Irrotational flows for Chaplygin gas. Conical waves

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1 Steady and self-similar flows of a compressible gas

Let $d = 2$ or 3 be the space dimension. The Euler equations of an isentropic compressible fluid governs the mass density $\rho > 0$ and the velocity field u. The pressure p is a prescribed smooth function of ρ. We assume that $\partial p/\partial\rho > 0$, the square root $c(\rho)$ being the sound speed.

$$\partial_t\rho + \operatorname{div}(\rho u) = 0, \tag{1}$$

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p(\rho) = 0. \tag{2}$$

Remark that real flows should be non-isentropic. The constancy of the entropy is not compatible once shock waves develop. It is only expected that the amplitude of oscillations for the entropy are of cubic order with respect to those of the flow. There is one notable exception, that of a Chaplygin gas, whose pressure has the form $p(\rho, s) = g(s) - f(s)/\rho$; then the entropy remains constant even beyond shock formation, if it was at initial time. This is proved in [5] for $d = 2$; see [6] for $d = 3$.

Irrotational flows. Assuming that the flow is $\mathcal{C}^2$ and isentropic, the vorticity $\omega := \operatorname{curl} u$ is transported according to

$$(\partial_t + u\cdot\nabla)\frac{\omega}{\rho} = (\frac{\omega}{\rho}\cdot\nabla)u. \tag{3}$$

If the initial velocity is irrotational ($\omega(\cdot, 0) \equiv 0$), the flow remains irrotational. Once again, this conclusion is incorrect for realistic flows once shock waves appear, but it is expected to be a good approximation if the flow has a mild amplitude. But for a *Chaplygin gas*, steady or self-similar irrotational flows are perfectly valid solutions of the full Euler system [5, 6].

For an isentropic, irrotational flow, we introduce the velocity potential ψ by $u = \nabla\psi$. It is governed by the Bernoulli equation

$$\partial_t\psi + \frac{1}{2}|\nabla\psi|^2 + \imath(\rho) = 0, \tag{4}$$

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where

$$\imath(\rho) := \int^{\rho} \frac{c^2(\mu)}{\mu}\, d\mu$$

is the enthalpy. The flow is determined by initial and/or boundary conditions for ρ and ψ, together with the PDEs (1,4), where the former can be written in the form

$$\partial_t \rho + \mathrm{div}(\rho \nabla \psi) = 0. \tag{5}$$

1.1 Steady flows

For an irrotational steady flow, $\partial_t \psi$ is a constant. We have therefore the Bernoulli equation

$$\frac{1}{2}|\nabla \psi|^2 + \imath(\rho) = \kappa, \tag{6}$$

with κ a constant. Since $\imath' = c^2/\rho$ is positive, this can be resolved in terms of the density:

$$\rho = h(\frac{1}{2}|\nabla \psi|^2 - \kappa).$$

Then the system of PDEs reduces to a single second-order equation

$$\mathrm{div}\left(h(\frac{1}{2}|\nabla \psi|^2 - \kappa)\nabla \psi \right) = 0. \tag{7}$$

Jump relations. Let the flow be discontinuous across a hypersurface with unit normal ν, moving at normal speed σ. The Rankine-Hugoniot condition for mass conservation writes

$$[\rho(u \cdot \nu - \sigma)] = 0. \tag{8}$$

Equation (8) allows us to define a net flow

$$j := \rho(u \cdot \nu - \sigma),$$

which has the same value on both sides.

The rest of the jump relations, as well as the nature of discontinuities, depends on the model under consideration. In the isentropic case, we have both acoustic waves (shocks) in which $j \neq 0$, and vortex sheets, along which $j = 0$. In the irrotational case, we only have acoustic waves. Across an isentropic shock, we have

$$j[u] + [p]\nu = 0, \tag{9}$$

which is equivalent to the combination of

$$[u_T] = 0 \tag{10}$$

and

$$j^2 \left[\frac{1}{\rho} \right] + [p] = 0. \tag{11}$$

For an irrotational shock, we only have (10), which amounts to writing the continuity of ψ.

1.2 Self-similar flows

The Euler system being a system of first-order conservation laws, it is invariant under space-time dilations. Therefore, dilation-invariant initial data (that is data that are positively homogeneous of degree zero) yield dilation-invariant solutions, if the Cauchy problem has a well-posedness property. It is thus interesting to look for solutions that depend only upon the ratio

$$y := \frac{x}{t}\,.$$

The Cauchy problem for a dilation-invariant data $U(\cdot,0)$ is called the *Riemann problem.*

When acting on a function that depends only upon y, the time derivative is equivalent to the operator $-t^{-1}y\cdot\nabla_y$. The Euler equations takes the new form

$$\begin{aligned} -y\cdot\nabla_y\rho+\mathrm{div}_y(\rho u) &= 0,\\ -(y\cdot\nabla_y)(\rho u)+\mathrm{div}_y(\rho u\otimes u)+\nabla p &= 0,\\ -y\cdot\nabla_y E+\mathrm{div}_y((E+p)u) &= 0, \end{aligned}$$

This system looks more complicated than the original one, but can be written in a simpler form by employing the so-called *pseudo-velocity*

$$v(y) := u(y) - y,$$

where we notice that both u and y have the dimension of a velocity. We obtain

$$\mathrm{div}_y(\rho v)+d\rho = 0, \tag{12}$$

$$\mathrm{div}_y(\rho v\otimes v)+(d+1)\rho v+\nabla p(\rho,s) = 0, \tag{13}$$

$$\mathrm{div}_y((E[v]+p)v)+\rho|v|^2+d(E[v]+p) = 0, \tag{14}$$

where d stands for the dimension of the physical space. Its value, an integer between 1 and 3, depends on the applications we have in mind. Equation (14) above involves the quantity

$$E[v] := \frac{1}{2}\rho|v|^2+p(\rho,s)$$

which is obtained by replacing u by v in $E =: E[u]$.

Still, self-similar flows may be isentropic and even irrotational. In the latter situation, the potential ψ of u is replaced by $\Phi=\psi-\frac{t}{2}|y|^2$, that of v,

$$v = \nabla_x\Phi.$$

Remark that Φ is positively homogeneous of degree one. We prefer instead to use $\phi(y) := t^{-1}\Phi$, for which we have $v=\nabla_y\phi$. It satisfies the modified Bernoulli identity

$$\frac{1}{2}|\nabla\phi|^2+\phi+\imath(\rho)=0, \tag{15}$$

where now the constant of integration has been absorbed into ϕ. We may recover the density in terms of the potential,

$$\rho = h(\frac{1}{2}|\nabla\phi|^2 + \phi),$$

and rewrite the system as a second-order PDE

$$\text{div}\left(h(\frac{1}{2}|\nabla\phi|^2 + \phi)\nabla\phi\right) + dh(\frac{1}{2}|\nabla\phi|^2 + \phi) = 0. \tag{16}$$

Initial data. Self-similar flows are generated by self-similar data $U(\cdot, 0) =: U^0$ that are constant along rays around the origin. This yields "boundary conditions at infinity" when solving (12,13,14), because of $U(x,t) = U(x/t)$,

$$\lim_{\mu\to+\infty} U(\mu y) = U^0(y). \tag{17}$$

Jump relations. Because the principal part of (16) is the same as that of (7), the jump relations are very similar to the steady irrotational case. They consist on the one hand of the continuity of the potential, from which we derive

$$[v_T] = 0, \tag{18}$$

and on the other hand of

$$[\rho v \cdot \nu] = 0. \tag{19}$$

1.3 Chaplygin gas

At constant entropy, a Chaplygin gas obeys the equation of state

$$p(\rho, s) = p_\infty - \frac{a^2}{\rho}, \qquad a > 0. \tag{20}$$

The sound speed is $c = a/\rho$. The acoustic fields are linearly degenerate, in the terminology of hyperbolic systems. Two extreme regimes are excluded *a priori* for physical reasons. On the one hand, a small density would yield a negative pressure. On the other hand the pressure saturates at very high densities, which allows the mass to concentrate along codimension-one subsets. Therefore the Chaplygin equation of state is used only so far as the density remains in a suitable compact interval of $(0, +\infty)$.

Jump relations. In an acoustic wave, property (11) yields $(j^2 - a^2)[\frac{1}{\rho}] = 0$. Since the density is discontinuous across such a wave, we deduce

$$j^2 = a^2. \tag{21}$$

It is remarkable that in a Chaplygin gas, the net flow does not depend on the strength of the discontinuity, but always equals $\pm a$. Using the definition of j, this gives on both sides of the wave

$$(u_{\pm} \cdot \nu - s)^2 = a^2/\rho_{\pm}^2, \tag{22}$$

which tells us that acoustic waves are sonic: $|u \cdot \nu - s| = c$. This property is specific to the Chaplygin gas.

Irrotational flows are true flows. For general flows, there is no reason why an irrotational data would produce an irrotational flow if there are discontinuities. However, the situation is favourable for a steady isentropic flow.

Theorem 1.1 (See [6]) *Consider a steady shock wave in a Chaplygin gas. Let Γ be the surface of discontinuity and ν its unit normal. Let us decompose the vorticity into normal and tangential components along the shock locus:*

$$\omega = \omega_T + (\omega \cdot \nu)\nu.$$

Then we have the jump relations

$$[\omega \cdot \nu] = 0, \qquad \left[\frac{\omega_T}{\rho}\right] = 0. \tag{23}$$

In particular, if the flow is irrotational on one side of the shock, it is irrotational on the other side.

In conclusion, piecewise smooth irrotational steady flows are not only good approximations of Euler flows, but they are genuine flows of the full gas dynamics system with a Chaplygin equation of state. According to [6], this is true also for self-similar flows.

Second-order equations for a Chaplygin gas. Because of $\imath = -a^2/(2\rho^2)$, we have

$$h(z) = \frac{a}{\sqrt{2z}}\,. \tag{24}$$

The potential equations (7) and (16) therefore rewrite, respectively,

$$\operatorname{div}\frac{\nabla\psi}{\sqrt{|\nabla\psi|^2 - 2\kappa}} = 0, \tag{25}$$

$$\operatorname{div}\frac{\nabla\phi}{\sqrt{2\phi + |\nabla\phi|^2}} + \frac{d}{\sqrt{2\phi + |\nabla\phi|^2}} = 0. \tag{26}$$

2 Second-order equations and their types

For irrotational models, the potential of a steady flow is governed by the second-order equation (7), and that of a self-similar flow by (16). We now describe a third kind of wave.

2.1 Conical waves.

It may well be the case that a flow is simultaneously steady and self-similar. This happens when the initial data (ρ^0, u^0) is homogeneous of degree zero in $\mathbb{R}^d$, if it is a steady solution. Then its potential ψ is a solution of (25), homogeneous of degree one. Its pseudo-potential at unit time is

$$\phi(y,1) = \psi(y) - \frac{1}{2}|y|^2.$$

Let (r,ω) be the spherical coordinates in $\mathbb{R}^3$, with $r>0$, ω in the unit sphere, and $x = r\omega$. Let us write $\psi(x) = r\theta(\omega)$. The domain of ψ is a cone $K = [0,\infty)\times D$. The variational formulation of (7) is

$$\int_K h(\frac{1}{2}|\nabla\psi|^2 - \kappa)\nabla\psi\cdot\nabla\chi\,dx = 0, \qquad \forall\chi\in\mathcal{D}(K). \tag{27}$$

Since $\mathcal{D}(K) = \mathcal{D}(]0+\infty[)\otimes\mathcal{D}(D)$, it is enough to choose test functions χ of the form $f(r)g(\omega)$. With $|\nabla\psi|^2 = \theta^2 + |\nabla_\omega\theta|^2$ and

$$\nabla\psi\cdot\nabla\chi = \theta f'g + \frac{f}{r}\nabla_\omega\theta\cdot\nabla_\omega g,$$

we find

$$\int_0^{+\infty} r^2dr\int_D h\left(\frac{\theta^2+|\nabla_\omega\theta|^2}{2} - \kappa\right)(f'\theta g + \frac{f}{r}\nabla_\omega\theta\cdot\nabla_\omega g)\,d\omega = 0$$

for all test f and g. Integrating by parts in r, we see that this formulation does not depend on f at all, and reduces to

$$\int_D h\left(\frac{\theta^2+|\nabla_\omega\theta|^2}{2} - \kappa\right)(-2\theta g + \nabla_\omega\theta\cdot\nabla_\omega g)\,d\omega = 0, \qquad \forall g\in\mathcal{D}(D).$$

This yields the following second order PDE over a spherical domain D:

$$\mathrm{div}_\omega(h(\cdots)\nabla_\omega\theta) + 2\theta h\left(\frac{\theta^2+|\nabla_\omega\theta|^2}{2} - \kappa\right) = 0, \qquad \omega\in D, \tag{28}$$

where the divergence operator stands for the adjoint of $-\nabla_\omega$ over the unit sphere.

2.2 The type of the PDEs

The analysis of a Cauchy problem or of a boundary value problem depends crucially on the type of the PDE (or system of) under consideration. We see below that the transition between hyperbolicity and ellipticity occurs at different thresholds for different kind of waves: steady, self-similar and conical ones.

For evolutionary PDEs, hyperbolicity is the algebraic property that makes the Cauchy problem well-posed. The full Euler system is hyperbolic. In every direction $\xi \in \mathbf{S}^{d-1}$, infinitesimal waves propagate at velocities $u \cdot \xi$ and $u \cdot \xi \pm c(\rho, s)$, with $c = a/\rho$ for a Chaplygin gas. The latter are velocities of acoustic waves, whereas $u \cdot \xi$ is the velocity of contact discontinuities and vorticity waves, which are always linearly degenerate. For irrotational flows, there remains only acoustic waves. They are genuinely nonlinear for most equations of state, but are linearly degenerate for a Chaplygin gas.

Steady irrotational flow. Let us devellop (7) as a quasilinear equation

$$h(\frac{1}{2}|\nabla\psi|^2 - \kappa)\Delta\psi + h'(\frac{1}{2}|\nabla\psi|^2 - \kappa)\mathrm{D}^2\psi(\nabla\psi, \nabla\psi) = 0,$$

whose principal symbol is

$$P(x;\xi) = h(\cdots)|\xi|^2 + h'(\cdots)(\xi \cdot \nabla\psi)^2, \qquad \xi \in \mathbb{R}^d. \tag{29}$$

The eigenvalues of this quadratic form are h (multiplicity $d-1$) and $h + |\nabla\psi|^2 h'$ (simple). Remembering that h is positive (it is the value of a density), we see that the equation is elliptic (respectively hyperbolic) if $h + |\nabla\psi|^2 h'$ is positive (resp. negative). From the definition, we have $\imath(h(B)) = -B$ and therefore $h'\imath' = -1$, that is $h'c^2 = -\rho = -h$. Hence the type depends of the sign of $(|\nabla\psi|^2 - c^2)h'$. Thanks to $\imath' > 0$, we have $h' < 0$ and therefore only the sign of $|\nabla\psi|^2 - c^2$ matters. Because of $\nabla\psi = u$, we see that (7) is elliptic (resp. hyperbolic) if the flow is subsonic (resp. supersonic), meaning that $|u| < c$ (resp. $|u| > c$).

Self-similar irrotational flow. Again, equation (16) is equivalent to a quasilinear one

$$h(\frac{1}{2}|\nabla\phi|^2 + \phi)\Delta\phi + h'(\frac{1}{2}|\nabla\phi|^2 + \phi)\mathrm{D}^2\phi(\nabla\phi, \nabla\phi) + dh(\cdots) + |\nabla\phi|^2 h'(\cdots) = 0,$$

whose symbol has a similar expression, but with ψ replaced by ϕ, and $\frac{1}{2}|\nabla\psi|^2 - \kappa$ by $\frac{1}{2}|\nabla\phi|^2 + \phi$,

$$P(x;\xi) = h(\cdots)|\xi|^2 + h'(\cdots)(\xi \cdot \nabla\phi)^2, \qquad \xi \in \mathbb{R}^d.$$

With the same calculation as above, we find that only the sign of $|\nabla\phi|^2 - c^2$ matters. This yields the conclusion that the equation is elliptic (resp. hyperbolic) equation if the flow is *pseudo*-subsonic (resp. pseudo-supersonic), meaning that $|v| < c$ (resp. $|v| > c$).

Conical flow. Since a conical flow is both steady and self-similar, whereas the types of equations (7) and (16) follow distinct rules, we must anticipate that the type for conical waves follows a third one. The symbol of the principal part in (28) is nothing but the restriction of the symbol P of (7) to the tangent space defined by $\xi \cdot \omega = 0$. This gives here

$$P(\omega;\xi) = h(\cdots)|\xi|^2 + h'(\cdots)(\xi \cdot \nabla_\omega \theta)^2,$$

where the argument of h is now $\frac{1}{2}(\theta^2 + |\nabla_\omega \theta|^2) - \kappa$. The eigenvalues are h and $h + |\nabla_\omega \theta|^2 h'$, the latter being $\rho(1 - c^{-2}|\nabla_\omega \theta|^2)$. The equation (28) is therefore elliptic whenever

$$|\nabla_\omega \theta| < c(\rho). \tag{30}$$

In terms of the velocity, this means that $|u_\omega| < c(\rho)$, where $u = u_r \vec{e}_r + u_\omega$ is the decomposition into radial and tangential components. It may happen that $|u_\omega| < c(\rho) < |u|$, in which case the flow is supersonic while equation (28) is elliptic !

About the notion of sub/super-sonic flows. The discussion above shows that the notion of sub-/super-sonicity is not universal. On the contrary, it depends a lot upon the context. For instance, the collection of steady flows is stable under the Galilean changes of variables, but their type (subS or superS) is not. For a steady shock, it is natural to choose a frame attached to the shock; then only the normal component of the velocity is Galilean-invariant. It is known that the flow is superS on one side of the shock and subS on the other, except for a Chaplygin gas, where they are both sonic, because of degeneracy. For a self-similar flow, the relevant notion is that of pseudo-sub/super-sonic flow. Finally, for conical flows, the relevant notion is yet another one. This variety of criteria is due to the fact that in the various settings, the relations among the space-time coordinates are different. The moral is that *before speaking about sonicity, one has to* identify these relations, or equivalently to *describe the invariance group of the problem.*

3 The role of special waves

3.1 Self-similar waves

The study of the Riemann problem in one space dimension goes back to Riemann himself in 1860, and a general theory for small data was established by Lax in 1957. The two-dimensional analysis is much less understood, apart from the scalar case and irrotational flows. Let us only mention the breakthrough done by Chen and Feldman [2]. In two space dimensions, they proved the existence of a potential flow in the classical problem of regular reflexion against a solid wedge. One difficulty of this problem is that the (pseudo-)subsonic region is bordered by a reflected shock (which plays the role of a free boundary), and by a sonic line that is characteristic (where ellipticity degenerates).

Chaplygin gas. In this case, the whole boundary is pseudo-sonic; it is characteristic but it is no longer a free boundary. This situation was treated in [5]. The subsonic domain is convex, in the strong sense that its inward curvature is stritly positive. In practice, the boundary is piecwise $\mathcal{C}^2$. The strong convexity turns out to be a necessary and sufficient condition for the existence to the characteristic Dirichlet problem encountered in the construction of the flow.

Theorem 3.1 (D. S. [5]) *Let Ω be a convex open domain in $\mathbb{R}^2$, with piecewise $\mathcal{C}^2$ boundary. Assume that the inward curvature is uniformly positive and bounded. Then there exists a unique positive solution $\phi \in \mathcal{C}^\infty(\Omega) \cap Lip(\overline{\Omega})$ of (26) with $d = 2$, with the boundary condition $\phi = 0$.*

Let us recall that $\phi = 0$ is precisely the value at which ellipticity is lost. Whether ϕ is smooth up to the boundary is an open problem .

3.2 Steady waves

Steady waves are governed by the restricted system of conservation laws $\mathrm{Div}_x F(U) = 0$. There does not exist a general theory because such a system may have a moving type, hyperbolic or elliptic according to the solution itself. However, if in gas dynamics we consider flows with a sufficiently high velocity in some direction, say x_1, then the system of the steady Euler equations is hyperbolic in this direction. This happens when the first component of the velocity is larger than the sound speed. Taking x_1 as an artificial time variable, we can construct steady solutions by solving Cauchy problems with data at $x_1 = 0$. Because we wish to treat only forward problems, the best strategy is usually to take an explicit, simple solution for $x_1 \leq 0$, and then to solve the system for $x_1 > 0$. This is an *interaction problem*, where incoming waves are given and outgoing waves are the unknown.

In two space dimensions, an explicit pattern at $x_1 < 0$ is given by two approaching planar shock waves that meet at $x_1 = 0$, say at the origin. The flow at $x_1 = 0$ is made of two constant states $U_\pm$, and there remains to solve a one-dimensional Riemann problem, in the (x_1, x_2)-plane instead of the (t, x)-plane. This is something that we can do, for instance for small incident strengths, thanks to Lax's Theorem. Because the flow is self-similar and steady, this is an example of conical wave, a rather simple one indeed.

If moreover the flow is irrotational, there are *only two* outgoing waves, which are often shocks (for instance if the strengths of the incoming waves are of the same size). Such a flow is piecewise constant and takes four distinct values separated by half-lines merging at the origin. The outgoing shocks are determined by a *shock polar analysis*, for which we refer to [3]. For a Chaplygin gas, the construction of the emerging shocks has a very nice geometrical interpretation [5].

The situation described above does occur in the resolution of 2-dimensional Riemann problems. Let us assume for instance that the initial data $U(x, 0)$ is constant in each coordinate quadrant, the jump of the velocity being purely normal, in order to satisfy irrotationality. Then each initial discontinuity along a half-axis is resolved by a backward and a forward waves, propagating in the normal direction. Up to a Galilean change of variables, we may assume that the forward shock S_{1f+} in the upper half-plane, is a steady shock. Meanwhile, we may assume that the forward shock S_{2f+} in the right half-plane is steady too. Notice that S_{1f+} "moves to the right" and S_{2f+} "moves up". Therefore S_{1f+} and S_{2f+} move towards each other, and an interaction takes place, at the origin since these waves have been made steady. These incoming waves never reach the pseudo-subS zone. Instead the transmitted waves are candidates to reach it, although it is possible that two transmitted waves, produced by distinct wave interactions, interact with each other somewhere in the pseudo-superS zone.

What is important in this description is that for points $y = x/t$ in some conical neighbourhood $\mathcal{V}$ of the origin (the point where the interaction takes place), the dependence cone meets only two initial discontinuities, separating three states. The solution behaves exactly as if $U(x, 0)$ was equal to the planar interaction of two shocks. Therefore the flow in $\mathcal{V}$ is *equal* to this interaction: it is piecewise constant with straight shocks. In particular, the flow is constant behind the interaction.

Remark. When the wave strength in the Riemann data is small, the shock velocity is close to the sound speed $c = c(\rho)$. Therefore, after the shocks $S_{1,2f+}$ have been made steady, the fluid velocity is approximately $(c, c)^T$, whose norm is $c\sqrt{2}$. The flow is thus supersonic in the direction $(1, 1)$, as expected.

3.3 Conical waves in 3-D

The two-dimensional conical waves studied in the previous paragraph were solved by algebraic equations (shock polar analysis). We turn now to the three-dimensional situation, where we encounter a new phenomenon. The conical waves are no longer piecewise constant, hence their construction is much more elaborate.

Following the same strategy as above, we choose in the past ($x_1 < 0$) three incoming steady shock waves, which separate a central state U_0 from three surrounding states U_1, U_2, U_3. We assume that the gas velocity has a dominant component in the x_1 direction, so that the irrotational system is hyperbolic in this direction, and we may expect that the Cauchy problem be solvable. Again, this situation occurs in 3-D Riemann problems, where the data is constant in octants, and x_1 is the diagonal direction in an octant. The state U_0 is the data in the opposite octant. The latter is limited by three planes, which separate U_0 from constant states V_j ($j = 1, 2, 3$). At $t = 0$, each discontinuity $U_0|V_j$ is resolved by a 1-D Riemann problem. This RP produces a backward and a forward waves, with a constant state U_j in between. Up to a Galilean change of frame, we may assume that the backward waves, those between U_0 and the U_j's, are steady. It is their interaction which we are interested in. For data with small strength, the flow velocity is approximately a constant V such that $V \cdot \nu_i \sim c$, where ν_i is the unit normal to the shock betwwen U_0 and U_i. Since the vector $(1, 0, 0)^T$ is a convex combination of the ν_i's, we deduce $V_1 > c$, hence the flow is supersonic in the x_1 direction. We thus expect that there is a conical neighbourhood $\mathcal{V}$ of the origin in which the solution of the hyperbolic Cauchy problem depends only upon the data in $x_1 < 0$, but not upon that in $x_1 > 0$. Therefore, we do not need to know anything about the data, but the states $U_0, \dots, U_3$ and their separatrices.

We observe that a part of a 3-D RP is made of 2-D RPs, because away form the origin, the three planar shocks $U_0|U_j$ ($j = 1, 2, 3$) interact pairwise. We resolve these interactions with the help of the analysis made in Paragraph 3.2. If (i, j, k) denote the indices $(1, 2, 3)$ in some order, the interaction between $U_0|U_i$ and $U_0|U_j$, produces a transmitted state U^k, and we denote Π_i^k the plane supporting the shock between U_i and U^k.

It is worth to observe that if the system governing the flow was linear, then the Π_i^ks would be the continuations of the incident shock planes. In addition, the state U^k would be given by the formula $U^k = U_i + U_j - U_0$. To have the complete pattern of interaction, there would remain to extend the plane mentionned above, thus dividing the whole space in 8 cells, and to set $\hat{U} := U_1 + U_2 + U_3 - 2U_0$ in the remaining cell. Nothing deep here.

The situation in our fluid problem is much more complicated, due the nonlinearity of the system. Remark that the binary interactions described above take place in $x_1 < 0$, and therefore our "initial data" at $x_1 = 0$ consists of the six constant states U_1, U_2, U_3 and U^1, U^2, U^3. These states are separated by six half-lines emanating from the origin, which are the traces of the planes Π_j^k for $j \neq k$. The state U_0 is supported by a cell contained in $x_1 < 0$, thus we may forget it. What is important for the moment is that nearby states (for instance U_1 and U^2) are linked by a shock wave, thus we do not have to solve a 1-D RP between them. Instead, we remark that the planar shocks between U_1 and its nearby states U^2 and U^3 are approaching each other as x_1 increases. Therefore they interact along some line (the intersection of Π_1^2 and Π_1^3). This interaction produces two emerging shocks, along planes denoted by Π^{21} and Π^{31}, together with a transmitted state $\hat{U}^1$.

In a perfect world (and the linear world is perfect in this respect), the states $\hat{U}^j$ would coincide, and the planes Π^{ij} and Π^{ik} would coincide too, for each i. Instead, we may verify on concrete examples that $\hat{U}^1, \hat{U}^2$ and $\hat{U}^3$ are distinct, and that the Π^{ij} are six distinct planes, instead of only three. This means that gluing the ten states $U_0, U_1, \dots, U^2, \dots, \hat{U}^3$ and the fifteen corresponding shocks does not solve the problem.

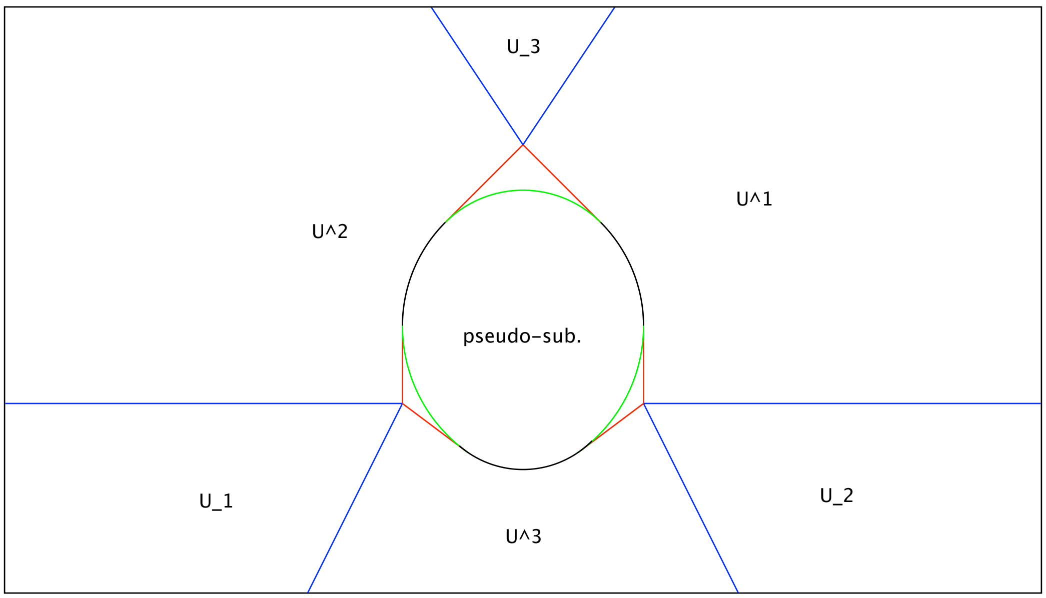
The reason of this failure is that we are solving a Cauchy problem for a hyperbolic system in the direction x_1, where the data has a higher-order singularity at the origin. Although the solution is rather simple, piecewise constant, away from the influence cone K of the singularity, there is no reason why it should be so everywhere. Instead, we expect that in K, the potential is a non-trivial solution of the governing PDE (28). The problem to solve has a free boundary in general. The cone K is bordered by the states $U^1, U^2, U^3, \hat{U}^1, \hat{U}^2, \hat{U}^3$, and ∂K must be determined with the help of the Rankine–Hugoniot conditions. The boundary is made of shock waves, and/or of pieces of characteristic cones.

For a Chaplygin gas, the acoustic fields are linearly degenerate. These waves are thus supported by the characteristic cones associated with the external states U^j and $\hat{U}^k$. We verify that the cones associated with nearby states (say U^i and $\hat{U}^j$ with $i \neq j$) are interiorly tangent to each other, and the plane Π^{jk} is tangent to them along their intersection line. Thus there exists a unique convex cone K whose boundary is made of pieces of these six cones. It is known *a priori*, and we have to solve an ordinary boundary-value problem, with the difficulty that ellipticity degenerates at the boundary.

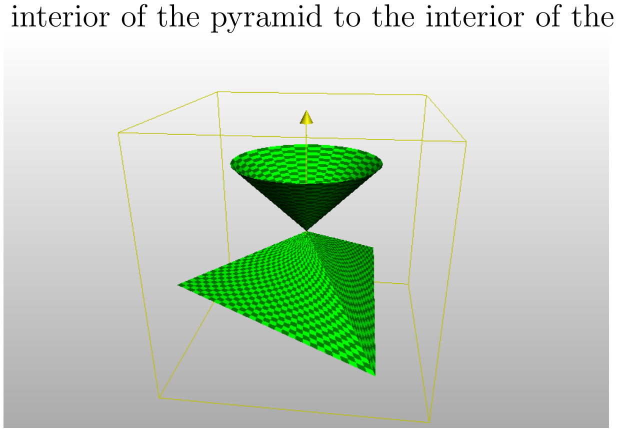
For a gas with a general equation of state, say a γ-law, the characteristic cones $K(U^i)$ and $K(\hat{U}^j)$ are not tangent to each other, and not characteristic. Therefore one needs at least three shocks among the six waves to form ∂K. We leave open the question of whether there are actually six shocks or if there are only three, associated with three sonic waves. The example of

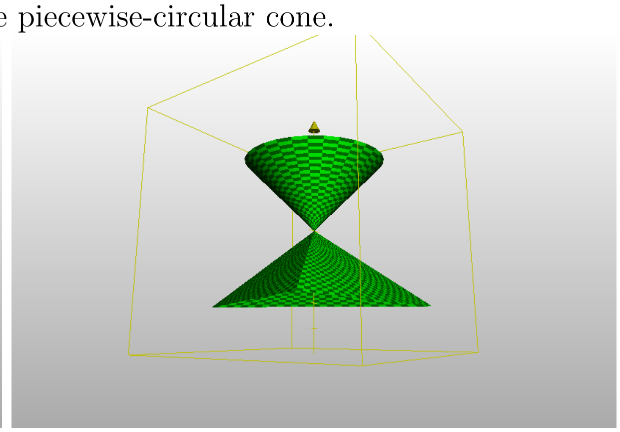
2-D shock reflexion against a solid wedge suggests that the waves separating K from the states $\hat{U}^j$ could be pseudo-sonic, whereas those between K and the states U^j would be shocks. If so, the former would be pieces of circular cones, whereas the latter would be free boundaries.

Pictures for a Chaplygin gas. Our first picture is a section in the plane $x_1 + x_2 + x_3 = 1$. It displays the states U_j and U^k. Semi-infinite lines are the traces of the planar shocks Π_j^k. Small segments are the traces of shocks Π^{jk}. The oval is the trace of the "sonic cone"; it is made of six arc of circles, tangent to the previous segments.



The next two pictures display to views of the incoming steady shocks (they form a pyramid) between U_0 and the U_j's, and the outgoing conical wave. The shocks $U_j \mapsto U^k$ and $U^k \mapsto \hat{U}^\ell$ are not shown. In both pictures, the conical wave expands upwards. The x_1 axis passes from the interior of the pyramid to the interior of the piecewise-circular cone.





4 Existence results for a Chaplygin gas

Theorems 4.1 and 4.2 are taken from [5]. Other statements of this section are taken from [6].

4.1 Self-similar flows

We are concerned with Equation (26), which we rewrite in quasilinear form

$$(2\phi + |\nabla\phi|^2)\Delta\phi - \mathrm{D}^2\phi(\nabla\phi,\nabla\phi) + 2d\phi + (d-1)|\nabla\phi|^2 = 0. \tag{31}$$

Recall that it is elliptic whenever $\phi > 0$, and hyperbolic if $\phi < 0$. In the course of the construction of solutions of the d-dimensional Riemann problem, we are interested in the Dirichlet problem in a bounded domain $D \subset \mathbb{R}^d$, the boundary condition being

$$\phi = 0 \qquad \text{on } \partial D. \tag{32}$$

Let us pretend that the construction of the outer, hyperbolic solution in $\mathbb{R}^d \setminus D$ is done *a priori*. For the 2-D Riemann problem, $(\rho, u)_{\rm out}$ is piecewise constant, obtained by resolving binary interactions. In 3-D, it is generally more complicated, because it involves conical waves, as well as other patterns. The boundary of D is the level set $\{\phi_{\rm out} = 0\}$. The following proposition tells us that D cannot be arbitrary.

Proposition 4.1 *The inward mean curvature H of ∂D is strictly positive. More precisely, it satisfies*

$$H\,\frac{\partial\phi_{\rm out}}{\partial\nu} = 1,$$

where the derivative is in the inward normal direction to ∂D.

In the 2-D case, we deduce that the connected components of D (D is usually connected) are strictly convex. In particular the boundary is Lipschitz. In 3-D, we say that D is strictly *mean-convex*. Like convexity, mean convexity can be defined without invoking the curvature of the boundary. It does not imply any kind of regularity, because the control of the mean curvature does not mean a control of the second fundamental form, see [1]. At a global level, it may happen that D is not simply connected, is knotted or has two linked connected components. However we are not aware of a Riemann problem for which D would be torus-shaped.

Two-dimensional results.

Theorem 4.1 ($d = 2$) *Let $D \subset \mathbb{R}^2$ be a bounded convex domain, whose boundary is piecewise $\mathcal{C}^2$ and the curvature satisfies $0 < \min\kappa$, $\max\kappa < +\infty$. Then the Dirichlet problem (31,32) admits a unique positive solution ϕ in $Lip(\bar{D}) \cap \mathcal{C}^\infty(D)$.*

The regularity of ϕ at the boundary remains an open problem.

Theorem 4.1 is used to complete the resolution of 2-D Riemann problems. The data being piecewise constant and separated by rays, the pseudo-supersonic flow can be constructed with the help of binary interactions. It is piecewise constant. This construction may reach an obstruction, because some binary interactions with large incident waves might have no solution. For moderate data however, the solution can be extended until one reaches a pseudo-sonic line, the zero set of $\phi_{\rm out}$. This level line is made of arcs of pseudo-sonic circles that are pairwise interiorly tangent. It encloses a strictly convex domain D, in which we apply Theorem 4.1. This yields an existence result.

Theorem 4.2 **($d = 2$, Chaplygin gas.)** *Given an irrotational Riemann data with discontinuities of moderate strength, the 2-D Riemann problem admits a solution. It consists of finitely many constant states and one non-constant, pseudo-subsonic state. Their domains are separated by straight shocks and by arcs of sonic circles.*

We emphasize that the only obstruction that could be met for strongly discontinuous initial data is that some binary interaction of planar shocks could have no solution. This is a drawback of the Chaplygin equation of state.

Three-dimensional results. In the 3-D case, we need to reformulate the hypothesis of mean convexity. Because $\phi_{\rm out}$ is not everywhere smooth, ∂D is not $\mathcal{C}^2$, we assume the following.

1. There exists a radius $\rho > 0$ such that, for every point $p \in \partial D$, there is a ball $B(z;\rho) \subset D$, such that $p \in \overline{B(z;\rho)}$. In particular, the sphere $S(z;\rho)$ is tangent to ∂D at p.

2. There is a constant $\alpha > 0$ such that the following holds. For every point $p \in \partial D$, there are orthogonal coordinates $x = (y, x_d)$, an $\epsilon > 0$, and a symmetric matrix S with $\operatorname{Tr} S \geq \alpha$, such that $p = (0, 0)$, and $D \cap B(p;\epsilon)$ is included in the set defined by

$$x_d > y^T S y.$$

When ∂D is of class C^2, Property 2 is equivalent to the uniform positivity of the mean curvature, whereas Property 1 is associated with an upper bound of the curvature tensor at p.

Theorem 4.3 **($d = 3$)** *Let $D \subset \mathbb{R}^3$ be a bounded mean-convex domain, in the sense of the hypotheses 1 and 2. Then the Dirichlet problem (31,32) admits a unique positive solution ϕ in the class $Lip(\bar{D}) \cap \mathcal{C}^\infty(D)$.*

The proofs of Theorems 4.1 and 4.2 are quite similar. They use the maximum principle satisfied by the auxiliary unknown $w := \sqrt{2\phi}$. We begin with an easy L^∞-estimate based on barrier functions. Being more clever, we establish a Lipschitz estimate at the boundary; this is where we use the mean convexity of D. Then we apply the maximum principle to an auxiliary quantity z to obtain interior Lipschitz estimates. The latter guarranty the uniform ellipticity of the principal part of (31), on every compact subdomain. This allows us to apply classical regularity theory. This provides enough compactness to pass to the limit in an approximation procedure. The boundary estimate is easy in 2-D, using the potentials of constant states as barrier functions, whereas it is very technical in 3-D. On the contrary, the interior estimate is easy in 3-D, where we may take $z = |\nabla\phi|^2$, whereas it is complicated in 2-D, where we need a $z = |\nabla\phi|^2 + 2\alpha\phi$ with some parameter $\alpha \in (2, 3)$.

The necessary and sufficient condition of mean convexity looks natural once we observe that the principal part of the PDE governing w is nothing but the mean-curvature operator

$$\mathrm{div}\frac{\nabla w}{\sqrt{1+|\nabla w|^2}}\,.$$

The mean convexity is known to be sufficient for the solvability of the Plateau problem with Dirichlet boundary condition $w = g$, and it is necessary in order that it be solvable for every continuous g. See [4] for details. Notice also that the above operator is of the form $\mathrm{div}A(\nabla w)$, hence is appropriate for applying the maximum principle, a crucial tool in the proof of the *a priori* estimates.

4.2 Conical waves

As explained in Paragraph 3.3, conical waves are used to resolve ternary interaction of steady planar waves. As such, they arise in some regions of the 3-D Riemann problem. The initial data of this RP consists in six states U^j and U_j ($j = 1, 2, 3$), with U^j and U_k separated by shock waves whenever $j \neq k$. In the x-space, shocks surrounding U_j are approaching each other as x_1 increases, and eventually interact, producing two planar shocks and a transmitted state $\hat{U}^j$. The plane supporting the shock between U^k and $\hat{U}^j$ is characteristic for both states, which means that it is tangent to the cones of equations

$$(|U|^2 - c(\rho)^2)|x|^2 = (U \cdot x)^2.$$

Because the tangential component of the velocity is continuous accross a shock, the plane and both cones are actually tangent along the same ray. We may form a conical surface Γ by gluing one piece of each of the six characteristic cones. This surface is piecewise $\mathcal{C}^2$ (except at the origin, of course) and encloses a convex cone $K \sim \mathbb{R}^+ \times \Omega$. Its curvature vanishes in the radial direction and is piecewise constant (thus uniformly positive) in the orthogonal direction.

The system to solve in Ω is formed of the PDE (28) with $h(z) = a/\sqrt{2z}$, together with the boundary condition $|\nabla\theta| = c$. From $h'(z)c^2 = -h(z)$, we deduce (because the gas is Chaplygin) $c^2 = 2z$, that is $c^2 \equiv \theta^2 + |\nabla\theta|^2 - 2\kappa$. Therefore the boundary condition is $\theta^2 = 2\kappa$. Since θ_{out} is positive on $\partial\Omega$, we can recast the boundary condition as

$$\theta = \sqrt{2\kappa} \qquad \text{on } \partial\Omega. \tag{33}$$

Inside Ω, the ellipticity tells us $\theta > \sqrt{2\kappa}$. The proof of the following result can be found in [6].

Theorem 4.4 *Let Ω be a spherical domain with piecewise $\mathcal{C}^2$ boundary and uniformly positive inward curvature (in particular, Ω is strictly contained in a half-sphere). Then the BVP (28,33) with $h(z) \equiv a/\sqrt{2z}$ admits a unique solution $\theta \in Lip(\bar{\Omega}) \cap \mathcal{C}^\infty(\Omega)$.*

As a corollary, the ternary interaction problem between three steady shocks of moderate strengths in a Chaplygin gas admits a solution.

The proof of the solvability of the BVP in every "strongly convex" Ω works more generally for an equation of state $p(\rho) = p_0 - a^2\rho^{-1/\beta}$ with $\beta \in [0,1]$. The extreme points $\beta = 1$ and $\beta = 0$ correspond to either the Chaplygin gas or the incompressible gas, and we use a homotopy between both. Amazingly, the auxiliary unknown $z := \cosh^{-1}\theta$ satisfies the maximum principle. This is why we are able to establish *a priori* estimates that are uniform in s, ensuring the uniform ellipticity on compact subsets of Ω.

For $\beta = 0$, the PDE reduces to the linear equation $\Delta\theta_0 + 2\theta_0 = 0$. For a half-sphere defined by $\xi \cdot \omega > 0$ with $\xi \in \mathbf{S}^2$, the first eigenvalue of $-\Delta$ is $\lambda_1 = 2$; the corresponding eigenfunction is $\xi \cdot \omega$. Because λ_1 is a decreasing function of the domain, and $\bar{\Omega}$ is contained in a half-sphere, we have $\lambda_1(\Omega) > 2$, and the Dirichlet problem for θ_0 is uniquely solvable.

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