Improvement of Thm 2.3

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The notations are those of our paper [1].

Let $A \in L^1(\Omega)$ be a positive semi-definite symmetric tensor over a bounded open domain Ω, with the property that $\mathrm{Div} A$ is a bounded measure over Ω. It admits a well-defined trace $A\vec{n}$ which belongs to the dual space of $\mathrm{Lip}(\partial\Omega)$. We have the formula

$$\int_\Omega (\mathrm{Tr}\,(A\nabla\phi) + \phi \mathrm{Div} A)\,dx = \langle A\vec{n}, \phi|_{\partial\Omega}\rangle, \qquad \forall \phi \in \mathrm{Lip}(\Omega)^d.$$

Let us assume that $A\vec{n}$ is actually a bounded measure over $\partial\Omega$, so that the extension of A by 0_n (still denoted A) to the complement of Ω, has the property that $\mathrm{Div} A$ is a bounded measure over $\mathbb{R}^d$.

Let us choose $\rho \in \mathcal{D}^+(\mathbb{R}^d)$ such that $\int \rho(x)\,dx = 1$, with support in the unit ball. Then $A^\varepsilon := \rho_\varepsilon * A$, where $\rho_\varepsilon(x) = \varepsilon^{-d}\rho(x/\varepsilon)$, converges towards A in $L^1(\mathbb{R}^d)$. In addition, $\mathrm{Div} A^\varepsilon = \rho_\varepsilon * \mathrm{Div} A$ converges to $\mathrm{Div} A$ in the sense of measure and the L^1-norm of $\mathrm{Div} A^\varepsilon$ tends to the total mass of $\mathrm{Div} A$. Of course, A^ε is of $\mathcal{C}^\infty$ class.

The last approximation consists in adding of constant positive tensor, that is defining $A^\varepsilon_\eta = \eta I_d + A^\varepsilon$. We know have a uniformly positive smooth symmetric tensor in $\mathbb{R}^d$.

Let us choose a ball B_R that contains $\Omega + B_1$. On the boundary, we have $|A^\varepsilon_\eta \vec{n}| \equiv \eta$. Applying Thm 2.3 of [1] to A^ε_η in B_R, for $\varepsilon \in (0,1)$, we have

$$\int_{B_R} (\det A^\varepsilon_\eta(x))^{\frac{1}{d-1}}\,dx \le \frac{1}{d|S^{d-1}|^{\frac{1}{d-1}}} \left(\eta|\partial B_R| + \|\mathrm{Div} A^\varepsilon\|_{\mathcal{M}(B_R)}\right)^{\frac{d}{d-1}}. \tag{1}$$

Letting $\eta \to 0+$, we obtain

$$\int_{B_R} (\det A^\varepsilon(x))^{\frac{1}{d-1}}\,dx \le \frac{1}{d|S^{d-1}|^{\frac{1}{d-1}}} \|\mathrm{Div} A^\varepsilon\|^{\frac{d}{d-1}}_{\mathcal{M}(B_R)}. \tag{2}$$

When $\varepsilon \to 0+$, the right hand side converges towards the right-hand side in (11), Theorem 2.3. Meanwhile $A^\varepsilon(x)$ converges almost everywhere to $A(x)$ and therefore $(\det A^\varepsilon(x))^{\frac{1}{d-1}}$ converges towards $(\det A(x))^{\frac{1}{d-1}}$. By the Fatou Lemma, we deduce that $(\det A)^{\frac{1}{d-1}}$ is integrable and satisfies the same inequality as in Theorem 2.3. This theorem is therefore improved in the following way

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Theorem 0.1 *Let Ω be an open subset in $\mathbb{R}^d$, with Lipschitz boundary. Let $A \in L^1(\Omega)$ be a symmetric, non-negative tensor, whose row-wise divergence is a bounded measure over Ω. Assume that its normal trace $A\vec{n}$ (which is well-defined in the dual of* $\mathrm{Lip}(\partial\Omega)$*) is actually a bounded measure over $\partial\Omega$. Then $(\det A)^{\frac{1}{d-1}}$ is integrable and we have the inequality*

$$\int_\Omega (\det A(x))^{\frac{1}{d-1}} dx \leq \frac{1}{d|S^{d-1}|^{\frac{1}{d-1}}} \left(\|A\vec{n}\|_{\mathcal{M}(\partial\Omega)} + \|\mathrm{Div} A\|_{\mathcal{M}(\Omega)} \right)^{\frac{d}{d-1}}. \tag{3}$$

Comments

- We have established the gain of integrability, which was left open in [1]. We also weaken the assumption about the normal trace, asking that it be only a bounded measure, instead of being integrable.
- We also get rid of the convexity assumption about Ω. Only the convexity of B_R was used, when we applied Theorem 2.3 to the pair $(B_R, A^\varepsilon_\eta)$. This strategy was suggested to me, in the context of the application of Theorem 2.3 to the isoperimetric inequality, by Guido de Philippis.
- The assumption about the normal trace is of similar nature as that about the divergence. It amounts to saying that the divergence extension by 0_n is a bounded measure.
- The isoperimetric inequality follows now from theorem 0.1 for arbitrary domains, and not only for convex ones, by choosing $A \equiv I_d$.

References

[1] D. Serre. Divergence-free positive symmetric tensors and fluid dynamics. *Accepted, Annales de l'Institu Henri Poincaré.*