Who Won the 1989 Tour de France?

by Denis Serre

The Tour de France is the most popular cyclist race in the world. Every year, it lasts three weeks in July. The race is split into about 21 day-long segments called "stages." You've probably heard of seven-time champion Lance Armstrong, but the first US cyclist to win the Tour de France was Greg Lemond. Lemond won three tours (in 1986, 1989, and 1990) and was very popular among the French audience. His victory in the 1989 Tour was amazing. After more than 3,200 kilometers, Greg won by a mere 8 seconds (!) ahead of Laurent Fignon (winner in 1983 and 1984). This was the closest margin in history.

Well, I just gave you the answer to the title question, so why did I ask it? Let me say a few words about the actual and official running times. To rank the riders, one records the time they spend on each stage. The official records are in seconds. Compared to a 100 meter sprint where times are given in hundredths of a second, this looks coarse. However, it makes sense because the stages last several hours and it is very unlikely that hundredths, or even tenths, of a second will separate racers. Of course, this means that the actual time a racer takes to cover a stage is different from his official time. According to the official rules of the Tour de France, the official time is obtained by rounding the actual time *down* to the nearest second. Saying that a cyclist spent 1 h 23' 45" on a stage means that his actual time was greater than or equal to 1 h 23' 45" and less than 1 h 23' 46". After 20 stages, a rider might have been given an advantage of up to 20 seconds, whereas his opponent might not have been advantaged at all. Of course, this is very unlikely, because it is reasonable to assume that the round-down errors between 0 second and 1 second are equally likely. Hence, twenty round-downs would tend to average out so that the total discrepancy between official and actual time is likely to be close to 10 seconds. (This phenomenon is known in probability as the **Law of Large Numbers**.) Nevertheless, we can't completely discard the event where the cumulative round-downs flip the rank of two racers. If after 20 stages rider G is *officially* ahead of L by less than 20 seconds, there is a possibility that L *actually* rode faster than G! But how likely is this possibility?

So, back to the title question, while Lemond *officially* went faster than Fignon, because he won by a mere 8 seconds, it is possible that Fignon *actually* was the faster cyclist. The question I want to address is, how likely is it that the official result was wrong?

Before we answer, the above description of how the racers are ranked must be completed with the following rule: When two riders finish a stage in the same peloton[1], one declares that they spent equal times on that stage. This may seem arbitrary, but it is wise. In case of a mass finish, with more than 150 riders arriving together, the length of the peloton could be 30 meters or more, and the last rider finishes a few seconds later than the first one. It would be dangerous if so many cyclists make a mad dash to finish first from this group. Hence, the rule.

Going back to the 1989 Tour, Greg and Laurent arrived in the same group 11 times. On these occasions, we are not concerned with the round-downs as far as our question goes because the rule tells us that they both rode these stages in equal time. Still, there remain 10 stages where they arrived in different pelotons, sometimes Greg ahead, sometimes Laurent. Each one could have been given an advantage of up to 10 seconds. Now let us suppose that Greg got an advantage of g seconds and Laurent got an advantage of l seconds. If $g - l > 8$, then Greg's final win would have been the result of round-downs. This is possible but very unlikely. The question I want to address is: How unlikely is the result due to round-downs?

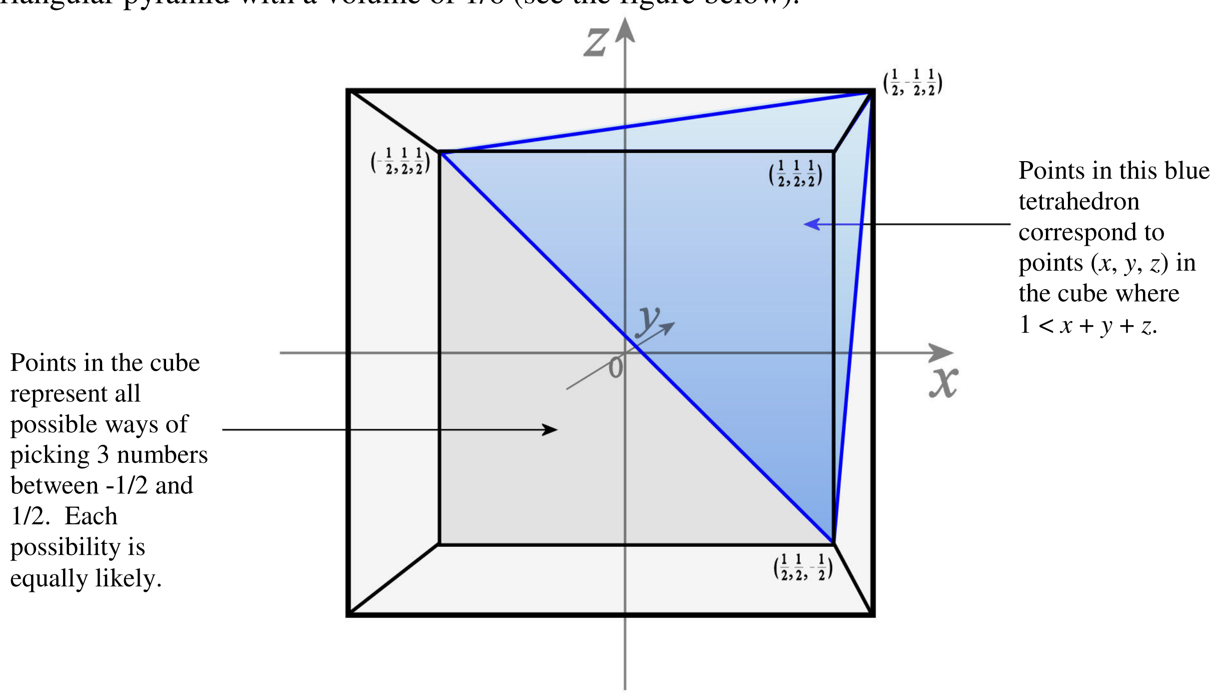
---

[1] A **peloton** is a group of cyclists riding together in a close formation like a flock of birds.

Transforming the Problem. If you already see how the problem amounts to computing the volume of a certain 20-dimensional polyhedron, you might enjoy trying to determine the exact answer yourself and skip to the end to check your answer. But if not, I will try to show how this problem relates to geometry by transforming this problem to an equivalent, but more symmetric problem that generalizes in a way that permits us to see more easily how the geometry arises.

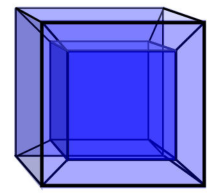
Our original problem involves 20 round-off errors between 0 and 1 and determining the probability that the sum of the first 10 minus the sum of the last 10 exceeds 8. If we subtract 1/2 from each of the 20 round-offs, we can see this is equivalent to determining the probability that the sum of the first 10 minus the sum of the last 10 still exceeds 8. Finally, if we negate the last 10 of these numbers bewteen -1/2 and 1/2, we end up asking what the probability is that the sum of 20 random numbers between -1/2 and 1/2 add up to a quantity greater than 8. In other words, we have a special case of the general problem: Given N numbers selected uniformly at random from between -1/2 and 1/2, what is the probability that their sum exceeds T? Our original question is the case $N = 20$ and $T = 8$.

Simplified Example. Before understanding the general situation, or even our specific one, let us work out the simpler case $N = 3$ and $T = 1$. Let the three random numbers by x, y, and z. We can think of them as the coordinates of a point (x, y, z) which belongs to the unit cube with vertices $(\pm 1/2, \pm 1/2, \pm 1/2)$. Because each point in this cube is equally likely to be chosen, the probability that a point falls inside some subset of this cube is equal to the volume of the subset divided by the volume of the cube. Since the volume of this cube is 1, the probability is just the volume of the subset. In our example, the event is defined by $1 < x + y + z$ (the sum of the numbers exceeds 1). As a subset of the cube, this event is the tetrahedron whose vertices are (1/2, 1/2, -1/2), (1/2, -1/2, 1/2), (-1/2, 1/2, 1/2), and (1/2, 1/2, 1/2). Notice that this tetrahedron is a right triangular pyramid with a volume of 1/6 (see the figure below).



A cube is just one term in a sequence of analogous objects called **unit cubes of dimension N**. The unit cube of dimension 1 is a segment of length 1. The unit cube of dimension 2 corresponds to a unit square. The unit cube of dimension 3 is illustrated above. Although it becomes harder to imagine because we live in a world with 3 spatial dimensions, one can continue this process to make the unit cube of dimension 4.

In fact, when you looked at the figure on the previous page, you saw a cube. Yet, it is not really a 3-dimensional object because this magazine is printed on a sheet of paper, which is 2-dimensional. If we are able to represent a 3-dimensional object by means of a 2-dimensional figure, we can try to represent a 4-dimensional cube with a 3-dimensional construction. It will not be an actual 4-dimensional cube, but a projection of one to 3-dimensions just as a drawing of a cube is also a projection. It is actually possible to leap another dimension and draw the projection of a 4-dimensional cube on a plane (see right).



Now, an N-dimensional cube contains a generalization of the tetrahedron, a body whose vertices have coordinates $(\pm 1/2, \ldots, \pm 1/2)$ with at most one minus sign. There are $N + 1$ vertices and the body is known as an ***N*-simplex**[2]. Let's call this N-simplex S_N. Beginning with small dimensions, the S_1 happens to be the same as the 1-dimensional cube: a line segment of unit length. Next, S_2 is a right triangle with sides adjacent to the right angle of length 1. Thus the area of our 2-simplex is 1/2. The blue region in the figure on the previous page shows S_3, and so on. The fact that S_3 has volume 1/6 is a special case of the general formula for the volume of our N-simplex S_N:

$$\text{The volume of } S_N \text{ is equal to } \frac{1}{N!}.$$

Back to the 1989 Tour. The event we're considering is equivalent to the case $N = 20$ and $T = 8$:

$$x_1 + x_2 + \ldots + x_{19} + x_{20} > 8.$$

This is a subset of the 20-dimensional unit cube. It is a 20-dimensional polyhedron P whose vertices are the points $(\pm 1/2, \ldots, \pm 1/2)$ where at most two coordinates have a negative sign and the rest are positive. The probability that Laurent lost the Tour because of round-offs is the volume of P. Notice that the 20-dimensional unit cube still has volume one. We give below the exact volume of P, but let us begin with an upper bound. Even though P is not a 20-simplex, it is contained in the 20-simplex whose vertices are the points $(x_1, \ldots, x_{20})$ with all coordinates equal to 1/2 except possibly one which is equal to -3/2. Let's call this 20-simplex S. Thus, the volume of P is less than that of S. This means that the volume of P is less than $\frac{2^{20}}{20!} \approx 4.31 \times 10^{-13} =$ 0.000000000000431. (Note that S is a scaled version of S_{20}. It is scaled up by a factor of 2. Since it is a 20-dimensional object, its volume is therefore 2^{20} times the volume of S_{20}.)

This is less than a percent of a percent of a percent of a percent of a percent of a percent! In conclusion, as I might say in French, *Greg Lemond peut dormir sur ses deux oreilles*[3]. He definitely deserves his victory.

Exercise: Show that each vertex of P is either a vertex of S or the midpoint of one of its edges.

Exercise: The set P is the compliment of 20 unit simplices in the double simplex S. Hence its volume is exactly equal to $\frac{2^{20} - 20}{20!}$, so the upper bound given above is quite accurate!

---

[2] An N-simplex is an N-dimensional polyhedron with $N + 1$ vertices

[3] Literally, "Greg Lemond may sleep on his two ears."