rate for every data in L^2, thanks to an approximation argument: If $u_0 \in L^2$ is given, there exists a sequence u_{0m} in H^1, converging in L^2 towards u_0. Let u and u_m be the corresponding solutions of the Cauchy problem. By linearity, $u - u_m$ is also the solution associated to $u_0 - u_{0m}$. We thus have

$$\|u(t)\| \leq \|u(t) - u_m(t)\| + \|u_m(t)\| \leq \|u_0 - u_{0m}\| + O_m(t^{-1/2}) \stackrel{t\to+\infty}{\longrightarrow} \|u_0 - u_{0m}\|.$$

We deduce

$$\lim_{t\to+\infty} \|u(t)\| \leq \|u_0 - u_{0m}\|.$$

Taking the limit as $m \to +\infty$, we obtain

$$\lim_{t\to+\infty} \|u(t)\|_{L^2} = 0.$$

Proposition 6.1.1 *Let the strongly entropy-dissipative linear system admit a (non-homogeneous) non-negative quadratic Lyapunov function $\mathcal{Y}$, of which the dissipation rate $\mathcal{Q}_1$ dominates the L^2-norm. Then for every initial data u_0 in $L^2(\mathbb{R}^d)$, the solution $u(t)$ converges to zero in L^2 as $t \to +\infty$.*

Warning. The principal part may be of order one (relaxation, elliptic coupling) or two (viscosity), but the method described above is valid only in the latter case. It has a counterpart in the former and we shall describe the situation in details for the relaxation. What is true is that under an assumption, known as the *Kawashima assumption*, the conclusion of the proposition always hold. Namely every solution of the Cauchy problem, with an L^2 initial data, decays to zero in that space. Additionally, there is no rate for this decay, unless one assumes a higher regularity for the data u_0, or unless the dissipation $\mathcal{Q}_0$ already dominates[1] the L^2-norm. As a matter of fact, was there an estimate

$$\|u(t)\| \leq \phi(t)\|u_0\|$$

with $\phi(+\infty) = 0$, we should have $\|S_t\| \leq \phi(t)$, and S_T would be a strict contraction for some large T. But then we should have $\|S_t\| \leq ce^{-\omega t}$ with $\omega > 0$, thus an exponential decay rate, which is never true unless $\mathcal{Q}_0$ dominates the L^2-norm. In relaxation, this means that B has its spectrum of positive real part; in particular, B is non-singular. In viscous models, the exponential decay never happens because the essential spectrum of $B(\nabla_x) - A(\nabla_x)$ reaches the imaginary axis at the origin.

[1] This very strong dissipation is of small interest. Then the system of PDEs behaves like an ODE with a strongly stable equilibrium. The solution decays even exponentially.

6.1.1 A necessary condition

Since the non-dissipative part of the system under consideration is assumed to be entropy-conservative, we shall have counter-examples to the asymptotic decay whenever there exists a non-trivial solution u for which the dissipative term does not dissipate[2] at all. Since solutions of linear systems with constant coefficients are superposition of planar waves, it is sufficient to consider a one-dimensional system, either

$$\partial_t u + A(\xi)\partial_x u + Bu = 0, \tag{6.3}$$

or

$$\partial_t u + A(\xi)\partial_x u = B(\xi)\partial_x^2 u, \tag{6.4}$$

where ξ is a fixed vector, say a unit vector. These equations govern the evolution of solutions associated to planar data: if u_0 depends only on $x \cdot \xi$, one expects that u depends only on $x \cdot \xi$ and t. Herebove, one denotes abusively x the scalar coordinate $x \cdot \xi$.

Recall that strong dissipativy yields a term $\|Bu\|^2$ (respectively $\|Bu_x\|^2$) in the *a priori* estimate. Thus a solution is non-dissipative if it satisfies $Bu \equiv 0$ and then

$$\partial_t u + A(\xi)\partial_x u = 0. \tag{6.5}$$

But taking successive commutators of B and $\partial_t u + A(\xi)\partial_x$, we obtain $BA^j u \equiv 0$ for every $j \in \mathbb{N}$ ($0 \le j \le n-1$ is enough, because of the Cayley–Hamilton Theorem). This suggests to consider initial data u_0 with values in the intersection $\mathcal{N}$ of the kernels of $B, BA, \dots, BA^{n-1}$. Since $\mathcal{N}$ is stable under A, the corresponding solution of (6.5) is valued in $\mathcal{N}$. Since $\mathcal{N}$ is included in $\ker B$, this shows that u solves either of (6.3) and (6.4). If $\mathcal{N}$ is non-trivial, we thus obtain counter-examples to decay. Whence our necessary condition that $\mathcal{N} = \{0\}$. This is the *Kawashima condition*, although it was not written that way by Kawashima himself.

Remark. Even if our system does not have an entropy-dissipation structure, the fact that $\mathcal{N}$ is non-trivial automatically provides a counter-example to the L^2-decay. Thus our condition is necessary in a very general setting.

6.1.2 Kawashima *vs* Kalman's condition

Let us give an equivalent form of the necessary condition found above.

Proposition 6.1.2 *The following statements (*Kawashima condition*) are equivalent:*

1. *There holds*
$$\bigcap_{j\ge 0} \ker(BA^j) = \{0\}.$$

[2]Remember that, according to the Lasalle's Principle, we expect that such solutions carry the time asymptotics of our Cauchy problem. Somehow, Kawashima's argument is a way to justify this principle.

2. *There holds*

$$\bigcap_{j=0}^{n-1} \ker(BA^j) = \{0\}.$$

3. *There is no eigenvector of A in $\ker B$.*

Proof
As mentionned above, 1) and 2) are equivalent because of to the Cayley–Hamilton Theorem. If there is an eigenvector of A in $\ker B$, then it belongs to the intersections of kernels. Thus 2) implies 3).

Finally, let us assume that $\mathcal{N} \neq \{0\}$. Since this is an invariant subspace upon A, an eigenvector of the restriction of A to $\mathcal{N}$ is an eigenvector of A, which belongs to $\ker B$. Thus 3) implies 2).

∎

We notice that these equivalent conditions are nothing but the *Kalman's condition*, under which the ODE system

$$\dot{y} = A^T y + B^T z, \tag{6.6}$$

$y(t)$ being the state and $z(t)$ the control, is *controlable*. Kalman's condition is usually written

$$\mathop{+}_{j=0}^{n-1} R(M^j N) = \mathbb{R}^n,$$

with M and N the matrices of the system (here $M = A^T$ and $N = B^T$)), but this is equivalent to point 3). Mind however that in control theory, N does not need to be a square matrix, it may be rectangular instead.

6.1.3 Compensating matrix

We shall see in the sequel that the necessary Kawashima's condition is also a sufficient one for the asymptotic decay of every L^2 solution of the Cauchy problem. This uses a so-called *compensating matrix*, provided by the following abstract result.

Lemma 6.1.1 (Shizuta & Kawashima [49]) *Let $M \in \mathbf{M}_n(\mathbb{R})$ be a symmetric matrix and Π be a subspace of $\mathbb{R}^n$ which does not contain an eigenvector of M (this is Kawashima's condition if $M = A$ and $\Pi = \ker B$ with A and B as above).*

Then there exists a skew-symmetric matrix K with the property that the quadratic form induced by the symmetric matrix $MK - KM$, has a positive definite restriction to Π:

$$(v \in \Pi,\, v \neq 0) \Longrightarrow (v^T M K v > 0).$$

Proof
We take the proof from the appendix of [40]. We argue by induction on the number e of distinct eigenvalues of M. If $e = 1$, then $M = \lambda I_n$, thus $\Pi = \{0\}$ and there is nothing to prove.

If $e \geq 2$, we choose an eigenvalue λ and we define $F = R(M - \lambda I_n)$. We have

$$\mathbb{R}^n = F \oplus \ker(M - \lambda I_n).$$

To this direct sum we associate the decomposition $x = x_F + x_\lambda$.

By the induction hypothesis, there exists an alternate form $a : F \times F \to \mathbb{R}$ with the property that

$$(x \in \Pi \cap F,\, x \neq 0) \Longrightarrow (a(x, Mx) > 0).$$

We are going to extend a to $\mathbb{R}^n \times \mathbb{R}^n$ as an alternate form $\hat{a}$. Since F is an M-invariant subspace, we may look for a formula

$$\hat{a}(x, y) = a(x_F, y_F) + \rho \left(b(x_\lambda, y_F) - b(y_\lambda, x_F) \right),$$

where $b : \ker(M - \lambda I_n) \times F \to \mathbb{R}$ is bilinear.

By assumption $\Pi \cap \ker(M - \lambda I_n) = \{0\}$. Thus there exists a subspace P such that

$$\Pi \subset P \text{ and } \mathbb{R}^n = P \oplus \ker(M - \lambda I_n).$$

Then P has an equation of the form $x_\lambda = \theta(x_F)$, with θ linear. Since $M - \lambda I_n : F \to F$ is well-defined and invertible, we denote by σ the inverse of this restriction. Then choosing some Euclidian structure on $\mathbb{R}^n$, we set

$$b(x, y) := (x, \theta \circ \sigma(y)), \qquad \forall x \in \ker(M - \lambda I_n),\, y \in F.$$

By definition, we have

$$b(x, My - \lambda y) = (\theta(y), x), \qquad x \in \ker(M - \lambda I_n),\, y \in F.$$

Therefore, if $x \in P$:

$$b(x_\lambda, Mx_F - \lambda x_F) = |\theta(x_F)|^2 = |x_\lambda|^2.$$

The restriction to P of the quadratic form $q_b : x \mapsto b(x_\lambda, Mx_F - \lambda x_F)$ is thus positive semi-definite, with kernel $F \cap P$. The same is true when replacing P by its subspace Π. Since $q_a : x \mapsto a(x, Mx)$ has a positive definite restriction to $\Pi \cap F$, Lemma 6.1.2 tells us that one can choose $\rho > 0$ such that $q_a + \rho q_b$ be positive definite over Π. However, we have for every $x \in \Pi$

$$\hat{a}(x, Mx) = q_a(x) + \rho q_b(x),$$

because $(Mx)_\lambda = \lambda x_\lambda$. Thus we have proved that $x \mapsto \hat{a}(x, Mx)$ has a positive definite restriction to Π. The alternate matrix K associated to $\hat{a}$ has the properties announced in the Lemma.

∎

In the proof above, we have used a classical result about the construction of positive definite quadratic form, which will be invoqued several times in the sequel:

Lemma 6.1.2 *Let G be a real vector space and q_0, q_1 be two quadratic forms on G. We assume that q_1 is positive semi-definite, and that the restriction of q_0 to $\ker q_1$ is positive definite. Then there exists a positive number ρ such that $q_0 + \rho q_1$ is positive definite.*

The proof is left to the reader.

Remark that Lemma 6.1.1 is actually an equivalence, for if Π contains an eigenvector w of M, then $v^T MKv = \lambda v^T Kv = 0$ for every skew-symmetric matrix K.

Back to controllability. We come back to the control problem $\dot{y} = My + Nz$ where M and N are matrices, M being square. Typically, M is skew-symmetric, thus the solutions of the uncontrolled system $\dot{y} = My$ neither decay nor blow up. One general purpose of controlability is to find an operator (here, a matrix) L such that if we determine the control parameter z in terms of the state[3] y by $z = Ly$, then the solutions must decay to zero. A basic problem is thus whether there exists a matrix L such that the spectrum of $M + NL$ has a negative real part. the answer turns out to be a Kawashima-like condition: The whole space $\mathbb{R}^n$ is spanned by the ranges of the matrices $M^j N$, $0 \le j \le n-1$.

6.2 Time asymptotics in relaxation

We assume that the relaxation model (5.4,5.5) is strongly entropy-dissipative and that it satisfies Kawashima's condition. This amounts to saying that $A(\xi)$ does not have an eigenvector such that $w = 0$.

We begin with the case of one space-dimension (6.3). With a suitable change of unknowns, we may assume that the dissipative entropy is $\eta(u) = |u|^2$, that is A is symmetric and $(Bu, u) \ge \omega |Bu|^2$ where $\omega > 0$. Our starting point is the balance law

$$\partial_t |u|^2 + \partial_x (u^T A u) + 2\omega |Bu|^2 \le 0.$$

This can be applied to u_x too, since it solves the same system:

$$\partial_t |u_x|^2 + \partial_x (u_x^T A u_x) + 2\omega |Bu_x|^2 \le 0.$$

Of course, if B is singular, as it should be, we cannot apply the Gronwall Lemma. If $u_0 \in L^2$ has a compact support (this restriction can be removed), we know that $t \mapsto \|u(t)\|_{L^2}$ is non-increasing and that

$$2\omega \int_0^{+\infty} \|Bu(t)\|_{L^2}^2 dt \le \|u_0\|_{L^2}^2,$$

but these informations cannot be matched together to prove the decay to zero as $t \to +\infty$. What we need is proving that $t \mapsto \|u(t)\|_{L^2}^2$ is integrable. This is where we use the Kawashima condition.

We thus look for a generalized dissipative entropy, of the form

$$e(u, u_x) := |u|^2 + \alpha (K u_x, u) + \beta |u_x|^2, \qquad \mathcal{Y}[u] := \int_{\mathbb{R}} e(u)\, dx, \tag{6.7}$$

where the constants α and $\beta > 0$ will be chosen in such a way that on the one hand $e(u, u')$ is a positive semi-definite quadratic form, and on the other hand the dissipation rate q_1 of e (dissipativity means that q_1 is a positive semi-definite quadratic form) dominates $|u_x|^2$. We point out that this is a behavior different from that in the viscous case, in which q_1 dominates $|u|^2$.

[3] This is a rather simplified point of view. In pratice, only some coordinates Py of y are observable, and z can only be determined in terms of Py.

In the definition of e, the symmetric part of K is useless, since its integral over $\mathbb{R}$ will vanish for every $u \in H^1$. This is why we content ourselves with a skew-symmetric matrix K. We now compute the time derivative of $\mathcal{Y}[u(t)]$ with the help of (6.3). By symmetry, we have

$$\frac{d}{dt}\int_{\mathbb{R}}(Ku_x, u)\,dx = 2\int_{\mathbb{R}}(Ku_x, \partial_t u)\,dx = -2\int_{\mathbb{R}}(Ku_x, Au_x + Bu)\,dx.$$

Thus

$$\frac{d}{dt}\mathcal{Y}[u(t)] + 2\omega(\|Bu\|^2 + \beta\|Bu_x\|^2) + 2\alpha\langle Ku_x, Au_x + Bu\rangle \le 0,$$

where the norm and the brackets are those of L^2. From Lemma 6.1.1 and the Kawashima condition, we may choose K so that the form $z \mapsto z^T AKz$ has a positive definite restriction over $\ker B$. With Lemma 6.1.2, we thus know that the form $\omega\beta|Bz|^2 + \alpha z^T AKz$ is positive definite provided the ratio α/β is positive and large enough, say $\alpha > c_1\beta$. Under this condition, we have $\omega\beta|Bz|^2 + \alpha z^T AKz \ge c_2\beta|z|^2$ with $c_2 > 0$. Then the dissipation term has the following lower bound:

$$\begin{aligned}
\omega(|Bu|^2 + \beta|Bu_x|^2) + \alpha(Ku_x, Au_x + Bu) &\ge \omega|Bu|^2 + c_2\beta|u_x|^2 + \alpha(Ku_x, Bu) \\
&\ge \omega|Bu|^2 + c_2\beta|u_x|^2 - \alpha\|K\|\,|u_x|\,|Bu| \\
&\ge c_2\beta|u_x|^2 - \frac{1}{4\omega}(\alpha\|K\|\,|u_x|)^2.
\end{aligned}$$

This lower bound is positive definite in u_x provided $\alpha^2 < c_3\beta$ for some $c_3 > 0$. This analysis tells us that for some choices of (α, β), the dissipation $\mathcal{Q}_1$ dominates $\|u_x\|^2$.

There remains to check whether $\mathcal{Y}$ is convex. Since $e(u, z) \ge |u|^2 + \beta|z|^2 - \alpha\|K\|\,|u|\,|z|$, convexity certainly holds true if $\beta > (\alpha\|K\|)^2/4$. In conclusion, there are two positive numbers c_1 and c_4 such that if $\alpha > c_1\beta$ and $\alpha^2 < c_4\beta$, then, on the one hand $\mathcal{Y}$ is convex, and on the other hand the dissipation dominates $\|u_x\|^2$. Such a choice of (α, β) is possible: for instance, take $\alpha = 2c_1\beta$ with $0 < \beta \ll 1$. We thus have a dissipation inequality of the form

$$\frac{d}{dt}\mathcal{Y}[u(t)] + \delta\|u_x\|^2 \le 0$$

in which δ is positive and $\mathcal{Y}$ is convex and continous over H^1. We deduce that, if $u_0 \in H^1(\mathbb{R})$, then

$$\int_0^{+\infty} \|u_x(t)\|_{L^2}^2 dt \le c_5\|u_0\|_{H^1}^2. \tag{6.8}$$

Since we also know that $t \mapsto \|u_x(t)\|$ is non-increasing (dissipation for u_x, the solution associated to the data $\partial_x u_0$), we deduce that

$$\|u_x(t)\|_{L^2} \le \|u_0\|_{H^1}\sqrt{\frac{c_5}{t}}. \tag{6.9}$$

This proves that, for an H^1-initial data, the solution tends to zero in the homogeneous space $\dot H^1(\mathbb{R})$, the completion of $\mathcal{D}(\mathbb{R})$ for the norm $z \mapsto \|z_x\|_{L^2}$.

6.2.1 Decay for L^2 data

Proving the L^2-decay is more subtle, even for H^1-data. We shall prove it actually for every L^2-data. We begin with the case where our data u_0 is the derivative of an H^1 function U_0, $u_0 = \partial_x U_0$. Denoting by U the solution of the Cauchy problem associated to the data U_0, we have $u = \partial_x U$, which we know to decay to zero in L^2, by the analysis above. In order to extend this result to every data in L^2, we use as usual the contractivity of the semi-group $(S_t)_{t\geq 0}$, together with the fact that $\partial_x H^1(\mathbb{R})$ is dense in $L^2(\mathbb{R})$. Finally, we have

Theorem 6.2.1 *Let the linear one-dimensional system*

$$\partial_t u + A\partial_x u + Bu = 0$$

be strongly entropy-dissipative. Assume that it satisfies the Kawashima condition.

Then, for every initial data u_0 in $L^2(\mathbb{R})$, the solution u of the Cauchy problem satisfies

$$\lim_{t\to+\infty} \|u(t)\|_{L^2} = 0.$$

If $u_0 \in \partial_x H^1(\mathbb{R})$, then $t \mapsto \|u(t)\|_{L^2}$ is both non-increasing and square-integrable. In particular, we have a decay rate:

$$\|u(t)\|_{L^2} = O(t^{-1/2}).$$

Remark. Somehow the assumption $u_0 \in \partial_x H^1(\mathbb{R})$ tells that the total mass of the data is zero. However, this total mass in meaningless, since the form $z \mapsto \int z\,dx$ is not defined on this space. We shall give a more accurate result in a moment.

The multidimensional case. If $d \geq 2$, we can employ the one-dimensional strategy, with the following adaptation. First, work at the level of the Fourier transform. Then in each direction of the frequency, find a compensating matrix $K(\xi)$. It can be taken homogeneous of degree one, in such a way that $|\xi|^{-1}\|K(\xi)\|$ is uniformly bounded. Continuity is not always possible, because of topological obstructions, the projective space $\mathbb{P}_{d-1}(\mathbb{R})$ having a non-trivial topology. As above, we may choose positive numbers α and β such that

$$\mathcal{Y}[u] := \int_{\mathbb{R}^d} (|u|^2 + \alpha(K(\nabla_x)u, A(\nabla_x)u) + \beta|\nabla_x u|^2)\,dx$$

is convex, and its dissipation dominates the L^2-norm of $\nabla_x u$. The rest is as above. Therefore the Theorem 6.2.1 is valid in every space dimension. We point out that in this analysis, $K(\nabla_x)$ is usually a pseudo-differential operator. It has constant coefficients and is homogeneous of order one.

Important remark. So far we have concentrated on linear systems, having in mind that most of the techniques remain valid in a nonlinear setting, provided one work in H^s, $s > 1 + d/2$, instead[4] of L^2. The price is technical, with a number of commutators to handle properly.

[4] This is not always needed. See [40], where the authors work directly in $\partial_x H^1(\mathbb{R})$, a subspace of L^2.

However, an argument will almost always be missing when a system is non-linear, that is the contractivity of the semi-group. For this reason, one always need some amount of regularity to prove the decay as time goes to $+\infty$, and such results are always obtained with some explicit decay rate. For instance, there is no way to extend to L^2 a decay result valid in $\partial_x H^1$.

Miscellanea

A lot of comments can be made about both the method and the results given above. To begin with, we establish an accurate decay result, as promised before.

An accurate decay result. Even if u_0 is smooth with compact support, the solution cannot decay faster than $t^{-1/2}$ in the L^2-norm if the total mass of ℓu_0 is non zero for some $\ell \perp R(B)$. As a matter of fact, the integral

$$m_0 := \int_{\mathbb{R}} \ell u(x,t)\,dx$$

is independent of time. Since the support of the solution grows at velocity $a := a_+ - a_-$, where $a_\pm$ are the extreme eigenvalues of A, the Cauchy–Schwarz inequality implies

$$|m_0| \leq \sqrt{|\mathrm{Supp} u(t)|}\, \|\ell u(t)\|_{L^2} \leq \sqrt{a(c+t)}\, \|u(t)\|_{L^2}.$$

This suggest that the decay of u is not the same for all its components. The above calculation shows that, whenever the total mass of u_0 does not belong to $R(B)$, we have

$$\int_0^{+\infty} \|u(t)\|_{L^2}^2 dt = +\infty.$$

This is in constrast with the fact that

$$\int_0^{+\infty} \|Bu(t)\|_{L^2}^2 dt = \frac{1}{2\omega} \|u_0\|_{L^2}^2 < +\infty.$$

This is also in contrast with the fact (Theorem 6.2.1) that if the integral of u_0 is zero, then

$$\int_0^{+\infty} \|u(t)\|_{L^2}^2 dt < +\infty.$$

We unify these observations in the following statement.

Theorem 6.2.2 *Assume a square-integrable initial data of the form $u_0 = z_0 + \partial_x \phi$ with $\phi \in H^1(\mathbb{R})$ and z_0 taking its values in $R(B)$. Then the solution of the Cauchy problem has the decay rate*

$$\|u(t)\|_{L^2} = O(t^{-1/2}).$$

Note that the assumption amounts to saying that u_0 is in L^2, with its v-component in $\partial_x H^1(\mathbb{R})$.

Proof

We build a coarse approximate solution z for the component z_0 of the data, then we solve the problem for $U := u - z$. Our z is given as the solution of the heat-like equation

$$z_t - z_{xx} + Bz = 0, \qquad z(0,) = z_0(x).$$

Since z_0 takes values in $R(B)$, this holds true for z too. Thus we have $z = (0, w)^T$ where w is the solution of the Cauchy problem

$$w_t - w_{xx} + bw = 0, \qquad (0, w(0,x))^T = z_0(x).$$

Recalling that B is strongly dissipative, we have $(bw, w) \geq \omega |bw|^2$ with $\omega > 0$. However, since b is non-singular, this means that $bw, w) \geq \nu |w|^2$ with $\nu > 0$. A standard energy estimates thus shows that $\|w(t)\|_{L^2}$ decays exponentially fast. Additionally, the L^2-norm of every derivative, for instance of w_x and of w_{xx} decay exponentially too.

Let us now write the problem solved by $U := u - z$. On the one hand, its initial value is zero. On the other hand, one has

$$U_t + AU_x + BU = f := -\begin{pmatrix} Dw_x \\ Fw_x + w_{xx} \end{pmatrix}.$$

As shown above, f and its derivative decay exponentially in L^2 as $t \to +\infty$. Importantly, $f = g_x$ is the derivative of a function g having the same behavior.

We now employ the Duhamel Principle:

$$U(t) = \int_0^t S_{t-s} f(s)\, ds = \int_0^t S_{t-s} g_x(s)\, ds.$$

From (6.9), we know that

$$\|S_t a_x\|_{L^2} \leq \frac{c_0}{\sqrt{t}} \|a\|_{H^1}.$$

We therefore have

$$\|U(t)\|_{L^2} \leq c_0 \int_0^t \|g(s)\|_{H^1} \frac{ds}{\sqrt{t-s}} \leq c_1 \int_0^t e^{-\nu s} \frac{ds}{\sqrt{t-s}}.$$

The right-hand side is known to decay at the rate $t^{-1/2}$.

■

The main idea in the proof above is that the decay obtained for H^1 data in Theorem 6.2.1 is still valid when the equation contains a forcing term. We leave the reader establishing a decay result for entropy-dissipative systems with a suitably decaying forcing term. Just use the Duhamel principle and the decay rate $t^{-1/2}$ of $S_t a_x$ when $a \in H^1$.

Symmetrization. We know show that a generalized entropy of the form $\mathcal{Y}$ as above can be used to give a generalized symmetrization. Recall that if a linear first-order system

$$\partial_t u + A \partial_x u = 0$$

admits a quadratic entropy $\eta(u) = \frac{1}{2}u^T A_0 u$, associated to an entropy-flux $q(u) := \frac{1}{2}u^T A_1 u$, then we have $A_1 = A_0 A$, and the system is equivalent to

$$A_0 \partial_t u + A_1 \partial_x u = 0,$$

where both A_0 and A_1 are symmetric matrices.

In our relaxation framework, where we have a strong dissipation and the Kawashima–Shizuta property, our system admits a generalized entropy $\mathcal{Y}[u] := \langle A_0 u, u\rangle$, with A_0 a differential operator, which is positive definite, densely defined and self-adjoint. It is associated to a dissipation $\langle Du, u\rangle$, where D has the same properties. In short, every solution with a data $u_0 \in H^s$, s large enough, satisfies

$$\frac{d}{dt}\langle A_0 u, u\rangle + \langle Du, u\rangle = 0. \tag{6.10}$$

System (6.3) can then be rewritten as

$$A_0 \partial_t u + A_1 \partial_x u + Du = 0,$$

with

$$A_1 := \mathrm{Re}(A_0 A) := \frac{1}{2}(A_0 A + A^T A_0), \qquad D := \frac{1}{2}(A_0 A - A^T A_0) + \mathrm{Re}(A_0 B)$$

are symmetric operators.

To summarize, the search of a generalized entropy $\mathcal{Y}$ as above amounts to finding a symmetric, positive definite, differential operator A_0, such that the operator D defined above is also positive definite.

We point out that both $\mathcal{Y}$ and D define Hilbert subspaces of $L^2(\mathbb{R})$, but that these spaces are not equal in general. Thus a Gronwall estimate cannot be applied to (6.10). We rather have, for smooth enough data, the properties that $t \mapsto \mathcal{Y}[u(t)]$ is non-increasing, and $t \mapsto \langle Du(t), u(t)\rangle$ is square-integrable. Applying $\mathcal{Y}$ and/or D to derivatives of u, we obtain that beyond some appropriate order k_0, the function $t \mapsto \mathcal{Y}[\partial_x^k u(t)]$ is simultaneously non-increasing and square-integrable. Therefore, with enough initial regularity, we have an upper bound

$$\mathcal{Y}[\partial_x^k u(t)] \leq \frac{c[u_0]}{t}.$$

This strategy is extremely general and can be applied to other types of dissipation, for instance viscosity, coupling with elliptic equations, boundary conditions and so forth. The difficulty lies in the derivation of the operator A_0.

Important remark. In a non-linear framework, when the system is strongly entropy-dissipative and the Kawashima–Shizuta condition is fulfilled, such a symmetrization can be made, where the operators A_0 and D now depend on the state $u(x,t)$. Local existence can be established following the usual strategy of high-order estimates, using the generalized entropy instead. Of course, what we shall estimate is $\mathcal{Y}_{u(t)}[\partial_x^k u(t)]$ for k large enough. But something new arises: the control of the time-integral of $\langle D_{u(t)}\partial_x^k u(t), \partial_x^k u(t)\rangle$ can help us in proving the global existence when the initial data is small enough in some H^s space.

6.3 Time asymptotics for viscous models

Since most of the tools and arguments have already been developpped in the relaxation framework, this section can be shorten a lot. Let us begin with a one-dimensional problem

$$\partial_t u + A\partial_x u = B\partial_x^2 u. \tag{6.11}$$

We assume that this system is strongly entropy dissipative and we may fix the entropy to $\eta(u) = \frac{1}{2}|u|^2$. This means that A is symmetric and B satisfies an inequality

$$(Bz, z) \le \omega |Bz|^2, \qquad \forall z \in \mathbb{R}^n \quad (\omega > 0).$$

Additionally, we assume the Kawashima–Shizuta condition (recall what it means: no eigenvector of A can belong to $\ker B$).

We now look for a Lyapunov function $\mathcal{Y}$, defined as in (6.7). Since K is skew-symmetric, we have

$$\frac{d}{dt}\langle Ku_x, u\rangle = 2\langle Ku_x, \partial_t u\rangle = 2\langle Ku_x, Bu_{xx} - Au_x\rangle.$$

Therefore, we obtain

$$\frac{d}{dt}\mathcal{Y}[u(t)] + 2\omega(\|Bu_x\|^2 + \beta\|Bu_{xx}\|^2) - 2\alpha\langle Ku_x, Bu_{xx} - Au_x\rangle = 0.$$

As in Section 6.2, we can choose K such that the restriction of $z \mapsto (Kz, Az)$ to $\ker B$ is positive definite. Then we can choose $\alpha > 0$ so that the quadratic form $z \mapsto Q(z) := \omega|Bz|^2 + \alpha(Kz, Az)$ is positive definite over $\mathbb{R}^n$. Choosing then $\beta > 0$ large enough, the term $\alpha(Ku_x, Bu_{xx})$ can be absorbed by $2Q(u_x) + 2\omega|Bu_{xx}|^2$, thanks to the Cauchy–Schwarz inequality. A large β also makes $\mathcal{Y}$ a positive definite quadratic form. Finally, we obtain

$$\frac{d}{dt}\mathcal{Y}[u(t)] + \omega_1\|u_x\|^2 \le 0 \qquad (\omega_1 > 0).$$

This is the same as in the relaxation case. We therefore have statements similar to Theorem 6.2.1:

Theorem 6.3.1 *Let the linear one-dimensional system*

$$\partial_t u + A\partial_x u = Bu_{xx}$$

be strongly entropy-dissipative. Assume that it satisfies the Kawashima condition.

Then, for every initial data u_0 in $L^2(\mathbb{R})$, the solution u of the Cauchy problem satisfies

$$\lim_{t\to+\infty} \|u(t)\|_{L^2} = 0.$$

If $u_0 \in H^1(\mathbb{R})$, then

$$\|u_x(t)\|_{L^2} \le \frac{c_0}{\sqrt{t}}\|u_0\|_{H^1}. \tag{6.12}$$

In particular, if $u_0 \in \partial_x H^1(\mathbb{R})$, then $t \mapsto \|u(t)\|_{L^2}$ is both non-increasing and square-integrable. In particular, we have a decay rate:

$$\|u(t)\|_{L^2} = O(t^{-1/2}).$$

Of course, we have an extension of Theorem 6.3.1 to the multi-dimensional case. The proof is left to the reader.

Better decay in special situations. There are at least two interesting cases where the results of Theorem 6.3.1 can be improved. Let us begin with the heat equation (thus $A = 0$ and B is symmetric, positive definite). It is then classical that

$$\|Bu_{xx}\|_{L^2} \leq \frac{1}{t}\|u_0\|_{L^2}.$$

By interpolation, we find

$$\|u_x\|_{L^2} \leq \frac{c_0}{\sqrt{t}}\|u_0\|_{L^2},$$

where the decay of u_x is the same as in (6.12), but one needs u_0 in L^2 only, instead of H^1.

The second example is a scalar viscous conservation law:

$$\partial_t u + f(u)_x = u_{xx}.$$

Again, there is a dissipation, but now we have the competition between convection and dissipation, as in (6.11). It turns out that the nonlinear semi-group is L^1-contracting. Thus assuming $u_0 \in L^1(\mathbb{R})$, we have $\|u(t)\|_{L^1} \leq \|u_0\|_{L^1}$. Thanks to the Moser inequality $\|z\|_{L^2}^3 \leq c\|z\|_{L^1}^2\|z_x\|_{L^2}$, together with the entropy estimate for $\eta(u) = u^2$, we obtain classically a dispersion inequality

$$\|u(t)\|_{L^2} \leq \frac{c}{t^{1/4}}\|u_0\|_{L^1}.$$

In this result, we do have a decay rate in L^2, provided that the data is in L^1, in spite of the fact that the total mass is non-zero in general. If the mass vanishes, one may prove that $t^{1/4}\|u(t)\|_{L^2}$ tends to zero as $t \to +\infty$. We point out that the constant c above is an absolute constant: it does not depend on the flux f.

Chapter 7

Linear systems with variable coefficients

In practice, there are at least three sources of variable coefficients. On the one hand, the physical space can be inhomogenous, due to matter for instance, or to deformation. On the other hand, the well-posedness analysis of an initial-boundary value problem requires to partition the domain into elementary pieces, and to make straight the boundary; in this process, partition of unity and diffeormorphisms destroy the homogeneity if it was a property of the physical space. At last, nonlinear Cauchy problems are solved through a fixed point argument, in which each step consists in solving a linear problem where the coefficients are given by the approximate solution obtained at the previous step.

This chapter is devoted to the linear Cauchy problem, when the coefficients are not constant. Of course, Fourier transform does not apply properly in this context. We thus need an other tool. We shall use here a duality approach, as it is described in [3]. All we need are *a priori* estimates. Roughly speaking, an estimate for the direct problem yields a uniqueness statement, while an estimate for the adjoint problem yields an existence result. It is important in the theory that regularity propagate. Thus estimates must also be established for the derivatives of the unknown.

Since the theory is rather well-known for first-order systems of balance laws, whose principal part is Friedrichs-symmetrizable (see [3], Chapter 2), we focus on the viscous case. The class of operators that we consider is naturally related to the analysis made in Section 4.1. It is remarkable that, despite the fact that we shall obtain estimates of the same form for both the direct and the adjoint problem, there is a difference in the way the get them. This is due to the fact that this class is not invariant under duality.

The operators that we consider have the form

$$L = \partial_t + \sum_{\alpha=1}^{d} A^\alpha(x,t)\partial_\alpha + C(x,t) - \sum_{\alpha,\beta=1}^{d} \partial_\alpha(B^{\alpha\beta}(x,t)\partial_\beta), \tag{7.1}$$

where A^α, C and $B^{\alpha\beta}$ are matrix-valued, real. In this chapter, we content ourselves with the situation where these matrices are of $\mathcal{C}^\infty$ class. We suppose that all coefficients are uniformly bounded, as well as their derivatives. Above all, we assume the following properties

Symmetrization. There exists a $\mathcal{C}^\infty$ positive definite symmetric matrix $S_0(x,t)$ such that $S^\alpha := S_0 A^\alpha$ be symmetric.

Dissipation. There exists a positive constant $\omega > 0$ such that for every vector $X \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^d$, the following inequality holds

$$(S_0 X | B(\xi; x,t)X) \geq \omega \sum_\alpha |B^\alpha(\xi; x,t)X|^2. \tag{7.2}$$

Block structure. The range of $B(\xi; x,t)$ does not depend on the arguments as long as $\xi \neq 0$. In other words, one can choose a coordinate system in which

$$B(\xi; x,t) = \begin{pmatrix} 0_{p\times n} \\ b(\xi; x,t) \end{pmatrix}, \tag{7.3}$$

where b has full rank $n-p$ when $\xi \neq 0$.

The dissipation inequality recalls (4.5), and we have used the same notations for $B(\xi; \cdots)$ and $B^\alpha(\xi;\cdots)$. As long as estimates are concerned, it will be used in the form of a Gårding-like inequality: For every field of class $H^1(\mathbb{R}^d)^n$, there holds

$$\int_{\mathbb{R}^n} \sum_{\alpha,\beta} \langle S_0 \partial_\alpha u, B^{\alpha\beta}\partial_\beta u\rangle \geq \omega \sum_\alpha \left\| \sum_\beta B^{\alpha\beta}\partial_\beta u \right\|_{L^2}^2 - C\|u\|_{L^2}^2 := \omega \sum_\alpha \|B^\alpha \nabla u\|_{L^2}^2 - c_0 \|u\|_{L^2}^2, \tag{7.4}$$

with c_0 a constant independent from u. Note that the positive constant ω in (7.4) might be smaller than that in (7.2).

7.1 A priori estimate

In the following, we shall use the fact that $R(B(\xi))$ and $\ker B(\xi)$ are S_0-orthogonal in the following way. At first, there exists a constant c such that

$$\big(y \in R(B(\xi)) = 0 \times \mathbb{R}^{n-p}\big) \Longrightarrow \left(\forall z \in \mathbb{R}^n,\ |\xi| \cdot |\langle S_0 z, y\rangle| \leq c|y| \sum_\alpha |B^\alpha(\xi)z|\right). \tag{7.5}$$

This implies the following special estimate.

Proposition 7.1.1 *There exists a constant c such that, for every given $y \in L^2$ and $z \in H^1$ with the property that y takes values in $R(B(\xi)) = 0 \times \mathbb{R}^{n-p}$, then the following inequality holds:*

$$\int_{\mathbb{R}^d} |\langle S_0 \nabla z, y\rangle|\, dx \leq c\|y\|_{L^2} \left(\sum_\alpha \|B^\alpha \nabla z\|_{L^2} + \|z\|_{L^2}\right). \tag{7.6}$$

7.1.1 The L^2 estimate

Our *a priori* estimates are established for smooth, compactly supported fields, and then extended to fields belonging to the appropriate Sobolev space. We begin by multiplying Lu by S_0u:

$$\begin{aligned} 2\langle S_0u, Lu\rangle &= \partial_t\langle S_0u, u\rangle + \sum_\alpha \partial_\alpha\langle S^\alpha u, u\rangle + \langle Ru, u\rangle \\ &\quad -2\sum_{\alpha,\beta}\partial_\alpha\langle S_0u, B^{\alpha\beta}\partial_\beta u\rangle + 2\sum_{\alpha,\beta}\langle\partial_\alpha(S_0u), B^{\alpha\beta}\partial_\beta u\rangle, \end{aligned}$$

where

$$R := 2S_0C - \partial_tS_0 - \sum_\alpha \partial_\alpha S^\alpha.$$

We now integrate with respect to the space variable and obtain

$$\begin{aligned} 2\int_{\mathbb{R}^d}\langle S_0u, Lu\rangle dx &\geq \frac{d}{dt}\int_{\mathbb{R}^d}\langle S_0u, u\rangle dx + \int_{\mathbb{R}^d}\langle (R - 2c_0I_n)u, u\rangle dx \\ &\quad +2\int_{\mathbb{R}^d}\sum_\alpha\langle(\partial_\alpha S_0)u, B^\alpha\nabla u\rangle + 2\omega\sum_\alpha\|B^\alpha\nabla u\|_{L^2}^2. \end{aligned}$$

Applying the Cauchy–Schwarz inequality, there comes

$$2\int_{\mathbb{R}^d}\langle S_0u, Lu\rangle dx \geq \frac{d}{dt}\int_{\mathbb{R}^d}\langle S_0u, u\rangle dx + \omega\sum_\alpha\|B^\alpha\nabla u\|_{L^2}^2 - c_1\|u\|_{L^2}^2. \tag{7.7}$$

At last, there comes for every $\gamma > 0$

$$\frac{d}{dt}\int_{\mathbb{R}^d}\langle S_0u, u\rangle dx + \omega\sum_\alpha\|B^\alpha\nabla u\|_{L^2}^2 \leq \left(c_1 + \frac{1}{\gamma}\right)\|u\|_{L^2}^2 + \gamma\|Lu\|_{L^2}^2. \tag{7.8}$$

Of course, a density argument ensures that the inequality (7.8) is valid, in the distributional sense over an interval I, as soon as $u \in L^\infty_{loc}(I; L^2(\mathbb{R}^d))$, $\partial_tu \in L^2_{loc}(I; L^2(\mathbb{R}^d))$ and $B^\alpha\nabla u \in L^2_{loc}(I; H^1(\mathbb{R}^d))$. In particular, it provides a uniqueness result. For if $Lu = 0$ and $\lim_{t\to 0^+} u(t) = 0$ in the L^2-norm, then the inequality

$$\frac{d}{dt}\int_{\mathbb{R}^d}\langle S_0u, u\rangle dx \leq \left(c_1 + \frac{1}{\gamma}\right)\|u\|_{L^2}^2$$

implies $u \equiv 0$, since $\int_{\mathbb{R}^d}\langle S_0u, u\rangle dx$ is equivalent to $\|u\|_{L^2}^2$.

7.1.2 Derivative estimates

Let ∂_μ be a first-order space derivative, and let us denote $v_\mu := \partial_\mu u$. Then

$$Lv_\mu = \partial_\mu Lu + [L, \partial_\mu]u.$$

When applying (7.8) to v, the right-hand side involves the norm of Lv, where we should like to have $\partial_\mu Lu$ instead. Thus we need to estimate the L^2-norm of $[L, \partial_\mu]u$. This does not yield a nice estimate. Instead, we have to go back to (7.7) and therefore estimate the integral of $\langle S_0 v_\mu, [\partial_\mu, L]u\rangle$. One has

$$[\partial_\mu, L] = (\partial_\mu A^\alpha)\partial_\alpha + \partial_\mu C - \sum_\alpha \partial_\alpha [\partial_\mu, B^\alpha(\nabla)].$$

The first two operators in the right-hand side yield terms that are bilinear in u and $\mathbf{v} := (v_1, \dots, v_d)$. Thus let us focus on the higher-order term. After an integration by part, it is

$$\int_{\mathbb{R}^d} \langle \partial_\alpha (S_0 v_\mu), [\partial_\mu, B^\alpha(\nabla)]u\rangle \, dx.$$

The bracket is a first-order operator, thus is linear in $\mathbf{v}$. Therefore the only term that is not estimated in terms of $\mathbf{v}$ and u in L^2 is

$$\int_{\mathbb{R}^d} \langle S_0 \partial_\alpha v_\mu, [\partial_\mu, B^\alpha(\nabla)]u\rangle \, dx.$$

To estimate this term, we invoque Proposition 7.1.1. Since the range of $B(\xi)$ is independent of $\xi \neq 0$, we see that $y := [\partial_\mu, B^\alpha(\nabla)]u$ takes values in this subspace. Thus we may apply (7.6) to y and $z := \partial_\alpha v_\mu$. We obtain therefore the bound

$$c\|\mathbf{v}\|_{L^2} \left(\sum_\alpha \|B^\alpha \nabla v_\mu\|_{L^2} + \|v_\mu\|_{L^2} \right).$$

The conclusion of the previous calculation is that v_μ satisfies an inequality

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^d} \langle S_0 v_\mu, v_\mu\rangle dx + \omega \sum_\alpha \|B^\alpha \nabla v_\mu\|^2_{L^2} \quad &\leq \quad \left(c_2 + \frac{1}{\gamma}\right)\|v_\mu\|^2_{L^2} + \gamma\|\partial_\mu Lu\|^2_{L^2} \\ &\quad + c\|v_\mu\|_{L^2}\|u\|_{L^2} + c\|\mathbf{v}\|_{L^2}\sum_\alpha \|B^\alpha \nabla v_\mu\|_{L^2}. \end{aligned}$$

Using again the Cauchy–Schwarz inequality, we can balance the right-most term with the dissipation rate. At last, we have

$$\frac{d}{dt}\int_{\mathbb{R}^d} \langle S_0 v_\mu, v_\mu\rangle dx + \omega \sum_\alpha \|B^\alpha \nabla v_\mu\|^2_{L^2} \leq \left(c_2 + \frac{1}{\gamma}\right)\|v_\mu\|^2_{L^2} + \gamma\|\partial_\mu Lu\|^2_{L^2} + c\|u\|^2_{L^2}, \tag{7.9}$$

with a smaller but still positive ω.

We can proceed similarly for higher-order estimates, and we obtain, for every differential ∂^k of order $k \geq 1$:

$$\frac{d}{dt}\int_{\mathbb{R}^d} \langle S_0 \partial^k u, \partial^k u\rangle dx + \omega \sum_\alpha \|B^\alpha \nabla \partial^k u\|^2_{L^2} \leq c_\gamma \|\partial^k u\|^2_{L^2} + \gamma\|\partial^k Lu\|^2_{L^2} + c\|\nabla^{k-1}u\|^2_{L^2}. \tag{7.10}$$

7.2 The adjoint Cauchy problem

The adjoint operator is

$$L^* = -\partial_t - \sum_{\alpha=1}^{d} \partial_\alpha (A^\alpha(x,t)^T \dot{)} + C(x,t)^T - \sum_{\alpha,\beta=1}^{d} \partial_\beta (B^{\alpha\beta}(x,t)^T \partial_\alpha).$$

When dealing with the forward Cauchy problem for L, we look for estimates for an adjoint problem, which is the *backward* Cauchy problem for L^*. Making the change $t \longleftrightarrow T-t$, this amounts to consider the forward Cauchy problem for the operator

$$M = \partial_t - \sum_{\alpha=1}^{d} \partial_\alpha (A^\alpha(x,T-t)^T \dot{)} + C(x,T-t)^T - \sum_{\alpha,\beta=1}^{d} \partial_\beta (B^{\alpha\beta}(x,T-t)^T \partial_\alpha).$$

Such an M ressembles a lot L. First of all, it has a positive definite symmetrizer, here S_0^{-1}, instead of S_0. Next, the second-order term is S_0^{-1}-dissipative. To see this, let us take a vector X and set $Y := S_0^{-1}X$. Then

$$\langle S_0^{-1}X, B(\xi)^T X\rangle = \langle S_0 Y, B(\xi) Y\rangle \geq \omega \sum_\alpha |B^\alpha(\xi) Y|^2 = \omega \sum_\alpha |B^\alpha(\xi) S_0^{-1} X|^2.$$

The right-hand side is a (square of a) semi-norm in X. Its kernel is $S_0 \ker B(\xi)$. However, the S_0-orthogonality of $\ker B(\xi)$ and $R(B)$ yields

$$S_0 \ker B(\xi) = R(B)^\perp = \ker B(\xi)^T.$$

Let now define $\hat{B}^\alpha(\xi) := \sum_\beta \xi_\beta (B^{\beta\alpha})^T$. This plays the role of $B^\alpha(\xi)$ when dealing with M. We have obviously $\bigcap_\alpha \ker \hat{B}^\alpha(\xi) \subset \ker B(\xi)^T$. Since moreover

$$\left(\bigcap_\alpha \ker \hat{B}^\alpha(\xi)\right)^\perp = +_\alpha R\left(\hat{B}^\alpha(\xi)^T\right) = +_\alpha R(\sum_\beta \xi_\beta B^{\beta\alpha}) = \{0\} \times \mathbb{R}^{n-p},$$

we deduce the equality $\bigcap_\alpha \ker \hat{B}^\alpha(\xi) = \ker B(\xi)^T$. Therefore the semi-norm above is equivalent to

$$X \mapsto \sum_\alpha |\hat{B}^\alpha(\xi) X|^2.$$

In conclusion, we have the dissipation inequality for the operator M:

$$\langle S_0^{-1}X, B(\xi)^T X\rangle \geq \omega' \sum_\alpha |\hat{B}^\alpha(\xi) X|^2. \tag{7.11}$$

So there is a significant difference between the class of operators L and the class of the "dual" operators M. Yes indeed, a difference that lies in the block structure of the dissipative symbol $\hat{B}(\xi)$. Instead of having a fixed range $\{0\} \times \mathbb{R}^{n-p}$, it has a fixed kernel $\mathbb{R}^p \times \{0\}$. Therefore Proposition 7.1.1 is useless for the derivative estimates of the corresponding Cauchy problem.

TODO : les estimations ne se font pas tout à fait pareil.

7.3 Well-posedness of the Cauchy problem

Chapter 8

Nonlinear issues

We consider in this chapter entropy-dissipative systems of the nonlinear forms described in Chapter 4. We are primarily concerned with the existence of smooth solutions for smooth initial data. Roughly speaking, the well-posedness of the linearized problem implies the local existence, even though the proof is cumbersome. Global existence requires additionally the Kawashima condition *and* the smallness of the initial data, in the sense that u_0 is close to an equilibrium $\bar{u}$; then one also have the decay towards $\bar{u}$.

The situation for large initial data is more complex. Then the Kawashima property cannot compete efficiently with the nonlinearity of the system. Because of that, relaxation behaves very much like first-order systems of conservation laws, where shock formation is the rule, at least in one space dimension. On the other hand, the hyperbolic-parabolic Cauchy problem can display or not shock formation, depending on whether the characteristic fields of the underlying *reduced hyperbolic Cauchy problem* are genuinely nonlinear or linearly degenerate.

8.1 Local existence of smooth solutions

Local existence is essentially the nonlinear version of well-posedness studied in Section 5.1. In particular, it does not need at all the Kawashima condition.

8.1.1 Local existence in relaxation

We consider a general system of balance laws

$$\partial_t u + \sum_\alpha \partial_\alpha f^\alpha(u) + g(u) = 0, \tag{8.1}$$

with smooth fluxes f^α and source g. We assume that the system admits a strongly convex ($\mathrm{D}^2\eta > 0$) entropy η, with entropy flux $\vec{q}$. However, since the source term g is sub-leading (it is not in the principal part of the system), we *do not need that it be dissipative.*

We recall that if $g \equiv 0$ (system of *conservation laws*), then the Cauchy problem is locally well-posed in H^s for $s > 1 + d/2$ where d is the space dimension. It is worth to point out that the solutions under considerations are classical, in the sense that $H^s \subset \mathcal{C}^1$.

The method for conservation laws is to make an energy estimate, which uses the Hessian of the entropy to define the norm over H^s, if s is an integer. Notice that this norm depends on the solution itself, but is equivalent to the standard one. Its definition is

$$[v]_{s,u} := \left(\sum_{|\gamma|\le s} \int_{\mathbb{R}^d} \mathrm{D}^2\eta(u) v^{\otimes 2} dx \right)^{1/2}.$$

Let s be as above. For an integer k, one typically obtain the tame estimate

$$\frac{d}{dt}[u(t)]^2_{k,u(t)} \le h_k(\|u(t)\|_{H^s})\|u(t)\|^2_{H^k} \tag{8.2}$$

with h_k some polynomial function. For $k = s$, this gives a superlinear differential inequality in $[u(t)]^2_{s,u(t)}$, from which we derive an explicit upper bound of this quantity, which in turns gives an H^s-estimate of the solution on some time interval $[0, T)$. Of course, because of superlinearity, T is finite and depends on the H^s-norm of the data u_0. In practice, one first constructs approximate solutions $(u^k)_{k\in\mathbb{N}}$, to which the above estimate is easily adapted. Thus the sequence is bounded in $\mathcal{C}([0,T);H^s)$. Then the estimate (8.2) is used with $k = 0$ to prove the convergence in $\mathcal{C}([0,T);L^2)$. By interpolation, the convergence still hold in $\mathcal{C}([0,T);H^r)$ for every $0 \le r < s$. Using the approximate equation, the sequence converges also in $\mathcal{C}^1([0,T);H^r)$ for $0 \le r < s-1$. In particular, the convergence holds in $\mathcal{C}^1([0,T)\times\mathbb{R}^d)$. Thus we may pass to the limit in the approximate equation satisfied by the sequence. Therefore, the limit is a solution of the Cauchy problem. Uniqueness uses the same kind of estimate, though applied to the difference of two solutions. See your favourite book, for instance [10, 3].

How we get (8.2). The easiest estimate is that of the total entropy,

$$\frac{d}{dt}\int_{\mathbb{R}^d} \eta(u(t))\,dx = 0,$$

but it is of course not enough, since it gives an estimate of u only in some Lebesgue-Orlicz space and that does not allow us to pass to the limit in the non-linear term when dealing with a sequence of approximated solutions.

Let us have a look to the estimate of the first derivatives ($k = 1$). We select an index $1 \le \beta \le d$. The estimate (8.2) is derived from

$$\begin{aligned} &\partial_t \frac{1}{2}\mathrm{D}^2\eta(\partial_\beta u, \partial_\beta u) + \mathrm{D}^2\eta \mathrm{d}f^\alpha(\partial_\alpha\partial_\beta u, \partial_\beta u) \\ &+\mathrm{D}^2\eta((\partial_\beta \mathrm{d}f^\alpha)\partial_\alpha u, \partial_\beta u) + \frac{1}{2}\mathrm{D}^3\eta(\partial_\beta u, \partial_\beta u, \partial_\alpha f^\alpha) \;=\; 0, \end{aligned}$$

in which the summation over α is implicit (Einstein notation). Since η is an entropy, the matrix $\mathrm{D}^2\eta \mathrm{d}f^\alpha =: S^\alpha(u)$ is symmetric and thus the term involving the second derivatives can be rewritten as

$$\mathrm{D}^2\eta \mathrm{d}f^\alpha(\partial_\alpha\partial_\beta u, \partial_\beta u) = \partial_\alpha \frac{1}{2} S^\alpha(\partial_\beta u, \partial_\beta u) - \frac{1}{2}(\partial_\beta S^\alpha)(\partial_\beta u, \partial_\beta u).$$

Then, integrating in space and assuming for instance that $u(t)$ has a compact support (we may afford this property in the construction of the approximate solution), we obtain

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d} \mathrm{D}^2\eta(\partial_\beta u, \partial_\beta u)\,dx &= \int_{\mathbb{R}^d} \big\{(\partial_\beta S^\alpha)(\partial_\beta u, \partial_\beta u) - 2\mathrm{D}^2\eta((\partial_\beta \mathrm{d} f^\alpha)\partial_\alpha u, \partial_\beta u) \\ &\qquad - \mathrm{D}^3\eta(\partial_\beta u, \partial_\beta u, \partial_\alpha f^\alpha)\big\}\,dx =: \int_{\mathbb{R}^d} P_3(\nabla_x u)\,dx.\end{aligned}$$

Notice that the right-hand side depends only on the first derivatives of u, which is a good thing. It is however cubic in ∇u, preventing us to apply a Gronwall argument. This is why estimating higher derivatives is necessary.

We thus go to higher order estimates. Let $\gamma = (\gamma_1, \dots, \gamma_d) \in \mathbb{N}^d$ be a multi-index of length $k := \gamma_1 + \cdots + \gamma_d$. Let ∂^γ be the corresponding space derivative of order k. Doing as above, we find that

$$\partial_t \frac{1}{2}\mathrm{D}^2\eta(\partial^\gamma u, \partial^\gamma u) + \partial_\alpha \frac{1}{2}\mathrm{D}^2\eta \mathrm{d} f^\alpha(\partial^\gamma u, \partial^\gamma u) = P_k^{2k+1}[u],$$

where P_k^{2k+1} denotes a polynomial in the derivatives of u up to the k-th, of weight $2k+1$, meaning that each monomial involves a total of $2k+1$ derivatives. Examples in one space dimension: $u_x(\partial_x^k u)^2$, $u_{xx}\partial_x^{k-1}u\partial_x^k u$ or $u_x^3(\partial_x^{k-1}u)^2$. Again, integrating over $\mathbb{R}^d$, we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^d} \mathrm{D}^2\eta(\partial^\gamma u, \partial^\gamma u)\,dx =: \int_{\mathbb{R}^d} P_k^{2k+1}(\nabla_x u)\,dx. \tag{8.3}$$

We now apply Gagliardo-Nirenberg estimates to the right-hand side of (8.3), to get an upper bound of the form

$$c_k\left(\|u\|_{W^{1,\infty}}\right)\|u\|_{H^k}^2,$$

with c_k some increasing function. This implies (8.2) since $H^s(\mathbb{R}^d)$ is imbedded into $W^{1,\infty}(\mathbb{R}^d)$.

Adding a lower order term. This strategy adapts easily to the case of balance laws. Suppose for instance that $g(0) = 0$, in order that the full system (8.1) be compatible with spaces made of functions that vanish at infinity. Such functions thus tend to an equilibrium at infinity. Then the energy estimates differ from (8.2) only by terms that can easily be controlled. The first one is

$$\frac{d}{dt}\int_{\mathbb{R}^d} \eta(u(t))\,dx = \int_{\mathbb{R}^d} \mathrm{d}\eta(u(t))g(u(t))\,dx,$$

where we have normalized η (by the addition of an affine function) such that $\eta(0) = 0$ and $\mathrm{d}\eta(0) = 0$. As long as u remains bounded in L^∞ (this will be a by-product of the high-order estimates), the expression $\mathrm{d}\eta(z)g(z)$ is an $O(z^2)$, thus is controlled by $\eta(z)$ itself. Thus we have

$$\frac{d}{dt}\int_{\mathbb{R}^d} \eta(u(t))\,dx \le h(\|u\|_{L^\infty})\int_{\mathbb{R}^d} \eta(u(t))\,dx,$$

to which a Gronwall argument can be applied.

Let us have a look to the estimate of the first derivatives ($k = 1$). In presence of the source term g, the differential estimate becomes

$$\begin{aligned}\partial_t \frac{1}{2}\mathrm{D}^2\eta(\partial_\beta u, \partial_\beta u) \quad & +\mathrm{D}^2\eta \mathrm{d}f^\alpha(\partial_\beta u, \partial_\beta u) + \mathrm{D}^2\eta((\partial_\beta \mathrm{d}f^\alpha)\partial_\alpha u, \partial_\beta u) \\ & +\mathrm{D}^2\eta \mathrm{d}g(\partial_\beta u, \partial_\beta u) + \tfrac{1}{2}\mathrm{D}^3\eta(\partial_\beta u, \partial_\beta u, \partial_\alpha f^\alpha + g) \quad = 0,\end{aligned}$$

with two extra terms involving g. After integration, we obtain

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d} \mathrm{D}^2\eta(\partial_\beta u, \partial_\beta u)\, dx \quad = & \quad \int_{\mathbb{R}^d} P_3(\nabla_x u)\, dx \\ & - \int_{\mathbb{R}^d} \left\{2\mathrm{D}^2\eta \mathrm{d}g(\partial_\beta u, \partial_\beta u) + \mathrm{D}^3\eta(\partial_\beta u, \partial_\beta u, g)\right\} dx.\end{aligned}$$

Summing up over β, we obtain

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d} \mathrm{D}^2\eta(\nabla_x u)^{\otimes 2} dx \quad = & \quad \int_{\mathbb{R}^d} \mathbf{P}_3(\nabla_x u)\, dx \\ & - \int_{\mathbb{R}^d} \left\{2\mathrm{D}^2\eta \mathrm{d}g(\nabla_x u)^{\otimes 2} + \mathrm{D}^3\eta(\nabla_x u, \nabla_x u, g)\right\} dx.\end{aligned} \tag{8.4}$$

The new terms are quadratic in $\nabla_x u$. They are immediately handable for a Gronwall argument. Notice that (8.4) is still of the form (8.2) with $k = 1$. Likewise, the estimate at order k keeps the form (8.2), with a new h_k that involves g and its derivatives. Thus the local existence theory remains valid. We can therefore state

Theorem 8.1.1 *Let the system (8.1) have smooth fluxes f^α and source g, with $g(0) = 0$. Assume that there exists a strongly convex ($\mathrm{D}^2\eta > 0$) entropy η (here, dissipation is not needed). Then, given an initial data $u_0 \in H^s(\mathbb{R}^d)$ with $s > 1 + d/2$, there exists a time $T > 0$ and a unique solution u in the class*

$$\mathcal{C}(0, T; H^s) \cap \mathcal{C}^1(0, T; H^{s-1})$$

of the Cauchy problem.

The effect of dissipation. If the system (8.1 is strongly dissipative, then the quadratic terms in (8.4) are essentially non-positive. To speak precisely,, they equal

$$-2\int_{\mathbb{R}^d} \mathrm{D}^2\eta(0)\mathrm{d}g(0)(\nabla_x u, \nabla_x u)\, dx,$$

up to cubic terms (recall that $g(0) = 0$). However, strong dissipativity (see (4.24)) tells that this is less than or equal to

$$-2\omega\int_{\mathbb{R}^d} |\mathrm{d}g(0)\nabla_x u|^2 dx.$$

This observation is of no immediate help in general, and it does not help at all if we lack the Kawashima condition. However we shall see in Section 8.2 that under the latter, this dissipation can compete efficiently with the nonlinearity, just as in ordinary differential equations, to give global solutions for small data. For the moment, we content ourselves with the case of a *total dissipation*, meaning that $\mathrm{d}g(0)$ is non-singular. Under this assumption, we have now and inequality of the form

$$\frac{d}{dt}\int_{\mathbb{R}^d} \mathrm{D}^2\eta(\nabla_x u)^{\otimes 2}dx + \omega_1 \int_{\mathbb{R}^d} \mathrm{D}^2\eta(\nabla_x u)^{\otimes 2}dx \le \int_{\mathbb{R}^d} \mathbf{P}_3(\nabla_x u)\,dx.$$

Of course, a Gronwall argument cannot be applied yet, essentially because L^2 does not contain L^3. We can only derive

$$\frac{d}{dt}\int_{\mathbb{R}^d} \mathrm{D}^2\eta(\nabla_x u)^{\otimes 2}dx + \omega_1 \int_{\mathbb{R}^d} \mathrm{D}^2\eta(\nabla_x u)^{\otimes 2}dx \le \|\nabla_x u(t)\|_{L^\infty} \int_{\mathbb{R}^d} \mathrm{D}^2\eta(\nabla_x u)^{\otimes 2}dx,$$

and we cannot deduce a bound from this inequality. But if instead we estimate higher-order derivatives, say of order k, we have a similar inequality

$$\frac{d}{dt}\int_{\mathbb{R}^d} \mathrm{D}^2\eta(\nabla_x^k u)^{\otimes 2}dx + \omega_1 \int_{\mathbb{R}^d} \mathrm{D}^2\eta(\nabla_x^k u)^{\otimes 2}dx \le h_k(\|\nabla_x u(t)\|_{L^\infty}) \int_{\mathbb{R}^d} \mathrm{D}^2\eta(\nabla_x^k u)^{\otimes 2}dx,$$

with h_k some suitable polynomial satisfying $h_k(0) = 0$. If $k = s$ is as above, Sobolev embedding gives

$$\|\nabla_x u(t)\|_{L^\infty}^2 \le c_0 \|u(t)\|_{H^s}^2,$$

where the right-hand side can be bounded in terms of

$$N[u]^2 := \int_{\mathbb{R}^d} \eta(u)\,dx + \sum_{k=1}^{s} \int_{\mathbb{R}^d} \mathrm{D}^2\eta(\nabla_x^k u)^{\otimes 2}dx.$$

Summing over k our estimates, we obtain

$$\frac{d}{dt}N[u]^2 + \omega_1 N[u]^2 \le h(N[u])N[u]^2,$$

where the polynomial h vanishes at the origin. We are now in position to obtain an estimate, uniform in time, provided $N[u_0]$ is small enough. This will be the abovementioned smallness of the initial data. More precisely, let N_0 be the smallest positive root of the equation $h(\nu) = \omega_1$. If $N[u_0] < N_0$ then $N[u(t)]$ remains smaller than N_0 forever, and thus the smooth solution is global in time, according to the blow-up criteria (see [10]). In addition, the number $\omega_1 - h(N[u(t)])$ remains uniformly positive, and therefore $N[u(t)]$ decays to zero. This is the trend to equilibrium for small solutions. Notice that in this context of total dissipation, this decay is exponential, a behaviour that we now is false in the other cases, even in linear situations, even under the Kawashima condition. We may therefore anticipate that for realistic problems, a Gronwall argument will not apply and a more subttle mechanism will be involved.

8.1.2 Local existence in viscous models

Historically, this topic evolved independently form others because of its relevance in viscous gas dynamics. Since most papers dealt with a specific system, a general argument was not clearly worked out. The first abstract work in this context is certainly Kawashima's PhD Thesis [23].

A naive idea is the following: Consider a system of viscous conservation laws of the form

$$\partial_t v + \partial_\alpha f^\alpha(v,w) = 0, \tag{8.5}$$

$$\partial_t g^0(v,w) + \partial_\alpha g^\alpha(v,w) = \partial_\alpha(b^{\alpha\beta}(v,w)\partial_\beta w), \tag{8.6}$$

with the property that

- on the one hand, at constant $\bar{v}$, the system $\partial_t g^0(\bar{v},w) = \partial_\alpha(b^{\alpha\beta}(\bar{v},w)\partial_\beta w)$ is uniformly parabolic,
- on the other hand, at constant $\bar{w}$, the system $\partial_t v + \partial_\alpha f^\alpha(v,\bar{w}) = 0$ is Friedrichs symmetrizable (for instance it admits a strongly convex entropy),

and of course $\mathrm{d}_w g$ is non-singular. Then the Cauchy problem should be locally well-posed. This turns out to be true, according to Kawashima [23], Chapter II, at least if there exists a symmetric positive definite matrix $S_0(v,w)$ such that $S_0(\mathrm{d}_w g)^{-1}b(\xi)$ is symmetric for every $\xi \in \mathbb{R}^d$. We shall not give a detailed account of this result, because we are motivated by physical systems, which have an additional structure (strongly dissipative entropy), and because we are interested in global existence and asymptotic behaviour.

Systems with strong entropy dissipation. In practice, one thus make the additional, and physically reasonable, assumption that the first-order part of (8.5, 8.6) is strongly entropy-dissipative. Note that this implies the symmetrizability of the v-system mentionned above. It turns out that the dissipativity assumption is extremely useful, in that it cancels some bad term in the estimates, thanks to the fact (Proposition 4.1.1) that the kernel and the range of the viscosity matrix $B(\xi : u)$ are orthogonal with respect to $S_0(u) := D^2\eta(u)$ (η the strongly convex entropy). To show how the estimates go on, let us consider for the sake of simplicity a one-dimensional system of viscous conservation laws

$$\partial_t u + \partial_x f(u) = \partial_x(B(u)\partial_x u). \tag{8.7}$$

We assume that B has the standard block form

$$B(u) = \begin{pmatrix} 0 \\ b(u) \end{pmatrix},$$

which means that it has a fixed range, or in other words, the system contains p first-order conservation laws, like mass conservation, ... Strong dissipation means that $S_0(BX,X) \geq \omega|bX|^2$ with some $\omega(u) > 0$.

We examine the estimates beyond the standard energy/entropy one. In the calculations below, we denote $v_k := \partial_x^k u$. We start from

$$\partial_t v_k + \partial_x^k(\mathrm{d}f(u)u_x) = \partial_x^{k+1}(B(u)u_x),$$

which we multiply at left by $v_k^T S_0(u)$. We obtain

$$\partial_t \frac{1}{2} v_k^T S_0 v_k + \partial_x \frac{1}{2} v_k^T S v_k + \mathrm{Pol}_0(v_1, \dots, v_k) = v_k^T S_0 \partial_x^{k+1}(B(u)u_x) + \frac{1}{2} v_k^T (\partial_t S_0) v_k,$$

with a polynomial in $(v_1, \dots, v_k)$ in the left-hand side, depending nonlinearly on u itself of course. Such a polynomial is the same as in the case of first-order conservation laws and will be treated similarly. What we have to do now is to show that the right-hand side does not destroy the estimates. To do so, we first integrate over $\mathbb{R}$. With an integration by parts, we have

$$\partial_t \frac{1}{2} \int_{\mathbb{R}} v_k^T S_0 v_k dx + \int_{\mathbb{R}} \mathrm{Pol}_0(v_1, \dots, v_k)\, dx + \int_{\mathbb{R}} \partial_x (v_k^T S_0) \partial_x^k (B(u)u_x)\, dx = \int_{\mathbb{R}} \frac{1}{2} v_k^T (\partial_t S_0) v_k,$$

In the two last integrals, the only terms that cannot be incorporated into the polynomial of order k are respectively the integrals of

$$S_0(B v_{k+1}, v_{k+1}), \qquad v_k^T (\partial_x S_0) B v_{k+1}, \qquad v_{k+1}^T S_0 [\partial_x^k, B] u_x, \qquad \frac{1}{2} v_k^T (\mathrm{d}S_0 B v_2) v_k,$$

the last one only if $k = 1$. The first quantity is the dissipative term; it is bounded below by $\omega |B v_{k+1}|^2$. It serves to absorb the second term, and also the last one if $k = 2$, thanks to the Cauchy–Schwarz inequality. This is done up to the price of a polynomial term of order k and weight $2k + 2$. But the remarkable fact is that the third term can be absorbed the same way, after the following remark: the range of $[\partial_x^k, B]$ is contained in that of B, since the latter is independent of u. Thus the range of $S_0[\partial_x^k, B]$ is contained in that of $S_0 B$, the orthogonal to $\ker B$. This means[1] that the bilinear form $(y, z) \mapsto z^T S_0 [\partial_x^k, B] y$ depends only on y and Bz. Therefore $B v_{k+1}$ can be factored out in the third term and absorbed by the dissipation. We have finally

$$\frac{d}{dt} \frac{1}{2} \int_{\mathbb{R}} v_k^T S_0 v_k dx + \int_{\mathbb{R}} \mathrm{Pol}(v_1, \dots, v_k)\, dx + \omega_1 \int_{\mathbb{R}} |B v_{k+1}|^2 dx \leq 0,$$

with $\omega_1 > 0$ and Pol is a P_k^{2k+2}. There remains to see that the second integral above can be estimated in terms of the first one. As far as Pol_0 is concerned, this is part of the theory of first-order conservation laws, where the polynomial has weight $2k + 1$ and we have

$$\left| \int_{\mathbb{R}} \mathrm{Pol}_0(v_1, \dots, v_k)\, dx \right| \leq c_k \left(\|u\|_{W^{1,\infty}} \right) \|u\|_{H^s}^2.$$

The only change with the new terms appearing in the viscous case is that $\mathrm{Pol}_v := \mathrm{Pol} - \mathrm{Pol}_0$ is of weight $2k + 2$, and it satisfies a slightly weaker inequality:

$$\left| \int_{\mathbb{R}} \mathrm{Pol}_v(v_1, \dots, v_k)\, dx \right| \leq c_k \left(\|u\|_{W^{2,\infty}} \right) \|u\|_{H^s}^2. \tag{8.8}$$

[1] This is the argument employed for Proposition 7.1.1.

Notice that the calculations made above apply readily to the multi-dimensional case. The only change is that we have derivatives v_γ where γ are multi-indices, and k is their length. We thus obtain

$$\frac{d}{dt}\int_{\mathbb{R}}(\nabla_x^k u)^T S_0 \nabla_x^k u\,dx + 2\omega_1\int_{\mathbb{R}}|B\nabla_x^{k+1}u|^2 dx \le c_k\left(\|u\|_{W^{2,\infty}}\right)\|u\|_{H^s}^2. \tag{8.9}$$

For closing our estimate, we have to sum over $k = 0,\dots,s$ and to choose s so that $H^s(\mathbb{R}^d)$ embeds into $W^{2,\infty}$, which means $s > 2 + d/2$. Notice that in the inviscid case, we could take $s > 1 + d/2$. Then (8.9) yields a differential estimate $Y' \le C(Y)$ for

$$Y(t) := \int_{\mathbb{R}^d}\eta(u(x,t)\,dx + \sum_{k=1}^{s}\int_{\mathbb{R}}(\nabla_x^k u(x,t))^T S_0(u(x,t))\nabla_x^k u(x,t)\,dx$$

Applying Gronwall argument, we find an estimate in $\mathcal{C}(0,T;H^s)$ for $s > 2 + d/2$, where $T > 0$ depends on $\|u_0\|_{H^s}$.

The rest is technical but classical: Construction of a sequence of approximate solutions to which we can adapt the above estimates, and passage to the limit. In conclusion, we have the following existence result.

Theorem 8.1.2 *Consider a viscous system of conservation laws*

$$\partial_t u + \sum_\alpha \partial_\alpha f^\alpha(u) = \sum_{\alpha\beta}\partial_\alpha(B(u)^{\alpha\beta}\partial_\beta u). \tag{8.10}$$

Assume the following:

- *The maps $u \mapsto f^\alpha(u)$ and $u \mapsto B^{\alpha\beta}(u)$ are smooth,*
- *System (8.10) is strongly entropy-dissipative for some smooth strongly convex entropy.*
- *The range of the symbol matrix $B(\xi;u)$ does not depend neither on $\xi \ne 0$ in $\mathbb{R}^d$, nor on the state u.*

Then, given an initial data u_0 in $H^s(\mathbb{R}^d)$ with $s > 2 + d/2$, there exists $T > 0$ and a unique solution in the class

$$\mathcal{C}(0,T;H^s)\cap\mathcal{C}^1(0,T;H^{s-1}).$$

Remarks.

- Unlike several authors, we do not assume any symmetry property of the tensor $B(\xi;u)$.
- As in the relaxation context, the strong dissipativity does not ensure the global existence of smooth solution, even for small data. Such a result would require the Kawashima–Shizuta property. The latter was actually originally designed for this task.

8.2 Global existence of small and smooth solutions

TODO

This needs the Kawashima condition. See Hanouzet & Natalini [16] in one space variable, and Wen-An Yong [56] in several space variables. See [50] for the specific case of gas dynamics with friction. Ruggeri and Serre [40] prove decay even for weak entropy solutions of zero total mass in one space variable.

8.3 Shock formation

The question of global existence of smooth solutions is obviously related to that of shock formation. It is an interesting and fundamental remark that the strong dissipation and the Kawashima condition do not prevent shock breaking for general large initial data. Since an other mechanism can be at work in viscous models, we begin with an analysis for relaxation models.

8.3.1 Shocks *vs* relaxation

An example. The shock formation is so much in force that it happens even with a *total dissipation*. To see this, it will be enough to consider a scalar equation in one space dimension:

$$\partial_t u + \partial_x \frac{1}{2} u^2 + u = 0. \tag{8.11}$$

Let us introduce as usual the characteristic curves by $\dot{x} = u$, along which we have $\dot{u} = -u$, so that

$$\|u(t)\|_{L^\infty} \leq e^{-t} \|u_0\|_{L^\infty}.$$

Differentiating (8.11), we also have

$$\frac{d}{dt} u_x + u_x^2 + u_x = 0.$$

Thus the situation depends on the sign of $1 + \partial_x u_0$. If $\partial_x u_0 \geq -1$ everywhere (the smallness assumption), then $\|u_x(t)\|_{L^\infty} \leq \|\partial_x u_0\|_{L^\infty}$ for all time and we have a global smooth solution, which decays as time goes to $+\infty$. On the contrary, if $\partial_x u_0$ takes somewhere a value lower than -1, then u_x blows up un finite time. The singularity appears along the characteristics originated from the point where $\partial_x u_0$ reaches its minimum value.

General models. In a general relaxation model of the form (8.1), the shock formation for large data is associated to the genuine nonlinearity of the principal part $u \mapsto \partial_t u + \sum_\alpha \partial_x f^\alpha(u)$. For instance, the Euler equations for an isentropic fluid with friction is genuinely nonlinear for rather general equations of state:

$$\begin{aligned} \rho_t + \mathrm{div}(\rho v) &= 0, \\ (\rho v)_t + \mathrm{div}(\rho v \otimes v) + \nabla p(\rho) &= -\rho v. \end{aligned}$$

Since it is entropy-dissipative and since it satisfies the Kawashima property, small smooth data yield global solutions, as shown by Sideris & coll. [50]. However, larger data yield blow-up in finite time. This is proved by considering the evolution of appropriate integral quantities; again, see [50], Section 7.

Semi-linear models. When the principal part is linear, we say that the system is *semi-linear*. Then shocks don't form (they actually don't exist) and the solution remains smooth as long as it exists. If a blow-up occur, then it must concern the norm $\|u(t)\|_\infty$ instead of the norm $\|\nabla_x u\|_\infty$. To see this, just apply the Duhamel formula and pretend that the nonlinearity, a lower-order term, is a source.

Typical examples with a linear principal part occur in kinetic models for gases with a discrete set V of velocities. They write

$$(\partial_t + v \cdot \nabla_x) f_v = Q_v(f), \qquad \forall v \in V,$$

where Q_v are quadratic forms over $\mathbb{R}^V$. Among them, we find the Broadwell system where $V = \{\pm\vec{e}_\alpha\,;\, 1 \le \alpha \le d\}$:

$$(\partial_t \pm \partial_\alpha) f_{\alpha\pm} = Q(f) - f_{\alpha-} f_{\alpha+}, \qquad \forall \alpha = 1, \dots, d,\ \forall \pm = \pm 1, \qquad \left(Q(f) := \sum_\beta f_{\beta-} f_{\beta+} \right).$$

Kinetic models have the property that the natural domain $(\mathbb{R}^+)^V$ is positively invariant. Since it is not bounded, we may not conclude that the solution remains bounded and therefore is global. Under reasonnable assumptions upon the gain-loss source terms Q_v, the global existence is known for one-dimensional kinetic models (Tartar, Bony, Hamdache, Illner, Aregba-D. & Hanouzet). This remains an open problem in several space dimensions.

8.3.2 Shocks in viscous models

The situation is a little bit different from the previous paragraph. For if a viscous system is *totally dissipative*, then the solution remains smooth forever. A general result in this direction, at least in the case where the viscous term is linear, is given in [17] for one space variable and in [18] in arbitrary dimension. The latter paper contains a small gap that was filled in Chapter 6 of [43].

Opposite to this result, blow-up in finite time may occur when the dissipation is incomplete and for large data, as shown in the following example. We consider the system

$$\begin{aligned} v_t + \frac{1}{2}(v^2)_x + w_x &= 0, \\ w_t + v_x &= w_{xx}, \end{aligned}$$

which is strongly entropy-dissipative with entropy $\eta(u) = |u|^2$, and satisfies the Kawashima condition, since $(1,0)^T$ is not an eigenvector of the Jacobian matrix

$$\begin{pmatrix} v & 1 \\ 1 & 0 \end{pmatrix}.$$

We again consider characteristic lines that obey to the differential equation $\dot{x} = v$ (we shall explain this choice later on). We have $\dot{v} + w_x = 0$ and $\dot{v}_x + v_x^2 + w_{xx} = 0$. The latter may be rewritten

$$\dot{v}_x + v_x^2 + w_t + v_x = 0.$$

Combining these informations, we have the differential equation (along characteristics)

$$\frac{d}{dt}\left(v_x + w + \frac{1}{2}v^2\right) + v_x^2 + v_x = 0. \tag{8.12}$$

Let us begin with an L^∞ estimate of w. This will follow from the dissipation inequality

$$\sup_t \|u(t)\|_2^2 + \int_0^T \|w_x(\tau)\|_2^2 dt \leq \|u_0\|_2^2.$$

Duhamel's formula gives

$$w(t) = K^t * w_0 - \int_0^t (\partial_x K^s) * v(t-s)\, ds,$$

yielding the estimate

$$\|w(t)\|_\infty \leq \|w_0\|_\infty + c\int_0^t \|v(t-s)\|_2 \frac{ds}{s^{1/4}}.$$

After the dissipation estimates, this gives

$$\|w(t)\|_\infty \leq \|w_0\|_\infty + ct^{3/4}\|u_0\|_2. \tag{8.13}$$

Of course, the same calculation can be done over (τ, t) instead of $(0, t)$, and it gives

$$\|w(t)\|_\infty \leq \|w(\tau)\|_\infty + c(t-\tau)^{3/4}\|u_0\|_2, \qquad \forall \tau < t.$$

Let us take the fourth power and integrate this inequality over $(t-1, t)$. We obtain

$$\|w(t)\|_\infty^4 \leq 2\int_{t-1}^t \|w(\tau)\|_\infty^4 d\tau + c\|u_0\|_2^4 \leq 8\left(\sup_s \|w(s)\|_2^2\right)\int_{t-1}^t \|w_x(\tau)\|_2^2 d\tau + c\|u_0\|_2^4.$$

Finally, we obtain

$$\|w(t)\|_\infty \leq c\|u_0\|_2, \qquad \forall t \geq 1,$$

which gives, with the help of (8.13),

$$\sup_{t>0} \|w(t)\|_\infty \leq c(\|w_0\|_\infty + \|u_0\|_{L^2}) =: M. \tag{8.14}$$

We now make the hypothesis that the minimum of $\partial_x v_0 + w_0 + v_0^2/2$ is less than $-1-M$. Clearly, there exist such initial data, since M does not involve the derivative of v_0. For such an initial data, we consider the characteristic γ originated from a point where $\partial_x v_0 + w_0 + v_0^2/2 < -1-M$. Using (8.12) and (8.14), we find that $t \mapsto y(t) := v_x + w + \frac{1}{2}v^2 + 1 + M$ is non-increasing along γ, thus remains negative. More precisely, it satisfies a differential inequality of the Ricatti form $\dot{y} + \alpha y^2 \leq 0$, with $\alpha > 0$. Therefore $y(t)$ blows up in finite time. In other words, the existence time T^* is finite because of the blow up of v_x.

What is going on in the calculation above? The fundamental reason why some smooth solutions of the system considered above blow up in finite time is that its *reduced hyperbolic system* of conservation laws has a *genuinely nonlinear* characteristic field. The reduced system is that obtained after the following operations:

- retain only the first-order conservation laws (here $v_t + (v^2/2)_x + w_x = 0$),
- freeze the dissipated component, 'defined' as those that are constant in every field whose dissipation rate Bu_x vanishes identically[2] (here w).

The reduced hyperbolic system is a particular form of what is called *principal subsystem* by Boillat and Ruggeri. However, these authors start from symmetric hyperbolic systems, instead of dissipative ones.

In our example, what remains is the Burgers' equation $v_t + (v^2/2)_x = 0$, which is a paradigm for shock formation.

Recall that in the linear case

$$\partial_t u + \begin{pmatrix} C & D \\ E & F \end{pmatrix} u_x = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} u_{xx},$$

the reduced system is nothing but $v_t + Cv_x = 0$, which we have encountered in Section 5.2. We know that it is hyperbolic. More generally, the reduced system of an entropy-dissipative hyperbolic-parabolic system is a hyperbolic system of conservation laws.

Nonlinearity *vs* Kawashima. The genuine nonlinearity of the reduced system is independent from the Kawashima condition. The former may or may not be satisfied by systems obeying the latter. Let us take a two-fold example from gas dynamics. Our system governs one-dimensional flows:

$$\begin{aligned} \rho_t + (\rho v)_x &= 0, \\ (\rho v)_t + \big(\rho v^2 + p(\rho, e)\big)_x &= \nu v_{xx}, \\ \left(\frac{1}{2}\rho v^2 + \rho e\right)_t + \left(\left(\frac{1}{2}\rho v^2 + \rho e + p\right) v\right)_x &= \nu (v v_x)_x + \kappa \theta_{xx}, \end{aligned}$$

where θ is the temperature, νv_x the viscous stress and $\kappa\theta_x$ the heat flux, with $\kappa > 0$ and $\nu \geq 0$. We notice that this system is strongly entropy-dissipative for the physical entropy, and that it does satisfy the Kawashima condition.

If ν is positive, the reduced system is the transport equation $\rho_t + \bar{v}\rho_x = 0$, with $\bar{v}$ a constant. Since it does not develop shocks, we expect that smooth initial data of the whole system yield global smooth solutions, no matter the size of the data (disregarding the difficult problem of vacuum, which finds its origin in the singularity of the equation along $\rho = 0$). This has been proved under more and more reasonable assumptions, by Kazhikov, Shelukin, Hoff and Serre

[2] At this stage, it is not clear what are the *dissipated components*, and even whether this notion makes sense. This point is clarified in Paragraph 4.2.

in one space dimension, and by P.-L. Lions, Hoff and Feireisl in several space variables. So far, only the Navier-Stokes system has been treated. It is thus an open question whether the linearity or just the linear degeneracy of the reduced hyperbolic system ensures that smooth solutions extend as long as their L^∞-norm remains bounded. The analysis, often specific to the Navier-Stokes equations is so much involved[3] that it is unlikely to adapt to more general situations. For instance, it makes use of the so-called effective pressure ($F := p - \nu v_x$ in one-D), which does not have a clear counterpart in general.

On the contrary, if $\nu = 0$, then the reduced system is nothing but the Euler system for an isothermal gas. Since the latter is usually genuinely non-linear, we expect that for large data, the full system with heat diffusion only (no Newtonian viscosity) develops shock waves in finite time. This has been proved rigorously in [12]. The global existence of weak, entropy solutions for large data is still an open question.

8.3.3 Discontinuities in viscous models

The reduced hyperbolic system plays a crucial role when we deal with weak, say piecewise smooth, solutions. Because of the dissipation, $\nabla_x w$ is square-integrable in space and time, thus cannot present a discontinuity across a hypersurface. Let us denote by v the components of u that correspond to the first-order conservation laws. This means that the system (8.6) contains the subsystem

$$\partial_t v + \sum_\alpha \partial_\alpha g^\alpha(v, w) = 0. \tag{8.15}$$

The reduced hyperbolic system is nothing but

$$\partial_t v + \sum_\alpha \partial_\alpha g^\alpha(v, \bar{w}) = 0 \tag{8.16}$$

with $\bar{w}$ any constant vector. If u is a solution of (8.6), discontinuous across Σ, let us write the jump relations associated to (8.15). Herebelow, ξ denotes a unit normal vector to the spacial trace Σ_t, and σ is the normal velocity to which Σ_t travels. The brackets denote the jump of a quantity across Σ_t, and $g(\xi)$ equals as usual $\sum_\alpha \xi_\alpha g^\alpha$:

$$[g(v, w; \xi)] = \sigma[v]. \tag{8.17}$$

As mentionned above, the viscous dissipation implies $[w] = 0$, and therefore (8.17) is nothing but the Rankine–Hugoniot relations for the reduced system (8.16). Thus the discontinuities of (8.6) are given by those of (8.16).

A similar analysis can be made for the, less singular, piecewise-$\mathcal{C}^1$ solutions of (8.6). The result is now that the w component is $\mathcal{C}^1$, and that the jump in $\nabla_x v$ across Σ is governed by the same algebraic equations as if v was a solution of (8.16).

In both calculations above, w plays the role of a parameter. It is continuous (first case) or $\mathcal{C}^1$ (second case). Its value along Σ is that denoted $\bar{w}$ in (8.16). Since it may vary along Σ, the Rankine–Hugoniot relations (8.17) have variable coefficients.

[3] After all, it was one of the achievements for which P.-L. Lions was awarded a Fields medal.

Conclusion. In a hyperbolic-parabolic dissipative system, the hyperbolic features are carried by the reduced hyperbolic system (8.16). The characteristic speeds of (8.6) are nothing but those of (8.16). Likewise the reduced system gives the full description of the propagation of singularities. In particular, nonlinearity or linear degeneracy must be read at the level of (8.16).

Examples. Let us go back to the Navier-Stokes equation for a heat-conducting gas, with positive conductivity κ. If ν is positive, we have seen that the reduced system is a single transport equation $\rho_t + \bar{v}\rho_x = 0$. Thus discontinuities propagate at the material velocity v. In other words, they follow the particle path. Across a discontinuity, the velocity and the temperature remain continuous. Since the shock velocity v equals the characteristic speed, these "shocks" are actually *contact discontinuities*.

If instead $\nu = 0$ (Fourier–Euler system), the reduced system consists of the Euler equations for an isothermal gas. Thus we find genuine shocks, among which some are admissible and others are not, and we need an entropy inequality to make a selection, even for the solution of the Fourier–Euler system! The well-posedness of the Cauchy problem for general (thus large) data is still an open problem even in one space dimension, up to our knowledge. Of course, global solutions usually develop shock waves.

8.3.4 Hyperbolic-elliptic coupling

Hyperbolic-elliptic models behave very much like relaxation models, because the dissipative term is more regular than the fluxes $\partial_\alpha f^\alpha(u)$. Similarly to the cases studied above, small smooth initial data yield global smooth solutions, provided the Kawashima property holds. Likewise, solutions with large data usually develop shocks, provided the principal part, here $u \mapsto \partial_t u + \sum_\alpha \partial_\alpha f^\alpha(u)$ (the same as in relaxation), has genuinely nonlinear characteristic fields.

We shall not give much details and we content ourselves discussing the elementary example (2.13), which is totally dissipative. Not surprisingly, we define the characteristic curves by the ODE $\dot{x} = u$. Along a characteristic curve, we have $\dot{u} + u = K * u$, where $K(x) := \frac{1}{2}e^{-|x|}$. Since $\|K\|_1 = 1$, there holds $\|K * u\|_p \le \|u\|_p$. Choosing $p = \infty$, we deduce the *maximum principle*: $t \mapsto \|u(t)\|_\infty$ is non-increasing. This actually holds true for every p-norm of u, because of

$$\int_{\mathbb{R}} (K * u - u)|u|^{p-2}u\,dx \le 0, \qquad \forall u \in L^p.$$

We now turn towards the estimate of $v := u_x$. We have $\dot{v} + v^2 + v = K * v = K_x * u$, and the right-hand side is bounded, for instance by $\|u_0\|_\infty$. We infer that $t \mapsto \max\{\beta, \sup_x v(x,t)\}$ is non-increasing, where β is the positive root of $X^2 + X - \|u_0\|_\infty$.

Let us assume now that $\|u_0\|_\infty < 1/4$. Then the situation is two-fold. On the one hand, if $\inf_x u_0'$ is larger than or equal to δ, the smaller root of $X^2 + X + \|u_0\|_\infty$ (a smallness assumption), then $u_x \ge \delta$ holds true forever and therefore the smooth solution extends to all positive time, since u_x remains bounded. On the other hand, if $\inf_x u_0'$ is smaller than γ (a largeness assumption), the negative root of $X^2 + X - \|u_0\|_\infty$, then $t \mapsto \inf_x v(x,t)$ is decreasing and eventually tend to $-\infty$ in finite time. Thus the smooth solution blows up in finite time.

8.4 Shock profiles

SP may be continuous or discontinuous. This is related to the possibility of shock formation (large amplitude) or to the Kawashima condition (small amplitude).

TODO

Chapter 9

The singular limit

Lack of L^∞-estimates in general

Shi–Jin relaxation: invariant domains, adapted entropies, C-C.

TODO

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