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Yokohama Publishers
http://www.ybook.co.jp

Waves in Systems of Conservation Laws: Large *vs* Small Scales

Denis Serre

ABSTRACT. Several results about the stability of waves in systems of conservation/balance laws, disseminated in the litterature, obey to a common rule. The linear/spectral stability of the microscopic pattern (the internal structure of the wave) implies the well-posedness of a macroscopic Cauchy problem for an other system of conservation laws. The latter is often obtained by retaining only the conservation laws of the former system and dropping the higher order terms. But recent examples display a more complicated "average system".

1. Dispersive/Dissipative Mechanisms for Conservation Laws

In general, we are interested in systems of conservation laws

$$(1)\qquad \partial_t u + \mathrm{div}_x q = 0, \qquad u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, \qquad q = (q^\alpha)_{\alpha=1,\dots,d}, \quad q^\alpha(x,t) \in \mathbb{R}^n.$$

In an ideal modelling, the fluxes q^α are determined by vector fields: $q^\alpha = F^\alpha(u)$. However, in more realistic situations, there is a discrepancy between the actual fluxes q^α and the one *at equilibrium* $F^\alpha(u)$. Among the various forms of physical mechanisms, we know the following:

Dissipation: Here, $q^\alpha = Q^\alpha(u, \nabla_x u)$ where some entropy is dissipated. The mechanism is not reversible, contrary to that described by (1). For instance, one has $q = F(u) - \nabla_x u$ (artificial viscosity). An important example is given by the Navier–Stokes equations for a compressible fluid. It is the occasion to notice that even in presence of dissipation, the system (1) is not really parabolic, but displays both hyperbolic and parabolic features. This is the

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This research is partially supported by the European IHP project "HYKE", contract # HPRN–CT–2002-00282.

case in Navier–Stokes equations, where the law of conservation of mass does not involve any dissipation process.

Dispersion: The mechanism may be reversible, but the flux depends on some derivative of the unknown. Examples are $q = F(u) - \Delta_x u$ (KdV-type equation) or $q = F(u) - i\nabla_x u$ with $i = \sqrt{-1}$ (Schrödinger-type equation). Notice that both dispersive and dissipative terms may be of higher order. As far as fluid dynamics is concerned, Korteweg's capillarity gives an instance of dispersive term, a bit more complicated than that in KdV equation.

Relaxation: One may couple (1) to a system of balance laws that has the tendency to pull back the flux to the equilibrium flux. Relaxation models have been known for several decades (see [15]) and can be highly nonlinear. Let us just mention the simple Jin–Xin model, where (1) is coupled to

$$\partial_t q^\alpha + a^2 \partial_\alpha u = \frac{1}{\tau}(F^\alpha(u) - q^\alpha). \tag{2}$$

R-models: The "R" in this denomination may come from the radiative gas dynamics, or after the physicist Rosenau [11], who coined a model as a regularized version of the Chapman–Enskog expansion in hydrodynamics. Typically, $q = F(u) + \epsilon(u - K * u)$ where K is an integrable kernel with the properties that $K(-x) = K(x)$ and

$$\int_{\mathbb{R}^d} K(x)dx = 1, \qquad \int_{\mathbb{R}^d} (x \cdot \xi)^2 K(x)dx \geq 0 \quad (\forall \xi \in \mathbb{R}^d).$$

A special choice of the kernel amounts to coupling (1) with an elliptic equation:

$$\partial_t u + \mathrm{div}_x(F(u) - r) = 0, \qquad -\Delta r + r = \nabla u. \tag{3}$$

2. Travelling Waves

We are interested in planar travelling waves, namely solutions of (1) of the form

$$u(x,t) := U(x \cdot \nu - ct), \quad \|\nu\| = 1.$$

The wave travels in the direction ν at the velocity c. Up to a rotation, and to the choice of the moving frame, which change the fluxes of each model in an obvious way, we may assume that u is a steady solution that varies only along the first axis of coordinates:

$$u = U(x_1).$$

The field U is the *profile* of the wave. It is typically a solution of an ODE, the *profile equation*. For instance, in artificial viscosity,

$$U'' = (F^1 \circ U)'.$$

Because (1) is made of conservation laws, the corresponding ODEs may be integrated once ; in the above example, this means

$$U' = F \circ U + \mathrm{cst}.$$

Compatibility at $\pm\infty$ usually results in *Rankine–Hugoniot conditions*. Following our example, this reads

$$[F \circ U]_{-\infty}^{+\infty} = 0.$$

Notice that when (1) is coupled to other equations, the corresponding ODEs do not admit first integrals in general.

There are typically four types of interesting waves, depending on whether the physical domain is $\mathbb{R}^d$ or is the half-space $\mathbb{H} := (0, +\infty) \times \mathbb{R}^{d-1}$.

Shock front.: When the domain is $\mathbb{R}^d$, one asks U to have distinct limits at both ends:

$$U(-\infty) = a_l, \quad U(+\infty) = a_r \qquad (a_r \neq a_l).$$

As a solution of the profile equation, U is a heteroclinic orbit. Its convergence towards the end states $a_{r,l}$ is generically exponential. This generic behaviour is an important assumption in the so-called Gap Lemma.

Solitary wave.: Similar, but with equal limits ($a_r = a_l$). The profile is a homoclinic orbit. One may always choose $a_{r,l} = 0$.

Periodic wave.: The opposite behaviour is when the profile is periodic:

$$U(\cdot + Y) = U.$$

Contrary to the linear case, the period Y is not given *a priori* by the conservation laws. It is merely a feature of the wave itself.

Boundary layer.: When the domain is $\mathbb{H}$, the system (1) is supplemented by appropriate boundary conditions. The profile must fit them at $x_1 = 0$, and we ask it to have a limit at infinity:

$$U(+\infty) = a_r.$$

Of course, ODEs may have a lot of other interesting orbits besides periodic and hetero/homo-clinic ones. But the present list is rich enough to have a nice mathematical theory.

3. Stability Issues

In presence of a standing wave, a natural question is that of its stability, which can be raised at various mathematical levels. The most useful one is certainly that of *nonlinear stability*. When the domain is $\mathbb{R}^d$, our models are translational invariant and we only expect a kind of *orbital stability*, meaning that under a small initial disturbance, the solution of our PDEs is asymptotic to the set of waves $U(\cdot + h)$ as $h \in \mathbb{R}$. In other words, there is a function $h(t)$ such that $u(\cdot, t) - U(\cdot + h(t))$ tends to zero in an appropriate norm. The situation is often better, because of the conservation of mass (which holds true except in the initial-boundary value problem)

$$\frac{d}{dt}\int_{\mathbb{R}} (u(x,t) - U(x))dx = 0.$$

In the case of a shock front, this helps to proving that $h(t)$ has a limit. Therefore, the expected result is that there exists an h_0 such that

$$u(\cdot,t) \to U(\cdot + h_0).$$

We point out that the shift h_0 can be determined explicitly in terms of the mass of the disturbance.

Less complete information is given by *linear stability*. Linearization about the steady solution $U(x_1)$ yields a linear system

$$\partial_t v = Lv, \tag{4}$$

where v denotes the linearized unknowns (beware of the additional unknowns in relaxation or in R-models). In the case when the domain is the half-space, one needs also the linearized boundary conditions. Under rather natural assumptions, the corresponding Cauchy problem (or IBVP) generates a semi-group $\exp(tL)$ that is continous on scale of Banach spaces. Linear stability states that

$$\lim_{t\to+\infty} \|\exp(tL)\|_{\mathcal{L}(X,Y)} = 0 \tag{5}$$

for suitable Banach spaces X and Y. When the domain is $\mathbb{R}^n$, the translational invariance shows that $LU' = 0$, a fact that is not compatible with (5). Therefore one must read instead

$$\lim_{t\to+\infty} \|\exp(tL)\|_{\mathcal{L}(X/N,Y)} = 0 \tag{6}$$

where N is a finitely dimensional (generalized) kernel.

The weakest notion is that of *spectral (in)-stability*. This is the property that the spectrum of the operator L above is (is not) a subset of the left half-space $\{\lambda \in \mathbb{C}\,;\, \Re\lambda < 0\}$. For several reasons, the stability property does never hold in such a strong form. The fact that the physical domain is unbounded, plus the translational invariance at infinity, result in that the essential spectrum contains the origin. In rare cases (typically a scalar conservation law), the introduction of a weight can move the essential spectrum strictly to the left of the imaginary axis, but this is unlikely in general. This raises the difficulty of the lack of a *spectral gap*. For a shock front or a solitary wave, the translational invariance implies that zero itself is an eigenvalue, whose algebraic multiplicity may be relatively high and is related to the kernel N. Then another difficult appears, as this eigenvalue is embeded in the essential spectrum.

Because it has variable coefficients, it is impossible in general to describe completely the spectrum of L. Even determining whether spectral stability holds true is a task beyond our current knowledge, except in rare cases. We shall discuss in a moment the techniques that are employed to prove on the contrary that a wave is spectrally unstable. Let us mention for the moment the strategy that has been developped by Zumbrun and co-authors in the general stability problem. One starts with the assumption that the wave is spectrally stable, in a suitable sense. This requires the stability of the states at infinity if any, plus the fact that the origin is the only element in the spectrum to have a non-negative real part, and also that it does not degenerate as an eigenvalue (degeneracy is a typical cause of transition between stability and instability). Here is the most difficult part: One estimates the Green function for the operators $L-\lambda$ on relevant function spaces. One needs to cover not only complex values of λ at the right of the spectrum of L, but also in some neighbourhood of the origin. This means understanding the Green function in some domain where L is not a Fredholm operator ! This is done by analytic continuation, with an extension of the *gap lemma* of Gardner and Zumbrun [5]. Then the semi-group is reconstructed by a contour integral, from which one obtains estimates. Passing from the linear stability to the nonlinear orbital stability is somehow less difficult and consists in applying the linear estimates in a fixed point equation obtained from the Duhamel's principle. We refer to [17] for an overview of the method. This topic is in rapid progress. The simplified situations considered in the first articles have leaved the place to more realistic contexts. However, there remains several important open questions.

4. How to Prove Spectral Instability

If L was an operator over a finite dimensional space, its spectrum would be given as the zero set of its characteristic polynomial. It is not possible to define a similar object in infinite dimension, but something can be done, at least in a domain of the complex plane that contains the right half-plane. Observing first that the coefficients of L depend only on x_1 (the fact that they vary is the cause of our difficulties), we first perform a Fourier transform in $(x_2,\dots,x_n)$. This decouples our spectral problem into one-dimensional problems parametrized by the Fourier variable $\eta\in\mathbb{R}^{d-1}$:

$$(L_\eta-\lambda)w=0. \tag{7}$$

For fronts (namely shock fronts, solitary waves and boundary layers), a part Λ of the essential spectrum of L is given explicitly as the union of the spectra of the linearized operator at each far states $a_{r,l}$. Let Ω be the connected component of Λ^c containing large positive real numbers. Assuming the L^2-stability of the far states, Ω contains the closed right half-space, but the origin. More importantly, Ω does not meet the essential spectrum. In other words, each of the differential operators $L_\eta-\lambda$ ($\eta\in\mathbb{R}^{d-1}$ and $\lambda\in\Omega$) are Fredholm of index zero. Thus it becomes possible to apply a shooting method for the eigenvalue equation (7). The space $S(\lambda,\eta)$ of solutions of (7) that tend to zero at $+\infty$ (the stable space) depends analytically on λ. For shock fronts or solitary waves, there is a related unstable space $U(\lambda,\eta)$ (solutions that t end to zero at $-\infty$). The sum $\dim S(\lambda,\eta)+\dim U(\lambda,\eta)$ equals the dimension of the space of all solutions of (7). Hence for λ to be an eigenvalue, it is necessary and sufficient that $S(\lambda,\eta)$ and $U(\lambda,\eta)$ intersect non trivially. This can be encoded in an analytic equation $D(\lambda,\eta)=0$ where D is the Wronskian determinant of some analytically chosen bases of $S(\lambda,\eta)$ and $U(\lambda,\eta)$.

In the case of boundary layers, one has only to match $S(\lambda,\eta)$ with the space defined by the linearized boundary conditions. This yields an other determinant (smaller than the one defined above), still denoted by $D(\lambda,\eta)$.

The situation is slightly different in the periodic case, for then the domain of L_η is compact and its spectrum becomes discrete. A shooting method does not make sense any more. However, following Gardner [4], one may consider the monodromy matrix $M(\lambda,\eta)$ of the ODE (7), which maps the Cauchy data (at $x_1=0$) to the corresponding data after one period (at $x_1=Y$). Following Floquet's theory, λ is a spectral value of L over $L^p(\mathbb{R}^d)$ if and only if $M(\lambda,\eta)$ has an eigenvalue on the unit circle for some η. We therefore define a function $D(\lambda,\eta,\theta)$ over $\mathbb{C}\times\mathbb{R}^{d-1}\times\mathbb{R}$ by the formula

$$D(\lambda,\eta,\theta):=\det(M(\lambda,\eta)-e^{i\theta}I_N).$$

Then the spectrum of L is the λ-projection of the zero set of D. It is amazing that even in one space dimension ($d=1$) where η is not present, one has essentially the same complexity as we had in the case of a multi-dimensional front. For this reason, we restrict to one-dimensional periodic waves, so that $D=D(\lambda,\theta)$. Notice that we work on spaces $L^p(\mathbb{R}^d)$ or $L^p(\mathbb{R})$ that ignore the periodicity of the wave U. We are really interested in its stability upon localized initial disturbances. This kind of stability issue is called *modulational stability.*

The function D that was defined above is called an *Evans function* of the operator L. We point out that it is not uniquely defined ; it depends either on the choice of bases of $S(\lambda,\eta)$ and $U(\lambda,\eta)$, or on the choice of Cauchy data for (7). However, its zero set in Ω is uniquely defined. It can be proved also that the multiplicity of its zeroes does not depend on the particular choices. An important feature is that, L being real valued (but L_η is not when $\eta\neq0$), the Evans function may be chosen with the property

$$\overline{D(\lambda,\eta,\theta)}=D(\bar\lambda,-\eta,-\theta).$$

In particular, $D(\lambda,0)$ takes real values when $\lambda\in(0,+\infty)$. Notice that for fronts, the section $D(\cdot,0)$ encodes the stability properties of U under perturbations that depend only on x_1 (1-D (in-)stability). In the periodic case, it encodes the stability properties of U under perturbations that are Y-periodic (non-modulated (in-) stability).

Extension near the origin

A fundamental problem of the theory is the behavior od the Evans function near the origin ($\lambda=0$, $\eta=0$ or $\theta=0$). This corresponds to a large wave-length analysis of the stability (see next section). The first difficulty is to check whether D has some regularity there. This is a clear fact in the periodic case, since the monodromy matrix is defined for every parameters and depends analytically on them. Hence D is analytic on $\mathbb{C}\times\mathbb{R}$! In the three other situations, the definability of D is a difficult problem that was solved by Gardner and Zumbrun [5] (see also [16] for the multi-dimensional case) thanks to the Gap Lemma. Under the assumption that the profile tends exponentially fast to its end states (a generic fact), each section $D(\cdot,\eta)$ of the Evans function admits an analytic extension to a neighbourhood of zero. We warn the reader that this extension has not any more the meaning of a Wronskian between the stable and the unstable spaces of (7). We point out too that one obtains λ-regularity, but that η-regularity does not hold in general.

Next to the definition of D, a practical description of D is needed near the origin, which will be used in the subsequent computations. We point out that D is not smooth in general with respect to η at $\lambda = 0$; it displays typically a conical behaviour.

4.1. The stability index

Since $D(\cdot, 0)$ is a real-valued function on $(0,\infty)$, and since the behavior of D at the origin is more or less well understood, we may compare its sign for $\lambda \to 0+$ and for $\lambda \to +\infty$. The analysis near infinity is not too hard, as the dissipative/dispersive terms dominate. We point out that since D is not uniquely defined, these signs are not carved in the marble, and only the product of both really makes sense ; this product is called the *stability index* of L (or of the profile U). The knowledge of this index tells the parity of the number of unstable eigenvalues of L_0. Hence a necessary condition for stability is that the stability index be positive. In other words, a wave of negative index is certainly unstable.

The main difficulty is to find a continuity argument (a *homotopy* in terms of [1]) to compute this index. This was done for 2×2 parabolic systems in [5], then extended to $n \times n$ in [1]. The result is fully explicit when U is a profile for an *extreme* shock. The case of boundary layers is not so explicit, because we do not have such a good description of D at the origin. Several instances of dispersive equations, like KdV, have been treated with an argument to Arnol'd, which involves a Hamiltonian structure, but the present techniques would be handable there too. Finally, the formula for the stability index in the periodic case has been established by Oh & Zumbrun [9].

4.2. The large wave-length analysis

As explained above, D is smooth along rays. In particular, there exists an analytic function Δ defined for $\Re\lambda > 0$ and real η or θ, and an integer m, such that

$$D(\rho\lambda, \rho\eta) \sim \rho^m \Delta(\lambda, \eta),$$

as $\rho \to 0$ along $(0,+\infty)$. The function Δ is clearly homogeneous of degree m, and holomorphic in λ. It was observed in [16] that the vanishing of Δ somewhere implies that of D along some arc, and therefore yields spectral instability. This is a consequence of Rouché's theorem of persistence of zeros of holomorphic functions.

We deduce the following necessary condition for spectral stability, that *the function* Δ *must not vanish for* $\Re\lambda > 0$ *and real* η *or* θ. We now point out the remarkable fact that this means precisely that the Cauchy problem for some "limit system" is (weakly) linearly well-posed. Hence the general rule

> *Assume that the wave with profile U is spectrally stable. Then an appropriate "limit problem" is (weakly) linearly well-posed about the macrostate $\bar{u}$ associated to this wave.*

We notice that $m = 0$ for boundary layers in general, because there is no translational invariance and therefore no obvious element in $\ker L_0$, so that $D(0,0) \neq 0$.

5. The Limit Problem

An other approach to the large wavelength analysis is modulation theory (see [15]). It consists in rescaling the variables by $y := \epsilon x$ and $\tau := \epsilon t$, then in finding a "limit problem" which governs the envelop of the perturbed solution. Notice that $\partial_t = \epsilon\partial_\tau$ and $\partial_x = \epsilon\partial_y$, so that the higher derivatives become negligible except in the zone where $U(y_1/\epsilon)$ varies significantly. In the case of fronts, this means everywhere but on a codimension one subset (the shock front, the boundary). For periodic waves, this is just false everywhere !

Solitary waves. The simplest example is that of a solitary wave. After the change of scale, U looks like a very narrow pike and converges strongly in every L^p (but in L^∞) towards a constant state $\bar{u}$ equal to the far state.The limit problem is the hyperbolic one that is obtained by dropping the derivatives of order two or more in the conservation laws, and all derivatives in the balance laws. This is a first order system of conservation laws. Let us give a few examples of original and limit systems:

Viscous systems:

$$\partial_t u + \mathrm{div}_x F(u) = \sum_{\alpha,\beta} \partial_\alpha (B^{\alpha\beta}(u)\partial_\beta u) \qquad \longmapsto \qquad \partial_\tau u + \mathrm{div}_y F(u) = 0.$$

Relaxation:

$$\left.\begin{array}{rcl} \partial_t u + \mathrm{div}_x q & = & 0 \\ \partial_t q + A(\nabla_x) u & = & F(u) - q \end{array}\right\} \qquad \longmapsto \qquad \partial_\tau u + \mathrm{div}_y F(u) = 0.$$

We point out that the number of unknowns diminishes in the limit of a relaxation model. In particular, a part of the initial data must be dropped. A similar phenomenon will be encountered in the limit of IBVPs.

A necessary condition for stability of the solitary wave is that the far state $a_l = a_r$ be a stable equilibrium of (1), which implies in particular that the limit problem is weakly hyperbolic about $\bar{u}$.

Shock fronts. A less simple case is that of a shock front. After the change of scale, U looks like a step shock $\bar{u}$ between a_l and a_r, with front at $y_1 = 0$. The limit problem is the same as above, but for data close to $\bar{u}$ instead of a constant. It thus involves a free boundary (the shock location). The well-posedness is encoded in a *Lopatinskii determinant*, which turns out to be Δ, as shown in [16]. A necessary condition for well-posedness is precisely that Δ does not vanish for $\Re\lambda > 0$ and real η. This is actually necessary and sufficient for $\mathcal{C}^\infty$ well-posedness (a uniform version of this Lopatinskii condition amounts to well-posedness in H^s when the hyperbolicity is strict, as shown by Majda [8].) The conclusion of [16] is thus that the spectral stability of U implies the weak well-posedness of the first-order limit problem around the step shock $\bar{u}$.

We notice that the PDEs of the limit problem for the shock case depends only on the lower order terms in (1). For instance, in the vanishing viscosity method, it does not depend on the choice of the viscosity tensor. In relaxation models, it does not depend on the precise form of this dissipation process. One may ask whether the limit problem itself is independent of the original dissipation. This amounts to asking whether the admissibility criterion for discontinuities may vary according to the underlying model. It turns out that when $\bar{u}$ is a Lax shock and (1) dissipates somehow, the connection U is structurally stable in general ; then the admissibility conditions are that a discontinuity close to (a_l, a_r) is always admissible if $\bar{u}$ was, and small amplitude discontinuities close to either a_l or a_r are admissible if and only if they satisfy the so-called E-condition of T.-P. Liu and L. Hsiao. Thus the limit problem is completely described by its flux F, as far as small disturbances of $\bar{u}$ are concerned. The situation changes significantly when $\bar{u}$ is a non-Lax shock. For instance, if $\bar{u}$ is undercompressive, the admissible shocks close to $\bar{u}$ form a submanifold of the Hugoniot set, whose codimension equals $n+1$ minus the number of incoming characteristics. The important feature here is that this submanifold does depend on the profile equation and therefore upon the original system (1).

Boundary layers. A slightly more involved situation is observed for boundary layers. Since the variations of U are localized along the boundary, the PDEs in the limit system are obtained exactly as in the shock case above. However, the boundary conditions are a bit non trivial, as understood in Gisclon's thesis [6]. This may be seen through the fact that either the order of the limit system is strictly less than that of the original system (1), or that some unknowns have been dropped (if there was relaxation processes). In both situations, one needs less many boundary conditions than before. The general description of the role of the so-called "residual boundary conditions" in the stability of the layer is given in Rousset's paper [12]: Once again, the limit problem is a first-order initial-boundary value problem, whose (weak) well-posedness is precisely the fact that Δ does not vanish. We emphasize that unlike the Lax shock case, the limit problem does depend on the non-principal part, through the residual boundary condition.

Without going into the details, let us just describe what is the residual boundary condition in the viscous case. It is worth to notice that only the normal derivatives play a role in its derivation. Thus it is enough to consider a one-dimensional model

$$\partial_t u + \partial_x F(u) = \partial_x (B(u)\partial_x u).$$

The dissipation is characterized by the field of matrices $u \mapsto B(u)$. For the sake of simplicity, we shall assume that the system is parabolic. In particular, $B(u)$ is non-singular. It will be enough to assume that the boundary condition associated to (1) is of Dirichlet type, $u(0,t) = a$. The system admits steady solutions of the form $w(x)$ for which

$$B(w)w' = F(w) + \text{cst}.$$

If we ask that w has a limit $b \in \mathbb{R}^n$ as $x \to +\infty$, then the constant of integration is $-F(b)$, whence the ODE

$$w' = B(w)^{-1}(F(w) - F(b)). \tag{8}$$

Of course, it is not true in general that the solution of the Cauchy problem for (8) with data $w(0) = a$ tends to b at $+\infty$. We denote by $\mathcal{C}(a)$ the set of b's for which this holds true. The residual boundary condition writes

$$v(0) \in \mathcal{C}(a).$$

We point out that $w = U$ is a special solution of (8), so that we do have $a_r \in \mathcal{C}(a)$. We also emphasize that the set $\mathcal{C}(a)$ depends heavily on the underlying viscosity tensor. One task of the theory is to prove that $\mathcal{C}(a)$ is a submanifold through a_r, which has the transversality property for the 1-D limit IBVP to be locally well-posed.

Periodic waves. The complexity of the limit problem is even higher when U is periodic. The main reason is that there is no obvious terms in the PDEs (1) that could be neglected or dropped in the limit $\epsilon \to 0$. Also, the macrostate $\bar{u}$ is not a strong limit of $U(y_1/\epsilon)$ since the latter oscillates ; it must be some kind of weak limit. Because we are interested in modulational (in)stability, the relevant domain $\mathcal{P}$ of microstates consist of all periodic travelling waves with an arbitrary period and an arbitrary velocity (see [14]), *modulo* space shift (from a macroscopic point of view, a space shift is meaningless). Let n be the number of conservation laws in (1). If there is no additional conservation laws to be compatible with (1), then the set of genuinely space-periodic (meaning periodic and non constant) travelling waves is generically a manifold of dimension $n+2$, so that $\mathcal{P}$ is a manifold of dimension $n+1$. This reveals that the limit problem must be described by a first-order system of $n+1$ conservation laws in $n+1$ unknowns. This is a remarkable discrepancy from the front cases.

Of these $n+1$ conservation laws, n are obtained directly by averaging the original ones in (1): Every $\partial_t u_i + \partial_x F_i(u, \partial_x u, \dots) = 0$ yields $\partial_\tau \langle u_i \rangle + \partial_y \langle F_i(u, \partial_x u, \dots) \rangle = 0$ where the brackets denote the mean value of a periodic function. The resulting system of n conservation laws is not closed, because the average $\langle u \rangle$ does not determine the periodic travelling wave up to a space shift (n scalars do not determine $n+1$ unknowns). The last limit equation arises as $\partial_\tau \omega + \partial_y (c\omega) = 0$, where ω is the space frequency and c is the velocity of the waves in the variables (x,t). This is classical in averaging theory (see [15]). In the context of compressible fluid dynamics with a Van der Waals equation of state, the modulation theory was anticipated in [13], though the link between stability properties was not established at that time.

The important fact proved in [14] is naturally that the same rule as above holds true in the periodic situation. The (weak) well-posedness of the $(n+1) \times (n+1)$ limit problem (namely its weak hyperbolicity) is precisely the fact that Δ does not vanish for $\Re\lambda > 0$ and θ real. We notice that since D is analytic in (λ, θ), Δ is now a homogeneous polynomial. It can be written in terms of a characteristic polynomial

$$\Delta(\lambda, \theta) = \Gamma \det(\lambda \mathrm{d}F_0(\bar{u}) + i\theta \mathrm{d}F(\bar{u})),$$

where $\bar{u}$ is the state corresponding to U and F_0, F_1 are the fluxes in the limit problem $\partial_\tau F_0(v) + \partial_y F(v) = 0$. The Jacobian matrices are taken with respect to an arbitrary coordinate system on $\mathcal{P}$. Of course, the constant $\Gamma(\bar{u})$ is sensible to the choice of coordinates. It is non-zero whenever the orbit U has the transversality property which ensures that $\mathcal{P}$ is a submanifold.

The situation is not much different when (1) admits additional conservation laws. However, this implies in general that the dimension of $\mathcal{P}$ is larger than $n + 1$. For instance, in the Korteweg de Vries equation, for which $n = 1$, an open subset of the profile equation consists in periodic trajectories. This makes the dimension equal to 4, and that of $\mathcal{P}$ equal to 3, which is larger than $n + 1 = 2$. The limit system is obtained by averaging three conservation laws in general position, for instance. It is the well-known second system of the averaged-KdV hierarchy (the p-th consists in $2p - 1$ conservation laws, see [7]). Once again, the fact that $\Delta(\lambda, \theta)$ does not vanish for $\Re\lambda > 0$ and θ real means that the limit system is weakly hyperbolic. This latter fact was proved directly in [7].

6. Conclusion and Perspectives

The important rule described above is that, to a system of the form (1) with some given closure between q and u, and to some travelling wave $u = U(x_1)$ of simple form (homo/hetero-clinic connection, boundary layer, periodic orbit), there is a limit problem that is obtained after a rescaling. The natural phase space of this problem is a finite dimensional set whose description may be very easy (solitary waves, shock fronts), or quite complicated (boundary layers, periodic waves). The spectral stability of u is encoded into an Evans function $D(\lambda, \ldots)$, where λ is the Laplace frequency for the time and the dots account for the Fourier parameters. Weak spectral stability holds when D does not vanish when $\Re\lambda > 0$ and the others variables are real. The large wavelength analysis yields an asymptotics $D(\rho\lambda, \rho\eta) \sim \rho^m \Delta(\lambda, \eta)$ near the origin. Spectral stability implies the same non-vanishing property for Δ, by Rouché's theorem. This precisely means that the limit problem is weakly linearly well-posed around the macrostate $\bar{u}$ associated to u.

Such an analysis is versatile. For instance, it was adapted in [14] to discretized systems provided by numerical analysis. A natural scale is that given by the mesh size. R. Cavazzoni [2] has performed the general strategy in a model of gas dynamics that combines relaxation (of the equation of state) and Korteweg's capillarity. It would be interesting to extend it to more complicated waves, as quasi-periodic ones, when such waves are present. This would give an approach to the next systems in the KdV hierarchy. The goal here would be to construct an appropriate Evans function and then to make its Taylor expansion at the origin.

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Denis Serre
École Normale Supérieure de Lyon
UMPA (UMR 5669 CNRS)
46, allée d'Italie, F–69364 Lyon, cedex 07, France
E-mail address: serre@umpa.ens-lyon.fr