The hyperbolic/elliptic transition in the multi-dimensional Riemann Problem

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Abstract

For a continuous self-similar solution to a system of conservation laws, genuine nonlinearity yields Lipschitz continuity at points where the type of the governing system changes. This is a well-known fact in one space dimension, where a constant state $\bar{u}$ bifurcates towards a rarefaction wave at a point x/t that equals an eigenvalue $\lambda_j(\bar{u})$. We extend this observation to several space dimensions. The result generalizes a calculation that Bae, Chen and Feldman carried out in two space dimensions for an irrotational gas in their paper [3] (see their Theorem 4.2).

As a corollary, we find the astonishing fact that a genuinely 3-D rarefaction wave matches a constant state in a C^1-way, instead of a Lipschitz way!

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1 Introduction

We consider first-order hyperbolic systems of conservation laws, of the general form

$$\partial_t u + \sum_\alpha \partial_\alpha f^\alpha(u) = 0. \tag{1}$$

We are interested in the multi-dimensional Riemann problem, which is the search of self-similar solutions

$$u(x,t) = u\left(\frac{x}{t}\right).$$

Denoting $A^\alpha(u) = \mathrm{D}f^\alpha(u)$ the Jacobian matrices of the fluxes, the Riemann problem obeys the system

$$\sum_{\alpha=1}^{d} A^\alpha(u)\partial_\alpha u = (x\cdot\nabla)u, \tag{2}$$

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where we write x for x/t.

Even if (1) was hyperbolic in the direction of time, the steady-like system may have its own type, which can be hyperbolic, elliptic or something in between. In addition, because the space variable x enters in the coefficients through u itself and through the differential operator $x \cdot \nabla$, this type will in general depend on both x and the solution. Typically, when x is large, the transport operator $x \cdot \nabla$ dominates the left-hand side and System (2) is hyperbolic. But when one moves back to some compact zone of the space $\mathbb{R}^d$, the type will undergo a transition from the hyperbolic to some composite type, with a partial ellipticity. If the solution is constant $u \equiv \bar{u}$ in the hyperbolic region, it will likely bifurcate to a non-trivial one across a hypersurface Σ that is characteristic for the linearized equation

$$\sum_{\alpha=1}^{2} A^\alpha(\bar{u}, x)\partial_\alpha v = 0.$$

For definiteness, we shall say that Σ separates the ambiant space into two domains, the outer one in which $u \equiv \bar{u}$ and the type is hyperbolic, and the inner one where u becomes non constant and the type is composite. Because u is continuous across Σ, one usually speaks of *weak singularity.*

Our goal here is first to identify such hypersurfaces Σ (Section 3) and then to compute explicitly the Jacobian ∇u along the inner side of Σ. For the latter, we shall suppose that u is smooth in a neighbourhood of Σ, except for a jump in the first derivatives across Σ. When needed, we adopt the notation u_{in} to denote the restriction of u to the inner domain, so that u_{in} is smooth up to Σ. We stress that u is continuous and therefore $u_{in} \equiv \bar{u}$ along Σ. However, because we are interested in a bifurcation, we assume *a priori* that $\nabla u_{in} \neq 0$ along Σ (even though we shall ultimately find that this tensor vanishes when $d = 3$). As a matter of fact, if this Jacobian vanishes on Σ, we cannot immediately neglect the possibility that the solution be constant in the inner domain; this is a uniqueness question for a characteristic Cauchy problem. We shall discuss this point below in the specific context of gas dynamics under rotational symmetry.

In one space dimension ($d = 1$), this bifurcation is rather classical and is the object of a well established theory. We have an ODE instead of a PDE, $A(u, x)u' = 0$, which implies

$$\text{either} \quad u' = 0 \qquad \text{or} \quad \det A(u(x), x) = 0$$

The first of the above tells us that $u \equiv \bar{u}$ is locally a constant. As x varies, say increasingly from $-\infty$, the determinant $\det A(\bar{u}, x)$ will vanish at some point $\bar{x}$. There, a bifurcation may take place, towards a non-constant solution determined by the algebro-differential system

$$\det A(u(x), x) = 0 \qquad \text{and} \qquad u' \in \ker A(u, x).$$

This is the way rarefaction waves appear in the one-dimensional Riemann problem. By analogy, we shall sometimes speak of rarefaction waves when the dimension d is ≥ 2 and the solution experiences a weak singularity next to a constant state.

Notations and assumptions. As usual, the symbol of the differential operator $\sum_{\alpha=1}^d A^\alpha(u)\partial_\alpha$ is the matrix valued function

$$\xi \mapsto \sum_{\alpha=1}^{d} \xi_\alpha A^\alpha(u) =: A(u;\xi),$$

defined over $\mathbb{R}^d$. The original system is hyperbolic, which means that $A(u;\xi)$ is diagonalizable with real eigenvalues. We order the latter by $\lambda_1(u;\xi) \le \cdots \le \lambda_n(u;\xi)$. Each λ_j is a positively homogenenous function in ξ, of degree one.

Let us recall two old and fundamental results. On the one hand, Gårding proved in [5] that $\xi \mapsto \lambda_n(u;\xi)$ is a convex function. It cannot be strictly convex because of the homogeneity: it is linear along rays $\mathbb{R}^+\xi$. However, it is strictly convex in the directions transversal to the rays in most examples. This kind of strict convexity is associated with dispersive properties of the linearized operator $\partial_t + \sum_\alpha A^\alpha(\bar{u})\partial_\alpha$.

On the other hand, Boillat [2] proved that if some eigenvalue λ_j has locally a constant multiplicity $m \ge 2$, then the corresponding characteristic field is *linearly degenerate*. This means that $\mathrm{d}_\xi\lambda_j$ vanishes in the eigen-direction $\ker(A(u;\xi) - \lambda_j(u;\xi))$.

In the theory of hyperbolic conservation laws, it is well-known that the bifurcation towards a rarefaction wave is ensured by *genuine nonlinearity*. This notion is the opposite of linear degeneracy. Because the first transition towards partial ellipticity will be determined by λ_n, and because of Boillat's theorem, we must assume that

> **H1**: $\lambda_n(u;\xi)$ is a simple eigenvalue of $A(u;\xi)$ for every $\xi \neq 0$.

The simplicity ensures that λ_n is a smooth function of its arguments, away from $\xi = 0$. Then genuine non-linearity is

> **H2**: $\mathrm{d}_u\lambda_n(u;\xi) \cdot r_n(u;\xi) \neq 0$ for every $\xi \neq 0$, where $r_n(u;\xi)$ is a generator of the eigen-space $\ker(A(u;\xi) - \lambda_n(u;\xi))$.

Finally, Gårding's Theorem tells us that the Hessian $\mathrm{D}^2_\xi\lambda_n(u;\xi)$ is non-negative, and we ask

> **H3**: $\mathrm{D}^2_\xi\lambda_n(u;\xi)$ has rank $d-1$ for every $\xi \neq 0$: its kernel reduces to the line $\mathbb{R}\xi$.

The assumptions above are met by many of the physically interesting systems of conservation laws, including gas dynamics, nonlinear elasticity and Maxwell's equations of electro-magnetism with a nonlinear equation of state.

We may describe our analysis below as an attempt to analyze multi-dimensional rarefaction waves, in a rather general sense. There is however a fundamental difference between the cases $d = 1$ and $d \ge 2$. In several space dimensions, the inner problem is a PDE, which is not hyperbolic. Even the knowledge of the inner gradient along Σ is not sufficient to determine u_{in} uniquely. Instead, we need to know completely what this domain is, and what are the boundary conditions on the part of its boundary that is not included in Σ. In one space dimension, (2) is a system of ODEs and thus possesses a unique continuation principle, except at the bifurcation point $\bar{x}$.

The plan for the rest of this paper is as follows. In Section 2, we study the relatively simple case of radially symmetric gas dynamics. The system reduces to singular ODEs, which can be investigated by a bifurcation analysis. Therefore we not only calculate $(\nabla u)_{\rm in}$ along Σ, but we also prove that the corresponding solutions do exist. Most importantly, we prove that in dimension $d = 3$, non-constant solutions exist, despite the fact that their inner gradient vanishes at Σ. The fact that these solutions are non-unique is natural if we keep in mind that our analysis is only local while the Riemann Problem is global, obeying to a system that is partly elliptic in the subsonic domain.

Section 3 studies the geometry of characteristic hypersurfaces associated with a constant state $\bar{u}$. These are sets across which a weak singularity may occur. This description is reminiscent to an observation that Lax & Friedrichs [7] from $d = 2$ made in the case of the one-dimensional Cauchy problem. Section 4 shows that the hyperbolicity of (2) is lost precisely when the sign of $\lambda_n^*(\bar{u}; x)$ becomes negative.

Sections 5 and 6 establish our main results (Theorems 5.1 and 6.1) on the inner gradient in the two-dimensional and the higher-dimensional cases, respectively. In three space-dimensions, we meet the astonishing fact that the inner gradient must vanish along Σ and therefore a continuous piecewise smooth solution must actually be C^1 (Corollary 6.1). Thus the result obtained in Section 2 is neither specific to gas dynamics, nor to rotationally symmetric patterns.

Section 7 discusses an interesting consequence in a 2-D shock reflection problem; the solution whose existence was proved in [4] displays a boundary layer along the sonic line as the angle of incidence tends to $\frac{\pi}{2}$ (almost normal reflection).

2 Gas dynamics; radial flows

Let us investigate a quasi-one dimensional situation. This is a significantly simplified case, because self-similar solutions are functions of a single variable r. Thus they obey ordinary differential systems.

We start from the Euler equations of a barotropic gas

$$\begin{aligned} \partial_t \rho + \operatorname{div}(\rho v) &= 0, \\ \partial_t v + (v \cdot \nabla) v + \frac{c(\rho)^2}{\rho} \nabla \rho &= 0. \end{aligned}$$

In a radially symmetric flow, we have $\rho = \rho(r)$ and $v = w(r)\vec{e}_r$. The system reduces to

$$\begin{aligned} \partial_t \rho + \partial_r(\rho w) + \frac{d-1}{r} \rho w &= 0, \\ \partial_t w + w \partial_r w + \frac{c(\rho)^2}{\rho} \partial_r \rho &= 0. \end{aligned}$$

Now, if the flow is self-similar (it depends upon r/t, which we still denote r), there remains

$$\begin{aligned}(w-r)\rho' + \rho w' + \frac{d-1}{r}\,\rho w &= 0,\\ \frac{c(\rho)^2}{\rho}\,\rho' + (w-r)w' &= 0.\end{aligned}$$

If $d = 1$, this is nothing but the usual system governing the Riemann Problem. A rarefaction wave may originate at a point $\bar r = w \pm c(\rho)$, because then the matrix

$$\begin{pmatrix} w-\bar r & \rho \\ \frac{c(\rho)^2}{\rho} & w-\bar r \end{pmatrix}$$

is singular. More generally, if $d \geq 1$, we may have a transition from a constant state $\rho \equiv \bar\rho$, $w \equiv 0$ at the point $\bar r = c(\bar\rho)$; this is the point at which the differential system cannot be resolved as

$$\frac{d}{dr}\begin{pmatrix}\rho\\ w\end{pmatrix} = G(\rho, w).$$

Thus we start from a constant flow $(\bar\rho, 0)$ in the exterior domain $r \geq \bar r$, and we search for a solution in the interior, or at least in a corona $(\bar r - \epsilon, \bar r]$ with the properties that it be continuous across $\bar r$, and have regularity $C^2((\bar r - \epsilon, \bar r])$.

2.1 The inner derivative

We wish to determine the inner gradient

$$\frac{d}{dr}\begin{pmatrix}\rho\\ w\end{pmatrix}\Bigg|_{\bar r - 0}.$$

To this end, we first write the system at $\bar r$. As it turns out, both equations are equivalent and we get only one scalar information

$$\bar r \rho' = \bar\rho w'. \tag{3}$$

Next, we differentiate the system once, to obtain

$$\begin{aligned}(w-r)\rho'' + \rho w'' + (2w'-1)\rho' + \frac{d-1}{r}\,(\rho w)' - \frac{d-1}{r^2}\,\rho w &= 0,\\ \frac{c^2}{\rho}\,\rho'' + (w-r)w'' + (w'-1)w' + m(\rho)\rho'^2 &= 0,\end{aligned}$$

where $m = \frac{d}{d\rho}(c^2/\rho)$. We now set r to $\bar r$ and eliminate the second derivatives by multiplying the equations above by $\bar r$ and $\bar\rho$ respectively. There remains

$$(3\bar r^2 + \bar\rho^2 m(\bar\rho))\rho'^2 + (d-3)\bar\rho\bar r\rho' = 0.$$

This yields the alternative

$$\text{either } \rho' = 0 \quad \text{or } (3\bar{r}^2 + \bar{\rho}^2 m(\bar{\rho}))\rho' + (d-3)\bar{\rho}\bar{r} = 0.$$

The first possibility happens when we extend the solution by $(\rho, w) \equiv (\bar{\rho}, 0)$. The second one was expected as the bifurcation towards a rarefaction wave, as it happens in one space dimension. However, we observe that the inner gradient does depend upon the space dimension d. In particular, we have the only value $\rho' = 0$ when $d = 3$. It is hard to draw a rigorous consequence from this equality. At first glance it seems to suggest that a genuine 3-D rarefaction wave does not exist! We shall see however in the paragraph below that they do exist.

This phenomenon is not specific to gas dynamics and spherically symmetric solutions. This dependence upon d, and the fact that the inner gradient must vanish when $d = 3$ will be established in full generality in Theorem 6.1 and Corollary 6.1.

Nota: Let us mention an exceptional case, where the second term of the alternative does not imply $\rho' = 0$, because it writes $0\rho' = 0$. This happens when $3\bar{r}^2 + \bar{\rho}^2 m(\bar{\rho}) = 0$, where we recall $\bar{r} = c(\bar{\rho})$. When it occurs for every value of $\bar{\rho}$, we have

$$3c^2 + \rho^2 \frac{d}{d\rho}(c^2/\rho) \equiv 0.$$

This yields the equation of state $p(\rho) = p_\infty - a^2/\rho$, where a, p_∞ are positive constants. This is the so-called Chaplygin equation of state and the acoustic fields happen to be linearly degenerate. Thus we may anticipate that the genuine non-linearity will play a role in the analysis of the Riemann Problem for general systems in arbitrary dimension, as it did in one space dimension.

2.2 The perfect gas: a complete analysis

Let us assume the perfect gas law

$$p(\rho) = \frac{1}{\gamma}\rho^\gamma,$$

where the adiabatic parameter γ is a constant. We have $c = \rho^\theta$ with $\theta = \frac{\gamma-1}{2}$. Let us rewrite our system in terms of the unknowns (c, w) :

$$\begin{aligned} (w-r)c' + \theta c w' + \theta \frac{d-1}{r} c w &= 0, \\ \theta(w-r)w' + cc' &= 0. \end{aligned}$$

Let us define the auxiliary unknowns $z = \frac{w}{r}$ and $b = \frac{c}{r}$. We have

$$\begin{aligned} (z-1)(rb' + b) + \theta b(rz' + z) + \theta(d-1)bz &= 0, \\ \theta(z-1)(rz' + z) + b(rb' + b) &= 0. \end{aligned}$$

With the change of independent variable $s = \log r$, this is equivalent to

$$\begin{aligned} (z-1)(\dot{b} + b) + \theta b(\dot{z} + z) + \theta(d-1)bz &= 0, \\ \theta(z-1)(\dot{z} + z) + b(\dot{b} + b) &= 0. \end{aligned}$$

This amounts to writing

$$\frac{d}{ds}\begin{pmatrix} b \\ z \end{pmatrix} = \frac{1}{(z-1)^2-b^2}\,F(b,z), \qquad F(b,z) := \begin{pmatrix} b[b^2+(z-1)(\theta(1-d)z+1-z)] \\ z(db^2-(z-1)^2) \end{pmatrix}.$$

This is now an autonomous system. The bifurcation point corresponds to $(\bar{b},\bar{z})=(1,0)$; which is altogether a zero of F and of the denominator $\Delta=(z-1)^2-b^2$. We point out that the integral curves of $\frac{1}{\Delta}F$ are equal to those of F, with the following modifications:

- The orientation is the same if Δ is positive (near $(1,0)$, this means if $z-1+b<0$), but reversed if Δ is negative. Notice that these signs correspond respectively to the pseudo-supersonic / pseudo–subsonic cases.
- The flow of F corresponds to the independent variable y defined by

  $$((z-1)^2-b^2)\frac{d}{ds}=\frac{d}{dy}\,.$$

  When $y\to\pm\infty$, it may be that s tends to a finite limit. This is what we actually wish, because we are interested in the behaviour as $s\to\log\bar{r}=\log c(\bar{\rho})$.

In order to perform a local analysis, we differentiate F about $P:(1,0)$:

$$\mathrm{D}F(1,0)=\begin{pmatrix} 2 & \theta(d-1)+2 \\ 0 & d-1 \end{pmatrix}.$$

At first glance, P is repulsive, in terms of the variable y. A deeper analysis involves the space dimension. Let us point out that we always have the trajectory $z\equiv 0$, $b=b_0e^{-s}$, which crosses the equilibrium at $s_0=\log b_0$.

If $d=2$ **:** The other trajectories are tangent at P to $b-1+(\theta+2)z=0$. As $y\to-\infty$, a trajectory (b,z) tends to P and

$$F(b,z)\sim\begin{pmatrix} -(\theta+2)z \\ z \end{pmatrix}, \qquad (z-1)^2-b^2\sim 2(\theta+1)z.$$

This implies that s has a finite limit, which we may set to $\log c(\bar{\rho})$, and we have

$$\lim\frac{d}{ds}\begin{pmatrix} b \\ z \end{pmatrix}=\frac{1}{2(\theta+1)}\begin{pmatrix} -\theta-2 \\ 1 \end{pmatrix}.$$

If $d=3$ **:** The trajectories with $z\neq 0$ are such that $b-1\sim 2(\theta+1)yz$. This gives

$$F(b,z)\sim\begin{pmatrix} 4(\theta+1)yz \\ z \end{pmatrix}, \qquad (z-1)^2-b^2\sim -4(\theta+1)yz.$$

Therefore, we have

$$\lim\frac{d}{ds}\begin{pmatrix} b \\ z \end{pmatrix}=\begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

in terms of the primary variables (c,w), this means

$$\lim \frac{d}{dr}\begin{pmatrix} c \\ w \end{pmatrix} = 0. \tag{4}$$

We thus recover the fact the inner derivative vanishes at $\bar r$. But this explicit calculation tells us that this vanishing *is not a diagnosis of lack of pseudo-subsonic pattern.* As a matter of fact, there is a continuum of pseudo-subsonic flows in $(\bar r-\epsilon,\bar r)$. They correspond to those integral curves of F, converging to P as $y\to-\infty$, and contained in the half-plane $z+b>1$.

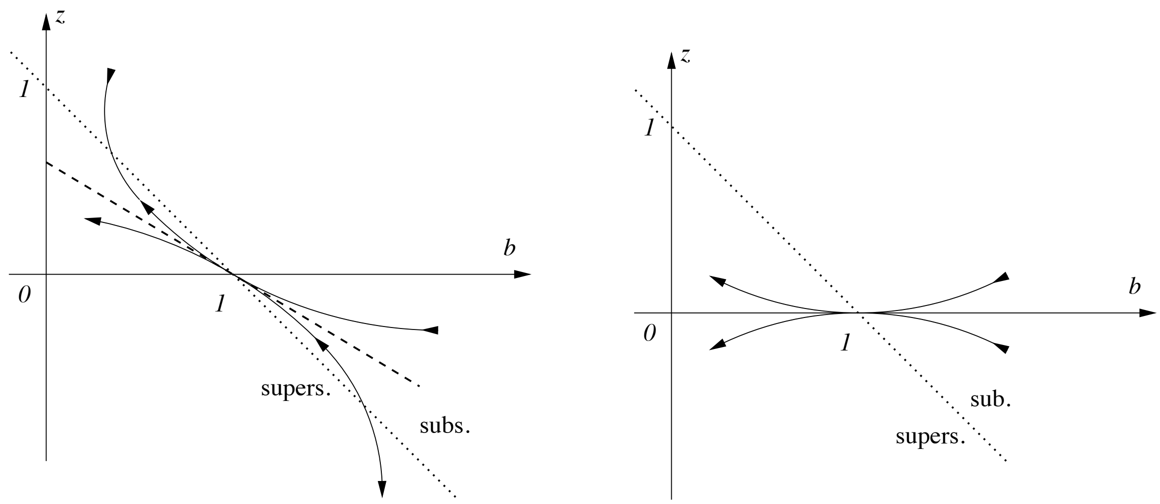

Figure 1: Phase portrait of the (b,z)-differential system; $d=2$ (left) and $d=3$ (right). The horizontal axis is always a trajectory, associated with the fluid at rest and at uniform density. Relevant trajectories tend towards the rest point $(1,0)$ from the pseudo-subsonic side $1<z+b$.

If $d\geq 4$ **:** This might not be a physically relevant situation, but it is at least mathematical meaningful. With one exception, we have $z=O((b-1)^{3/2})$, thus

$$F(b,z)\sim\begin{pmatrix} 2(b-1) \\ O((b-1)^{3/2}) \end{pmatrix},\qquad (z-1)^2-b^2\sim -2(b-1).$$

Hence

$$\lim \frac{d}{ds}\begin{pmatrix} b \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

This again yields (4).

The exception is the *fast* unstable manifold of P. Say that $d=4$ (the general case is a bit more complicated): we then have $z\sim z_0e^{3y}$ and $b-1\sim(3\theta+2)z$. Then $(z-1)^2-b^2\sim-6(\theta+1)z$, which yields

$$\lim \frac{d}{ds}\begin{pmatrix} b \\ z \end{pmatrix} = \frac{1}{2(\theta+1)}\begin{pmatrix} 3\theta+2 \\ -1 \end{pmatrix}.$$

In conclusion, we have a lot of flat pseudo-subsonic continuations, and just one non-flat one.

3 Characteristic hypersurfaces

Let us form the symbol

$$A(u;\xi) := \sum_\alpha \xi_\alpha A^\alpha(u)$$

and define

$$P(u,y;\xi) := \det(A(u;\xi) - (y\cdot\xi)I_n),$$

the scalar symbol of system (2). We make the assumption that the original system (1) is hyperbolic. Therefore $A(u;\xi)$ has real eigenvalues

$$\lambda_1(u;\xi) \le \cdots \le \lambda_n(u;\xi)$$

and we have

$$P(u,y;\xi) = \prod_{j=1}^{n} (\lambda_j(u;\xi) - y\cdot\xi).$$

Let Σ be a smooth hypersurface in $\mathbb{R}^d$, and u be a local solution of (2), continuous and piecewise smooth, such that $u \equiv \bar{u}$ on one side of Σ. We assume that Σ is the boundary of the set $\{u \equiv \bar{u}\}$. Because of the uniqueness property of the non-characteristic Cauchy problem, we know that Σ is characteristic: the unit normal ν at $x \in \Sigma$ satisfies $P(\bar{u}, x; \nu) = 0$. Therefore there exists an eigenvalue $\lambda = \lambda_j(\bar{u};\cdot)$ such that

$$x\cdot\nu = \lambda(\nu). \tag{5}$$

In order to avoid pathological situations, we assume that λ is a simple eigenvalue; then it is a smooth function of its arguments.

Let us define a hypersurface

$$\Gamma := \{\mathrm{d}_\xi\lambda(\xi)\,;\,\xi\in S^{d-1}\} = \{\mathrm{d}_\xi\lambda(\xi)\,;\,\xi\in\mathbb{R}^d\}.$$

If $\mathrm{D}^2_\xi\lambda$ has rank $d-1$, then Σ is smoothly embedded in $\mathbb{R}^d$, with normal vector ξ at $\mathrm{d}_\xi\lambda(\xi)$, because of the identity $\mathrm{D}^2_\xi\lambda(\xi)\xi = 0$ (by homogeneity). Therefore (5) amounts to saying that the tangent space $T_x\Sigma$ is tangent to Γ at $\mathrm{d}_\xi\lambda(\nu)$.

We recall that the curvature tensor of Σ at x is the symmetric map $S_x : \tau \mapsto \mathrm{d}\nu\cdot\tau$ over the tangent space. Differentiating (5) in a tangent direction τ, we obtain

$$(\mathrm{d}_\xi\lambda(\nu) - x)\cdot S_x\tau = 0.$$

Therefore $\mathrm{d}_\xi\lambda(\nu) - x$ is orthogonal to the range of S_x. Because of (5), it is orthogonal to ν too. Thus we find

$$\mathrm{d}_\xi\lambda(\nu) - x \in \ker S_x. \tag{6}$$

Since $\mathrm{d}_\xi\lambda(\nu) - x$ is a tangent vector, we may form the integral curve $s \mapsto x(s)$ of

$$\dot{x} = \mathrm{d}_\xi\lambda(\nu) - x.$$

Differentiating once, we have

$$\frac{d^2x}{ds^2} = \mathrm{D}^2_\xi\lambda(\nu) \cdot \dot{\nu} - \dot{x}.$$

However $\dot{\nu} = \mathrm{d}\nu \cdot \dot{x} = S\dot{x} = 0$, because of (6). Hence $\dot{x} \equiv e^{-s}\tau_0$. This shows that the integral curve is either a line (more precisely the line joining x to $\mathrm{d}_\xi\lambda(\nu)$) or a point, in which case it belongs to Γ. Finally, $\dot{\nu} = 0$ says that the normal, hence the tangent space, is constant along this line.

Proposition 3.1 *If $x \in \Sigma \setminus \Gamma$, then locally Σ contains a segment of the line between x and $\mathrm{d}_\xi\lambda(\nu)$. Along this line, the tangent space to Σ is constant, equal to the tangent space of Γ at $\mathrm{d}_\xi\lambda(\nu)$.*

Let us describe completely the situation in small space dimension:

- If $d = 2$, Σ is either a line, tangent to Γ, or an arc of Γ.
- If $d = 3$, Σ is either a plane (if $S = 0$), or a developpable surface (if $\ker S$ is one-dimensional), or a piece of Γ.

We remark that all the situations described above may happen. For instance, a rarefaction wave depending upon only one coordinate corresponds to the case where Σ is a hyperplane, that is $S = 0$. This leads us to the following

Definition 3.1 *We weak singularity across Σ is said to be m-dimensional if $\mathrm{rk} S = m - 1$. Equivalently, Σ is the envelop of an $(m-1)$-dimensional family of tangent hyperplanes to Γ.*

We focus in this article on the d-dimensional weak singularities. According to Proposition 3.1, such singularities (S being non-singular) must occur across Γ.

More precisely, we restrict to those waves for which $\lambda = \lambda_n$ is the largest eigenvalue. This makes sense because this is the first characteristic field met by the outer, constant state. As mentionned in the introduction, $\xi \mapsto \lambda_n$ is a convex function, homogeneous of degree one. The set Γ defined above is parametrized by $\nu \mapsto y(\nu) = \mathrm{d}_\xi\lambda_n(\bar{u};\nu)$, with $\nu \in S^{d-1}$. If λ_n is strictly convex in the directions transversal to ν (hypothesis (H3)), then this parametrization is injective and Γ is a smooth hypersurface. Differentiating in a tangent direction τ, we have

$$\tau = \mathrm{D}^2_\xi\lambda_n \cdot (\mathrm{d}\nu \cdot \tau).$$

In other words, S is the inverse of the restriction of $\mathrm{D}^2_\xi\lambda_n$ to $\nu^\perp$. Under assumption (H3), we deduce that Γ is a strictly convex hypersurface, in the sense that it is the boundary of a strictly convex domain. The latter domain can be described by

$$\lambda_n^*(\bar{u}; y) \leq 0, \tag{7}$$

where $\lambda_n^*(u;\cdot)$ is the Legendre–Fenchel conjugate of $\lambda_n(u;\cdot)$.

4 The type of (2)

The following result is classical. Its proof follows the arguments of [11], completed in [10] when some eigenvalues cross each other.

Proposition 4.1 *Let $y \in \mathbb{R}^d$ be given. Let us assume that there exists a $\xi_0 \in \mathbb{R}^d$ such that $y \cdot \xi_0 > \lambda_n(\xi_0)$ (in particular, $\xi_0 \neq 0$ can be taken in the unit sphere). Then the system (2) is hyperbolic at y, in the direction ξ_0.*

Sketch of the proof. We restrict to the case where (1) is strictly hyperbolic:

$$\lambda_1(u;\xi) < \cdots < \lambda_n(u;\xi),$$

so that the functions λ_j are smooth, analytic in $\xi \neq 0$. We have $P = \prod_j \phi_j$, with

$$\phi_j(y;\xi) = \lambda_j(u;\xi) - y \cdot \xi.$$

The hyperbolicity in direction ξ_0 is the fact that $P(y;\xi_0) \neq 0$ and for every ξ not parallel to ξ_0, the roots of the univariate polynomial $s \mapsto p(s) := P(y;\xi + s\xi_0)$ of degree n are real. It will be enough to prove that each factor $s \mapsto \phi_j(y;\xi + s\xi_0) =: \psi_j(s)$ has a real root. Because p has no more than n roots, we deduce the fact that each of the ψ_j's has only one zero s_j, which is non-degenerate. The existence of s_j follows from the intermediate value theorem and from $\psi_j(\pm\infty) = \mp\infty$. The latter comes from the asymptotics

$$\psi_j(s) \sim \left\{ \begin{array}{ll} s(\lambda_{n+1-j}(u;\xi_0) - y \cdot \xi_0), & \text{if} \quad x \to -\infty, \\ s(\lambda_j(u;\xi_0) - y \cdot \xi_0), & \text{if} \quad x \to +\infty. \end{array} \right.$$

∎

Proposition 4.1 tells us that if the system (2) is not hyperbolic, then necessarily, we have

$$\forall \xi \in \mathbb{R}^d, \quad \lambda_n(u;\xi) \geq y \cdot \xi.$$

This is exactly saying that $\lambda_n^*(u;y) \leq 0$.

To see that hyperbolicity is lost as y crosses Γ (associated with λ_n), we proceed as follow. The polynomial $P(y;\xi)$ splits as QR, with

$$Q(y;\xi) := (\lambda_n(\bar{u};\xi) - y \cdot \xi)(\lambda_1(\bar{u};\xi) - y \cdot \xi), \qquad R(y;\xi) := \prod_{j=2}^{n-1} (\lambda_j(\bar{u};\xi) - y \cdot \xi).$$

The factors are not polynomial in ξ, but they are homogeneous and smooth away from the origin (if λ_n is simple). We can view them as symbols of pseudo-differential operators. In a neighbourhood of $\bar{y} = \mathrm{d}\lambda_n(\bar{u};\nu) \in \Gamma$, the proof of Proposition 4.1 applies to R, which turns out to be hyperbolic in direction ν. Let us turn towards the factor Q, positively homogenenous of degree 2. Because of the formula $\lambda_1(\xi) = -\lambda_n(-\xi)$, $Q(y,\cdot)$ is actually homogenenous; roughly speaking, it behaves as a quadratic form.

Because λ_n is convex, we have

$$\lambda_n(\xi) \geq \lambda_n(\nu) + \bar{y} \cdot (\xi - \nu),$$

that is $\lambda_n(\xi) - \bar{y} \cdot \xi \geq 0$, with equality only if $\xi \in \mathbb{R}^+\nu$. Likewise, $\lambda_1(\xi) - \bar{y} \cdot \xi \leq 0$, with equality only if $\xi \in \mathbb{R}^-\nu$. Therefore $Q(\bar{y}; \cdot)$ is non-positive everywhere, and negative away $\mathbb{R}\nu$. It behaves like a semi-negative definite quadratic form whose kernel is $\mathbb{R}\nu$. To determine the behaviour of $Q(y; \cdot)$ when y is close to $\bar{y}$, we calculate the differential of $y \mapsto Q(y; \pm\nu)$ at $\bar{y}$. We have

$$\mathrm{d}_y Q(\bar{y}; \pm\nu)z = (\lambda_n(\bar{u}; \nu) - \lambda_1(\bar{u}; \nu))z \cdot \nu.$$

By the Implicit Functions Theorem, we find that there exists locally a smooth hypersurface Σ passing through $\bar{y}$, normal to ν, such that $Q(y; \cdot)$ is 'negative definite' if and only if y lies on the interior side of Σ, and takes some positive values when y is on the exterior side. Obviously, Σ coincides with Γ, because we already know that on it, $Q(y; \cdot)$ is non-positive but degenerate. Finally, we have proved that when $\lambda_n^*(y) < 0$ but y is close to Γ, the factor Q is negative definite, and therefore the corresponding pseudo-differential operator is elliptic.

Proposition 4.2 *Let us assume (H1,3). Then the operator $\sum_\alpha A^\alpha(\bar{u})\partial_\alpha - x \cdot \nabla$ is hyperbolic at points x such that $\lambda_n^*(\bar{u}; x) > 0$. At the boundary*

$$\Gamma = \{x\,;\, \lambda_n^*(\bar{u}; x) = 0\},$$

the operator experiences a change of type towards elliptic-hyperbolic (just elliptic if $n = 2$).

5 The inner gradient of the flow along Γ: 2-D case

We consider in this paragraph 2-dimensional self-similar solutions of (1). The analysis above leads us to focus on the hyperbolic/elliptic transition across the boundary Γ associated with λ_n. For the sake of simplicity, we drop the index n and just write λ instead of λ_n. Recall that Γ is parametrized by

$$y(\xi) = \mathrm{d}_\xi \lambda(\bar{u}; \xi). \tag{8}$$

We consider a solution of (2) in a neighbourhood of a point $x \in \Gamma$, which is constant ($u \equiv \bar{u}$) on the exterior side Ω_{out} of Γ, as well as non-constant on the interior side Ω_{in}. We know that the system (2) is hyperbolic in Ω_{out}, and anticipate that it be mixed elliptic-hyperbolic in Ω_{in}; at least, this is the case when we linearize about $\bar{u}$. If $x \in \Gamma$, we also denote $\tau(x)$ the unit tangent vector to Γ. The operator $(\tau \cdot \nabla)$ is arc-length derivative along Γ. The orientation of the vector fields is chosen so that the curvature κ satisfies

$$(\tau \cdot \nabla)\nu = \kappa\tau, \qquad (\tau \cdot \nabla)\tau = -\kappa\nu.$$

Our goal is to calculate, when possible, the gradient of u_{in}. We make the assumptions that u_{in} is twice differentiable up to Γ, and that u is continuous. In particular $u_{\text{in}} \equiv \bar{u}$ along Γ. For the sake of simplicity, we drop the index '*in*', as long as there is no ambiguity.

Differentiating along Γ, we obtain

$$(\tau\cdot\nabla)u\equiv 0. \tag{9}$$

Of course, the calculation of ∇u_{in} along Γ is not possible by just exploiting (2, 9) as a linear system:

$$\mathcal{A}\begin{pmatrix}\partial_1 u\\ \partial_2 u\end{pmatrix}=0,\qquad \mathcal{A}:=\begin{pmatrix}A^1(\bar u)-x_1I_n & A^2(\bar u)-x_2I_n\\ \tau_1 I_n & \tau_2 I_n\end{pmatrix}.$$

The matrix $\mathcal{A}$ has rank $2n-1$, with kernel spanned by $\nu\otimes r(\bar u;\nu)$, where $r(u;\xi)$ is the eigenvector of $A(u;\xi)$ associated with $\lambda(u,\xi)$. In the sequel, we also employ the eigenform $\ell(u;\xi)$. We thus have, with standard normalization

$$(A(u;\xi)-\lambda(u;\xi))r(u;\xi)=0,\qquad \ell(u;\xi)(A(u;\xi)-\lambda(u;\xi))=0,\qquad \ell\cdot r\equiv 1\quad(\lambda\text{ simple}). \tag{10}$$

The system above tells us that there exists a scalar function $\rho(x)$ over Γ, such that

$$\partial_\alpha u_{\rm in}=\rho\nu_\alpha r(\bar u;\nu),\qquad \alpha=1,2. \tag{11}$$

The co-kernel of $\mathcal{A}$ is made of a $2n$-form $(\ell(\bar u;\nu),m(\bar u;\nu))$, where ℓ has been defined above and m satisfies

$$\ell(A^\alpha(\bar u)-x_\alpha I_n)+\tau_\alpha m=0,\qquad \alpha=1,2. \tag{12}$$

The existence and uniqueness of m is ensured by the second of (10) with $\xi=\nu$, together with (8).

Before going further, we recall that $\mathrm{d}_\xi\lambda\cdot\zeta=\ell A(\bar u;\zeta)r$. With (8), this gives $x\cdot\tau=\ell A(\bar u;\tau)r$. Combining this with (12), we obtain

$$m\cdot r\equiv 0. \tag{13}$$

There remains to calculate ρ. To this end, we differentiate (2) in each direction x_β :

$$\sum_{\alpha=1}^{2}A^\alpha(u)\partial_\alpha\partial_\beta u-(x\cdot\nabla)\partial_\beta u=\partial_\beta u-\sum_{\alpha=1}^{2}(\mathrm{d}A^\alpha(u)\cdot\partial_\beta u)\partial_\alpha u=:f_\beta. \tag{14}$$

We therefore have

$$\mathcal{A}\begin{pmatrix}\partial_1\partial_\beta u\\ \partial_2\partial_\beta u\end{pmatrix}=\begin{pmatrix}f_\beta\\ (\tau\cdot\nabla)\partial_\beta u\end{pmatrix}.$$

Multiplying by the eigenform of $\mathcal{A}$, we deduce

$$\ell\cdot f_\beta+m\cdot(\tau\cdot\nabla)\partial_\beta u=0,\qquad \beta=1,2. \tag{15}$$

This system of two equations is equivalent to that formed by

$$\ell\cdot f(\nu)+m\cdot \mathrm{D}^2u(\tau,\nu)=0, \tag{16}$$
$$\ell\cdot f(\tau)+m\cdot \mathrm{D}^2u(\tau,\tau)=0, \tag{17}$$

with obvious notation $f(\xi) = \xi \cdot f$.

We point out that (17) is trivial, because of the following facts. First of all, $f(\tau) = 0$ along Γ because of (9). On the other hand, the derivation of (9) along Γ yields

$$\mathrm{D}^2 u(\tau, \tau) = \kappa(\nu \cdot \nabla)u. \tag{18}$$

Thus (17) reduces to $\kappa m \cdot (\nu \cdot \nabla)u = 0$, which is trivial thanks to (11) and (13).

There remains therefore only (16). With $(\nu \cdot \nabla)u = \rho r$, it gives

$$\rho(1 - \rho\, \ell(\mathrm{D}_u A(\bar{u}; \nu) \cdot r)r) + m \cdot \mathrm{D}^2 u(\tau, \nu) = 0,$$

or equivalently

$$\rho(1 - \rho\, \mathrm{d}_u \lambda(\bar{u}; \nu) \cdot r) + m \cdot \mathrm{D}^2 u(\tau, \nu) = 0. \tag{19}$$

To evaluate the last term, we differentiate the formula $(\nu \cdot \nabla)u = \rho r$ along Γ :

$$\mathrm{D}^2 u(\tau, \nu) + \kappa(\tau \cdot \nabla)u = (\tau \cdot \nabla)(\rho r),$$

that is

$$\mathrm{D}^2 u(\tau, \nu) = (\tau \cdot \nabla)(\rho r).$$

With $m \cdot r = 0$, we obtain

$$m \cdot \mathrm{D}^2 u(\tau, \nu) = \rho m \cdot (\tau \cdot \nabla)r = -\rho \ell \sum_\alpha (A^\alpha - x_\alpha)\partial_\alpha r.$$

Finally, we obtain the alternative that either $\rho = 0$ (which corresponds to the trivial solution $u_{in} \equiv \bar{u}$), or

$$\rho\, \mathrm{d}_u \lambda(\bar{u}; \nu) \cdot r = 1 - \ell \sum_\alpha (A^\alpha - x_\alpha)\partial_\alpha r. \tag{20}$$

We now simplify the right-hand side of (20). To do so, we remark that $\partial_\alpha = \tau_\alpha(\tau \cdot \nabla) + \nu_\alpha(\nu \cdot \nabla)$. Because of $\ell(\nu)(A(\nu) - x \cdot \nu) = 0$, there remains

$$\ell \sum_\alpha (A^\alpha - x_\alpha)\partial_\alpha r = \ell(\nu)(A(\tau) - x \cdot \tau)(\tau \cdot \nabla)r = \ell(\nu)(A(\tau) - \mathrm{d}_\xi \lambda \cdot \tau)(\tau \cdot \nabla)r.$$

Differentiating the identity $\ell(\nu)(A(\nu) - x \cdot \nu) = 0$, we have equivalently

$$\ell \sum_\alpha (A^\alpha - x_\alpha)\partial_\alpha r = -\mathrm{d}_\xi \ell \cdot \tau (A(\nu) - \lambda(\nu))(\tau \cdot \nabla)r.$$

We warn the reader that the operator *nabla* above is differentiation with respect to x, not to ξ, which involves the dependence of ν upon x. We pass from one to the other by the chain rule:

$$(\tau \cdot \nabla)r = \mathrm{d}_\xi r \cdot (\tau \cdot \nabla)\nu = \kappa \mathrm{d}_\xi r \cdot \tau,$$

hence the formula

$$\ell\sum_\alpha (A^\alpha - x_\alpha)\partial_\alpha r = -\kappa\left(\mathrm{d}_\xi\ell\cdot\tau\right)\left(A(\nu)-\lambda(\nu)\right)\left(\mathrm{d}_\xi r\cdot\tau\right),$$

or

$$\rho\,\mathrm{d}_u\lambda(\bar u;\nu)\cdot r = 1+\kappa\left(\mathrm{d}_\xi\ell\cdot\tau\right)\left(A(\nu)-\lambda(\nu)\right)\left(\mathrm{d}_\xi r\cdot\tau\right). \tag{21}$$

Finally we evaluate κ. Differentiating (8) with respect to arc-length, we have

$$\tau = \frac{dx}{ds} = \mathrm{D}_\xi^2\lambda\cdot\dot\nu = \kappa\,\mathrm{D}_\xi^2\lambda\cdot\tau,$$

which gives immediately

$$\kappa\,\mathrm{D}_\xi^2\lambda(\tau,\tau)=1.$$

However this second derivative can be computed from the identities (10). We have

$$(A(\tau)-\mathrm{d}_\xi\lambda\cdot\tau)r(\nu)+(A(\nu)-\lambda(\nu))\mathrm{d}_\xi r\cdot\tau=0,$$

then

$$-\mathrm{D}_\xi^2\lambda(\tau,\tau)r(\nu)+2(A(\tau)-\mathrm{d}_\xi\lambda\cdot\tau)\mathrm{d}_\xi r\cdot\tau+(A(\nu)-\lambda(\nu))\mathrm{D}_\xi^2 r(\tau,\tau)=0.$$

Multiplying at left by $\ell(\nu)$ gives

$$\mathrm{D}_\xi^2\lambda(\tau,\tau)=2\ell(\nu)(A(\tau)-\mathrm{d}_\xi\lambda\cdot\tau)\mathrm{d}_\xi r\cdot\tau=-2\left(\mathrm{d}_\xi\ell\cdot\tau\right)\left(A(\nu)-\lambda(\nu)\right)\left(\mathrm{d}_\xi r\cdot\tau\right).$$

Finally, we obtain the fundamental formula that

$$\rho\,\mathrm{d}_u\lambda(\bar u;\nu)\cdot r = 1-\frac12=\frac12\,.$$

Theorem 5.1 *Let u be a continuous, piecewise-C^2 solution of the 2-dimensional system (2), which is constant ($u\equiv\bar u$) on one side of its singular set Γ. We assume (H1,2,3), and that Γ is locally defined by $\lambda_n^*(\bar u;x)=0$.*

Then along Γ, we have either $\nabla u_{in}=0$ or

$$\nabla u_{in}|_\Gamma = \frac{1}{2\,\mathrm{d}_u\lambda_n(\bar u;\nu)\cdot r_n(\bar u;\nu)}\,\nu\otimes r_n(\bar u;\nu), \tag{22}$$

where ν is the normal to Γ.

Remarks about the linearly degenerate case. In [9], the 2-dimensional Riemann problem is considered for a gas obeying the Chaplygin equation of state. This is the case where the acoustic field is linearly degenerate ; hence (H2) is violated. In this situation, formula (22) suggests that the field (density, momentum) experiences an infinite normal derivative on the inner part of the sonic line. This article did not point out such a singularity. Instead, its Section 6.4 suggests a calculation which, in the case of a smooth outer flow, tells us that the inner flow would match the outer at infinite order! This is clearly in contradiction with (22). The explanation is that if one carries the calculations above in the linearly degenerate case, the equation in ρ becomes linear, instead of quadratic; therefore the alternative becomes hidden. This is why one retained only the trivial solution in [9]. Perhaps that was a bad choice, dictated by the obscure desire that the flow be smooth up to the boundary of the pseudo-subsonic domain D. Now that we suspect a wilder behaviour along Γ, it would be nice to perform numerical simulation in order to decide whether u_{in} has a gradient singularity along the sonic line or not. Finally, let us point out that the theorems of [9] remain correct; they only stated that the potential ϕ is Lipschitz over $\bar{D}$, being $\mathcal{C}^\infty$ in D. In terms of the flow itself, they tell us that the density and the velocity are bounded on D and smooth in the interior. They don't claim anything about the behaviour of ∇u_{in} (that is, of $\mathrm{D}^2\phi$) towards the boundary ∂D.

6 Multi-dimensional case

We show that Theorem 5.1 extends to higher space dimension, with a formula that depends upon d. The proof adapts as follows. We choose an orthonormal frame $(\tau^1, \dots, \tau^{d-1})$ of the tangent space $T_x\Gamma$. We still have $\nabla u_{in} = \rho\nu \otimes r(\bar{u}; \nu)$ for some scalar $\rho(x)$. We define

$$f_\beta := \sum_\alpha A^\alpha(u)\partial_\alpha\partial_\beta u - (x \cdot \nabla)\partial_\beta u = \partial_\beta u - \sum_\alpha (\mathrm{d}A^\alpha(u) \cdot \partial_\beta u)\partial_\alpha u.$$

More precisely, we consider $\ell \cdot f(\nu)$:

$$\begin{aligned} \ell \cdot f(\nu) &= \rho(1 - \rho \mathrm{d}_u\lambda(\bar{u}, \nu) \cdot r) \\ &= \sum_{\alpha,\beta} \nu_\beta p^\alpha \partial_\alpha\partial_\beta u, \end{aligned}$$

where $p^\alpha := \ell(A^\alpha - x_\alpha)$.

Lemma 6.1 *There exist vectors $m^1, \dots, m^{d-1}$ such that*

$$p^\alpha = \sum_\gamma \tau^\gamma_\alpha m^\gamma,$$

together with $m^\gamma \cdot r = 0$.

Proof
Let us consider the linear map

$$(m^1, \dots, m^{d-1}) \mapsto \left(\sum_\gamma \tau^\gamma_1 m^\gamma, \dots, \sum_\gamma \tau^\gamma_d m^\gamma \right),$$

defined over the space $(T\Gamma)^{d-1}$, and taking values in $(T\Gamma)^d$. Because the vectors τ^γ form a free family, the matrix $((\tau^\gamma_\alpha))$ has rank $d-1$. Therefore the linear map has rank $(d-1)\dim T\Gamma = (d-1)(n-1)$. Its image is contained into the subspace $X \subset (T\Gamma)^d$ formed of those $(q^1, \dots, q^d)$

such that $q(\nu)=0$. This subspace is precisely of dimension $(d-1)(n-1)$, and therefore it coincides with the range of the linear map.

Finally, we remark that $p\in X$: $p(\nu)=0$ is obvious and

$$p^\alpha\cdot r=\ell A^\alpha(\bar u)r-x_\alpha=\mathrm{d}_\xi\lambda(\bar u;\nu)\cdot\vec e^{\,\alpha}-x_\alpha=0.$$

∎

We have

$$\begin{aligned}\ell\cdot f(\nu) &= \sum_{\alpha,\beta,\gamma}\nu_\beta\tau_\alpha^\gamma m^\gamma\partial_\alpha\partial_\beta u=\sum_{\beta,\gamma}\nu_\beta m^\gamma(\tau^\gamma\cdot\nabla)\partial_\beta u=\sum_{\beta,\gamma}\nu_\beta m^\gamma(\tau^\gamma\cdot\nabla)(\rho\nu_\beta r)\\ &= \rho\sum_\gamma m^\gamma(\tau^\gamma\cdot\nabla)r=\rho\sum_\alpha p^\alpha\cdot\partial_\alpha r=\rho\ell\sum_\alpha(A^\alpha-x_\alpha)\partial_\alpha r.\end{aligned}$$

Therefore the formula (20) remains valid.

Now we use

$$\partial_\alpha=\nu_\alpha(\nu\cdot\nabla)+\sum_\gamma\tau_\alpha^\gamma(\tau^\gamma\cdot\nabla)$$

to get as above

$$\begin{aligned}\ell\sum_\alpha(A^\alpha-x_\alpha)\partial_\alpha r &= -\sum_\gamma \mathrm{d}_\xi\ell\cdot\tau^\gamma(A(\nu)-\lambda(\nu))(\tau^\gamma\cdot\nabla)r\\ &= -\sum_\gamma\left(\mathrm{d}_\xi\ell\cdot\tau^\gamma\right)\left(A(\nu)-\lambda(\nu)\right)\left(\mathrm{d}_\xi r\cdot(\tau^\gamma\cdot\nabla)\nu\right).\end{aligned}$$

We introduce the tensor κ by the formula $(\tau^\gamma\cdot\nabla)\nu=\sum_\delta\kappa_\delta^\gamma\tau^\delta$. This yields

$$\ell\sum_\alpha(A^\alpha-x_\alpha)\partial_\alpha r=-\sum_{\gamma,\delta}\kappa_\delta^\gamma\left(\mathrm{d}_\xi\ell\cdot\tau^\gamma\right)\left(A(\nu)-\lambda(\nu)\right)\left(\mathrm{d}_\xi r\cdot\tau^\delta\right).$$

Using again (8), we find that κ is the inverse of the $(d-1)\times(d-1)$ matrix whose entries are $\mathrm{D}^2\lambda(\tau^\gamma,\tau^\delta)$. In particular, κ is symmetric positive definite if λ is the maximal eigenvalue and is strongly convex in the directions transversal to ν. Finally, differentiation of (12) gives

$$\mathrm{D}^2\lambda(\zeta,\eta)=-\left(\mathrm{d}_\xi\ell\cdot\eta\right)\left(A(\nu)-\lambda(\nu)\right)\left(\mathrm{d}_\xi r\cdot\zeta\right)-\left(\mathrm{d}_\xi\ell\cdot\zeta\right)\left(A(\nu)-\lambda(\nu)\right)\left(\mathrm{d}_\xi r\cdot\eta\right).$$

We are now in the situation where

$$\kappa S=I_{d-1},\qquad S=M+M^T,\qquad S:=\mathrm{D}^2\lambda|_{T\Gamma}.$$

Therefore we have

$$\ell\sum_\alpha(A^\alpha-x_\alpha)\partial_\alpha r=\mathrm{Tr}(\kappa M)=\frac12\mathrm{Tr}(\kappa M+M^T\kappa)=\frac12\mathrm{Tr}(\kappa(M+M^T))=\frac12\mathrm{Tr}(\kappa S)=\frac{d-1}{2}.$$

This yields the following formula

Theorem 6.1 *Let u be a continuous, piecewise-C^2 solution of the d-dimensional system (2), which is constant ($u \equiv \bar{u}$) on one side of its singular set Γ. We assume (H1,2,3), and that Γ is locally defined by $\lambda_n^*(\bar{u}; x) = 0$.*

Then along Γ, we have either $\nabla u_{in} = 0$ or

$$\nabla u_{in}|_\Gamma = \frac{3-d}{2\,\mathrm{d}_u\lambda_n(\bar{u};\nu)\cdot r_n(\bar{u};\nu)}\,\nu \otimes r_n(\bar{u};\nu). \tag{23}$$

We recognize the Taylor expansion of a rarefaction wave in one space dimension, where we have $u' = \frac{1}{\mathrm{d}\lambda\cdot r}r$. But we obtain the amazing fact:

Corollary 6.1 *In the 3-dimensional Riemann problem, we have $\nabla u_{in} = 0$, even if $u_{in} \not\equiv \bar{u}$.*

Therefore, in 3 space dimensions, rarefaction waves have a C^1-matching with the neighbouring constant state.

(Non-)linearity *vs* spatial dimension. The table below summarizes the behaviour of the smoother waves that are genuinely d-dimensional; we thus exclude those waves that depend on fewer variables. The context is that of the Riemann problem, and the the outer state is a constant. We collect the results obtained in Sections 2 and 6 in the column associated with Genuinely Nonlinear characteristic fields. We compare them to the opposite situation of a linear system, say that of acoustics. It is interesting to observe the increase of regularity when going downwards (d increases) or rightward.

	Linear case	Genuinely Nonlinear
$d=1$	discontinuous (pure transport)	Lipschitz (Lax rarefaction wave)
$d=2$	Hölder continuity $C^{1/2}$ (see [6])	Lipschitz, but half-steeper profile
$d=3$	constancy ? Huyghens principle	C^1 (C^∞ ?) flat profile

7 The limit of the regular reflection at normal incidence

Let us consider a planar shock between two constant states, moving towards a solid wedge of aperture 2θ, with $\theta \le \pi/2$. The downstream state is at rest. This is the problem that Chen & Feldman [4] considered in the irrotational situation. When θ is large enough, we expect a *regular* reflection. See [8] for a description with pictures. For $t > 0$, the solution is self-similar (provided that it is unique). In the upper half-space (we have a symmetry with respect to the horizontal line), the incident shock I meets the wall at a point P where it is reflected as an other planar shock V. Between V, the wall and the sonic line, the flow is uniform: $U \equiv \bar{U}$. In particular the sonic line is a circle C of equation $|x-\bar{u}| = c(\bar{\rho})$, in accordance with (8). The calculation of V and $\bar{U}$ was carried out by von Neumann, using shock polar analysis.

Beyond C, the reflected shock is bended and becomes a free boundary F. Between F, the symmetry axis and the wall, the flow is pseudo-subsonic (system (2) is not hyperbolic). It is expected to be smooth up to C, even if it only Lipschitz across C. Experiments suggest that the factor ρ is non-zero, and therefore Theorem 5.1 gives

$$(\nabla U)_{\rm in} = \frac{1}{2{\rm d}_u\lambda(\bar{U};\nu)}\,\nu \otimes r(\bar{U};\nu). \tag{24}$$

Let us point out that this formula, which was established in this special context in [3], is valid in the irrotational case studied in [4], because even if we consider the isentropic Euler model, the flow remains uniform (thus irrotational) above C, and therefore remains irrotational in the influence domain of C, because a sonic line does not generate vorticity (remark that $r(\nu) = (\bar{\rho}, \bar{c}\nu)^T$ and thus $(\nabla u)_{\rm in} = {\rm cst}\,\nu\otimes\nu$ is symmetric), and because of the transport of the vorticity:

$$(u-x)\cdot\nabla\,\frac{\omega}{\rho} = 0.$$

Formula (24) depends of θ, because $\bar{U} = \bar{U}_\theta$ does. Let us consider the situation when $\theta \to \frac{\pi}{2} - 0$. Then $\bar{U}_\theta$ tends to a limit $\bar{U}_{\pi/2}$, which turns out to be different from the downstream state. Yet, the velocity $\bar{u}_{\pi/2}$ is zero. The formula (24) has a non-zero limit

$$\lim_{\theta\to\pi/2} (\nabla U_\theta)_{\rm in} = M := \frac{1}{2{\rm d}_u\lambda(\bar{U}_{\pi/2};\nu)}\,\nu\otimes r(\bar{U}_{\pi/2};\nu) \neq 0. \tag{25}$$

On the other hand, we expect that U_θ has a limit $U_{\pi/2}$ as $\theta \to \frac{\pi}{2} - 0$, this limit being a solution of the *normal* reflection problem. In [3], it is stated that $U_{\pi/2}$ *is the unique solution of the normal reflection.* If this is correct, then $U_{\pi/2}$ is constant in the subsonic zone. In particular

$$(\nabla U_{\pi/2})_{\rm in} \equiv 0.$$

This contrasts with Formula (25), where the limit of the inner gradient is non-zero. Of course, we must keep in mind that an inner gradient is a kind of limit (as x tends to Γ) and therefore we are comparing

$$\lim_{\theta\to\pi/2}\lim_{x\to x_\theta} \qquad vs \qquad \lim_{x\to x_{\pi/2}}\lim_{\theta\to\pi/2},$$

where x_θ belongs to the sonic circle C_θ and $x_\theta \to x_{\pi/2}$. It may well happen that both limits (inner and normal) do not commute, but it is worth emphasizing it. It might be the first time that such a discrepancy is observed in this context. We point out that this phenomenon is consistent with the $L^1_{\rm loc}$-convergence announced in the Theorem of [4] (2005).

If the above analysis is correct and if the regular reflection admits a solution that converges to the standard normal reflection as $\theta \to \frac{\pi}{2}$, then there is a **boundary layer** along the sonic line. The flow is close to a constant state (at rest), whereas its gradient *is not* small along C. This implies that ${\rm D}^2U$ does not remain uniformly bounded as $\theta \to \frac{\pi}{2}$.

References

[1] S. Benzoni-Gavage, D. Serre. Multi-dimensional hyperbolic partial differential equations. First order systems and applications. Oxford Mathematical Monographs, Oxford University Press (2007).

[2] G. Boillat. Chocs caractéristiques. *C. R. Acad. Sci. Paris, Série I*, **274** (1972), pp 1018–21.

[3] Myoungjean Bae, Gui-Qiang Chen, M. Feldman. Regularity of solutions to regular shock reflection for potential flow. *Invent. math.*, **175** (2009), pp 505–543.

[4] Gui-Qiang Chen, M. Feldman. Potential theory for shock reflection by a large-angle wedge. *Proc. Natl. Acad. Sci. USA*, **102** (2005), pp 15368–72. Global solution to shock reflection by large-angle wedges for potential flow. *Annals of Maths.*, (2010).

[5] L. Gårding. Linear hyperbolic partial differential equations with constant coefficients. *Acta Math.*, **85** (1951), pp 1–62. An inequality for hyperbolic polynomials. *J. Math. Mech.*, **8** (1959), pp 957–965.

[6] H. Gilquin, J. Laurens, C. Rosier. Multi-dimensional Riemann problems for linear hyperbolic systems. *RAIRO Modél. Math. et Anal. Numér.*, **30** (1996), pp 527–548.

[7] P. D. Lax. Hyperbolic systems of conservation laws. *Comm. Pure Appl. Math.*, **10** (1957), pp 537–566.

[8] D. Serre. Shock reflection in gas dynamics. In: Friedlander, S., Serre, D. (eds.) Handbook of Mathematical Fluid Dynamics, vol. 4, pp. 39–122. Elsevier, North-Holland (2007)

[9] D. Serre. Multi-dimensional shock interaction for a Chaplygin gas. *Arch. Rational Mech. Anal.*, **191** (2009), pp 539–577.

[10] D. Serre. Weyl and Lidskiĭ inequalities for general hyperbolic polynomials. *Chinese Annals of Maths. Ser. B*, **30** (2009), pp 785–802.

[11] H. F. Weinberger. Remark on the preceeding paper of Lax. *Comm. Pure Appl. Math.*, **11** (1958), pp 195–196.