Matrices : Theory & Applications
Errata & comments

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See also the **additional list** of exercises on
`http://www.umpa.ens-lyon.fr/ ~serre/extra.pdf`,
and the **solutions** to the exercises on
`http://www.umpa.ens-lyon.fr/ ~serre/exercises.pdf`.

Errata for the first edition

1. Page 17. In the last two displayed identities, the signs in the right-hand side should not be there.

2. In Exercise 19, page 35, question c), read X_j^{j-1} instead of X_j^j.

3. The discussion page 44, about the distance between spectra, could be put in a more rigorous style. If $n \geq 1$, let us consider an equivalence relation on $\mathbb{C}^n$, defined by $x\mathcal{R}y$ if and only if the coordinates of y are obtained from those of x by a permutation. Let denote by X_n the quotient of $\mathbb{C}^n$ by this relation. The set X_n differs from the set $\mathbb{C}^n$ of n-uplets, because its elements are non-ordered n-uplets. It also differs from the set of subsets of $\mathbb{C}$ of cardinality n, because elements such that (x, x, y) and (x, y, y) could be confused in the latter (and have only two elements), though they are distinct in X_3.

   The spectrum of an $n \times n$ matrix is clearly an element of X_n, since we do not pay attention to the order in which we write the eigenvalues, but we do care about their multiplicities. More generaly, the set of the zeroes of a polynomial of degree n is an element of X_n.

   The distance d is defined on X_n. It is obtained by taking the infimum of the ℓ^∞-distance between all the realizations of $A, B \in X_n$ as n-uplets.

4. Corollary 3.1.1, page 45, is obviously false. For instance, 0_n is diagonal, but every neighbourhood contains non trivial strictly upper triangular matrices. Such ones cannot be diagonalizable since their spectrum reduces to $\{0\}$.

One could say that the set of matrices with simple eigenvalues (they are diagonalizable, at least when $k = \mathbb{C}$), is open. One could as well say that the set of diagonalizable matrices is dense in $\boldsymbol{M}_n(\mathbb{C})$.

5. Page 45, Section 3.2, read $\|Mx\|_2 = \|M^*\|_2$, instead of $\|Ax\|_2 = \|A^*\|_2$, and $MM^* - M^*M$ instead of $AA^* - A^*A$.

6. In Theorem 3.2.1, page 46, we have actually an equivalence.

7. The proof of Theorem 4.5.1, page 72, though essentially correct, misses some topological issues. A connected component G of $\mathcal{G}$, a compact set of the plane, might not be simply connected. In such a case, one of its holes could contain an other connected component G' (or several ones). Then every Jordan curve encercling G will encercle such G'. The proof given in the book computes the number $N(G)$ of eigenvalues belonging to G and to every such G'. However, this is sufficient to conclude, using the following argument. Say that a connected component G is of level 0 if its holes do not contain any other component, and that it is of level ℓ if its holes contain a component of level $\ell - 1$, and every component contained in the holes of G is of level at most $\ell - 1$. When G is of level 0, the searched number is just $N(G)$. Then we compute inductively: The number of eigenvalues in a component G of level ℓ equals $N(G)$, minus the sum of the $N(G')$'s, over all the components G' contained in the holes of G, these being of level at most $\ell - 1$.

8. Page 83, in the proof of Theorem 5.3.1, the case $x > 0$ and $\|x\|_1 = 1$ actually yields $Ax > 0$, which is better than stated.

9. Page 100, right next to the proof of Proposition 6.1.3, read "the quadratic ring $\mathbb{Z}[\sqrt{14}]$ is a principal ideal domain, though not Euclidean."

10. Page 100, read "permutation matrices", instead of "transposition matrices".

11. Page 101, replace "From Proposition 6.1.2," by "Therefore".

12. In Theorem 6.3.2, page 104, the first sentence is incomplete. Read "with A_0, $A_1 \in \mathbf{GL}_n(k)$".

13. Exercise 5 of Chapter 6, "Lagrange's Theorem" is due to Wedderburn when $m = 1$ and to Guttman in the general case.

14. Exercise 8 of Chapter 6, read $K[X]$ instead of $\mathbb{C}[X]$.

15. Page 119, between (7.5) and (7.6), read $\mathrm{Sp}(M^l) = \log \mathrm{Sp}(H^l)$, instead of $\mathrm{Sp}(M^l) = \log \mathrm{Sp}(M^l)$.

16. In the proof of Lemma 7.4.2, read "The arcwise-connected set A contains the origin and the point $z = 1$, since..."

17. Pages 127–128, the computation of $\mathbf{SP}_n \cap \mathbf{O}_{2n}$ is much simpler if we write that $M^T = M^{-1}$ for $M \in \mathbf{O}_{2n}$. Then $M \in \mathbf{Sp}_n$ means $M^T JM = J$, that is $JM = MJ$, which gives immediately $D = A$ and $C = -B$. The same idea may be used in Sections 7.4 and 7.5.

18. Page 128, read "the map $M \mapsto A + iB$ is an isomorphism from $\mathbf{SP}_n \cap \mathbf{O}_{2n}$ onto $\mathbf{U}_n$."

19. For a general field of scalars, Corollary 7.6.1 is true and follows from Theorem 2.3.1.

20. Page 130, read "$(\ker M)^\perp$, a subspace of dimension r,".

21. Page 141, read "after summation from $k = 0$ to l".

22. In Proposition 8.1.3, one can take the slightly better

$$d_\alpha = 4 + \frac{3c_\alpha 2^\alpha}{2^{\alpha-1} - 1}.$$

23. Section 8.1. A thorough account of the complexity of matrix multiplication might be found in V. Pan, *How to multiply matrices faster.* Lecture Notes in Computer Science **179**. Springer–Verlag, Berlin (1984).

24. An easier proof of uniqueness in Proposition 8.3.1: once we have $Q = R$, with $Q \in \mathbf{U}_n$ and R upper triangular with a positive real diagonal, just say that $R^{-1} = Q^{-1} = Q^T = R^T$, where R^{-1} is upper and R^T is lower triangular. Therefore R^T, or R, is diagonal. Then $r_{jj} \in (0, +\infty)$ and $|r_{jj}| = 1$ give $r_{jj} = 1$, hence $R = I_n$.

25. Page 158, line 14, read "the case $\Delta(\rho(J)) \geq 0$".

26. Page 161, read "Hence $P(0) \neq 0$'' instead of "Hence $P(0) \neq 1$".

27. Page 164, line 6. Read $p_k = r_k + \cdots$ instead of $p_{k+1} = r_k + \cdots$.

28. Page 164, two lines after (9.8). Read $n\tau_{CG} \gg 1$ instead of $n\tau_{CG} \ll 1$.

29. In Theorem 10.1.1, the complexity is not $3n^3 + O(n^2)$, but $\frac{10}{3}n^3 + O(n^3)$.

30. Page 171, In the displayed formula for $X_\pm$, $\vec{q}$ should be $\mathbf{q}$.

31. Page 172. Formula (10.1) should be

$$P_{j+1}(X) = (X - m)P_j(X) - |a|^2 P_{j-1}(X).$$

32. Page 173, in the description of Sturm sequences, one should add the requirement that if $Q_0(c) = 0$, then $(x - c)Q_0(x)Q_1(x) > 0$ for x close to c.

33. Page 173, penultimate line. Read A_0 instead of A_1.

34. Page 176, read "Finally, the complexity of the *QR iteration on* a Hessenberg matrix ", instead of *factorization of.*

35. Page 178. Read "unitarily conjugated matrices", instead of 'unitary matrices conjugate".
36. Page 179, read "subspace $[\mathbf{e}^1, \dots, \mathbf{e}^{r-1}]$ spanned by $\mathbf{e}^1, \dots, \mathbf{e}^{r-1}$" (replace $\mathbf{e}^{n-p}$ by $\mathbf{e}^{r-1}$).
37. Page 181, line 13. The angles θ and $\theta + \pi$ give matrices K and K_+ differing from each other by signs. Actually, K_+ and K are conjugated through the symmetry

    $$x \mapsto (x_1, \dots, x_{p-1}, -x_p, x_{p+1}, \dots, x_{q-1}, -x_q, x_{q+1}, \dots, x_n).$$

    Such sign changes just propagate, but do not alter the method in a significant way. One can prove that adding π to some of the angles has the effect to conjugate the iterates by a matrix $P\Sigma$, where P is a permutation matrix and Σ is a diagonal matrix of signs. Due to the conjugation by P, the pairs (p, q) are modified. However these modifications are harmless ; for instance, the range of $(i, j) \mapsto |h_{ij}|$ remains the same. This is why we may restrict to angles $\theta \in (-\pi/2, \pi/2]$.
38. Page 183, read $k_{pp} - h_{pp} = -th_{pq}$ instead of $k_{pp} - h_{pp} = th_{pq}$.
39. In Theorem 10.3.2, page 183, we interpret (as other authors do) the estimate $\|E_{k+N}\| \leq c\|E_k\|^2$ for $N := n(n-1)/2$ as indicating that the Jacobi method is of order *two*, at least when the matrix has simple eigenvalues. If we say that a method is of order $\alpha > 1$ when the error ϵ_{k+1} at step $k+1$ is roughly ϵ_k^α, then the estimate rather tells that Jacobi's method is of order $2^{1/N}$. It is however valuable that this order is strictly larger than one. See the Additional Exercise 197 for an analysis of the exact order when $n = 3$ (slightly better than $2^{1/3}$.

    When performing the Jacobi method with a cyclic choice of the pair (p, q) (instead of the optimal choice), it is reasonnable to speak of the order per sweep. A *sweep* is a set of N consecutive iterations. Along a sweep, every off-diagonal entry has been settled to zero exactly once. The analysis of the order per sweep is still a research topic. For matrices with simple eigenvalues, this order is at least two and might be three. Thus the Jacobi method could be comparable with the QR algorithm.
40. Page 186, read ",is stable under M," instead of ",is stable under A,".

Errata for the second edition

1. Page 23, line 9, the word "orthogonal" must be followed by the word "group".
2. In the first line of Paragraph 3.2.1, $\hat{\text{i}}$ should rather be $\hat{\imath}$, as it is in the next displayed formula.
3. Page 45, in the middle of the page, the formula should read

    $$P_M(N) = \det \begin{pmatrix} N - m_{11}I_n & & & \\ & \ddots & -m_{ij}I_n & \\ & & \ddots & \\ & & & N - m_{nn}I_n \end{pmatrix} \in \mathbf{M}_n(K[N]).$$

The next sentence must read *The Cayley–Hamilton theorem expresses that this determinant is* 0_n *when* $N = M$.

In the first line of the proof, S is the adjugate of R^T.

4. Page 46, it is unfortunate that X denotes both the indeterminate an an eigenvector.
5. The last sentence of Paragraph 3.5.3 is a definition of semi-simplicity, apparently different from Definition 3.4. I should have proved their equivalence.
6. In the middle of page 57, read “is done *in* $O(n^4)$ operations ...”
7. Page 63. In 14.c, one actually asks for a relation between the eigenvalues of AB and those of BA.
8. Page 67. In Cauchy’ double alternant formula, read $b_\ell - b_k$ instead of $b_k - b_\ell$.
9. Page 67. In 30.a, read M instead of N.
10. Page 68. One can use 33.a to compute the characteristic polynomial of an $n \times n$ matrix within $O(n^3)$ operations.
11. In $\mathbf{M}_n(\mathbb{R})$, there is no counterpart of Proposition 5.3. If $x^T Mx = 0$ for all $x \in \mathbb{R}^n$, then M is skew-symmetric, and conversely.
12. Proposition 5.7 can be improved and completed in the following way. If π_A divides $f - g$, then $f(A) = g(A)$. On the other hand, $Af(BA) = f(AB)A$.
13. Page 99, bottom. The semi-major and semi-minor axes are $\sqrt{1+a^2}$ and a, respectively. The area of the ellipse equals $\frac{\pi}{4}(\det[B, B*])^{1/2}$.
14. The last inequality in the proof of Proposition 5.11 is $\|A\|_2 \leq 2w(A)$.
15. Page 108, Exercise 19. The conclusion is false in $\mathbf{M}_n(\mathbb{C})$ when $n \geq 3$.
16. Page 109, the claim that $\mathbf{H}_n \cdot \mathbf{H}_n = \mathbf{M}_n(\mathbb{C})$ is false. This starts as soon as $n = 1$, because then $\mathbf{H}_n = \mathbb{R}$. More generally, one always have $\mathrm{Tr}\,(HK) \in \mathbb{R}$. It turns out (Horn & Johnson, vol. I, Thm 4.1.7) that $\mathbf{H}_n \cdot \mathbf{H}_n$ is the set of complex matrices that are conjugated to a real matrix.

    The correct statement is $\mathbf{Sym}_n(\mathbb{R}) \cdot \mathbf{Sym}_n(\mathbb{R}) = \mathbf{M}_n(\mathbb{R})$.
17. Page 112, bottom -5/6 lines. Read $\lambda_n(M) \leq 0 \leq \lambda_1(M)$. Next line, read $M = 0_n$.