VISCOUS SYSTEM OF CONSERVATION LAWS: SINGULAR LIMITS

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Abstract. We continue our analysis of the Cauchy problem for viscous system of conservation, under natural assumptions. We examine in which way does the existence time depend upon the viscous tensor $B(u)$. In particular, we consider singular limits, where the rank of the symbol $B(u;\xi)$ drops at the limit. This covers a lot of situations, for instance that of the limit of the Navier-Stokes-Fourier system towards the Euler-Fourier system, or that of the vanishing viscosity.

We emphasize the symmetry of the dissipation tensor, an hypothesis which is reminiscent to the Onsager's reciprocity relations. We find it useful in this asymptotic context, when establishing uniform estimates.

Key words. Systems of conservation laws; dissipative structure; entropy; singular limit; Onsager's reciprocity relations.

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1. Main results.

1.1. Dissipative viscous systems of conservation laws.

We continue our work begun in [7]. We consider a system of PDEs in conservation form

$$\partial_t u + \mathrm{Div} f(u) = \mathrm{Div}(B(u)\nabla u) := \sum_{\alpha,\beta} \partial_\alpha (B^{\alpha\beta}(u)\partial_\beta u), \tag{1.1}$$

in which $u : (0,T) \times \mathbb{R}^d \to \mathcal{U}$ is the unknown. The phase space $\mathcal{U}$ is an open convex subset of $\mathbb{R}^n$. The symbol ∂_α denotes the partial derivative with respect to the coordinate x_α. The nonlinearities are encoded in the smooth functions

$$f : \mathcal{U} \to \mathbf{M}_{n\times d}(\mathbb{R}), \qquad B^{\alpha\beta} : \mathcal{U} \to \mathbf{M}_n(\mathbb{R}).$$

They describe in mathematical terms the kind of physics that is modelled by the PDEs (1.1).

In the notation $B\nabla u$ in (1.1), the tensor B plays the role of a linear map from the space of $n \times d$ matrices into itself. The operator Div is the row-wise divergence which, to a field of $n \times d$ matrices, associates a vector field of dimension n. When needed, we use also the ordinary divergence operator div, which applies to d-vector fields and produces scalar functions.

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A paradigm for the class of such systems is the Navier-Stokes-Fourier system for a compressible fluid, in which the components of the unknown are the mass density ρ, the linear momentum ρv (v the mean velocity of the fluid) and the mechanical energy per unit volume:

$$u =: \begin{pmatrix} \rho \\ \rho v \\ \frac{1}{2}\rho|v|^2 + \rho e \end{pmatrix}.$$

The flux f and the tensor B are given by

$$f(u) := \begin{pmatrix} \rho v^T \\ \rho v \otimes v + p(\rho, e) I_d \\ \left(\frac{1}{2}\rho|v|^2 + \rho e + p(\rho, e)\right) v^T \end{pmatrix}$$

and

$$B(u)\nabla u :=$$
$$\begin{pmatrix} 0 \\ \mathcal{T} := \mu(\rho, e)(\nabla v + (\nabla v)^T) + (\zeta(\rho, e) - 2\mu(\rho, e))(\operatorname{div} v) I_d \\ (\mathcal{T} v + \kappa(\rho, e)\nabla\theta)^T \end{pmatrix}.$$

Hereabove, the exponent T denotes transposition, while θ, the temperature, is a prescribed function of (ρ, e). The dissipation is due to Newtonian viscosity (coefficients ζ and μ), and to heat diffusion (coefficient κ).

Other examples come from electromagnetism (in material media), magnetohydrodynamics (see below), viscoelasticity, ... Let us mention also the multi-component gas dynamics with chemistry; see [2] for an accurate modelling.

The common features of these examples are four-fold:

1. On the one hand, the left-hand side of (1.1), which is a first-order system of conservation laws, admits an entropy-flux pair (η, q) in which η, the entropy, is strongly convex ($\mathrm{D}^2\eta > 0_n$) over $\mathcal{U}$. In other words, the system

$$\partial_t u + \operatorname{Div} f(u) = 0$$

implies formally

$$\partial_t \eta(u) + \operatorname{div} q(u) = 0.$$

Let us recall that the differentials of η, f^α and q^α satisfy the identity

$$\mathrm{d}q^\alpha = \mathrm{d}\eta\, \mathrm{d}f^\alpha.$$

2. On the next hand, the entropy η is *dissipated* by the right-hand side of (1.1). In mathematical terms, this means the following. Multiplying (1.1) by the differential $\mathrm{d}\eta$, we obtain

$$\partial_t \eta(u) + \operatorname{div} q(u) = \mathrm{d}\eta(u) \sum_{\alpha,\beta} \partial_\alpha (B^{\alpha\beta}(u)\partial_\beta u).$$

The right-hand side can be recast as the difference between a divergence and a quadratic expression in ∇u :

$$\sum_{\alpha,\beta} \partial_\alpha (\mathrm{d}\eta(u) B^{\alpha\beta}(u)\partial_\beta u) - \sum_{\alpha,\beta} \mathrm{D}^2\eta(u)(\partial_\alpha u, B^{\alpha\beta}(u)\partial_\beta u),$$

where we view $\mathrm{D}^2\eta(u)$ as a symmetric positive definite bilinear form. When u is a smooth solution, tending towards a constant $\bar{u}$ at infinity rapidly enough, we may assume that $\eta(\bar{u}) = 0$ and $\mathrm{d}\eta(\bar{u}) = 0$ (just subtract an affine function to η). Then an integration over $\mathbb{R}^d$ suppresses the divergence terms and yields the identity

$$\frac{d}{dt} \int_{\mathbb{R}^d} \eta(u)\,dx + \int_{\mathbb{R}^d} \sum_{\alpha,\beta} \mathrm{D}^2\eta(u)(\partial_\alpha u, B^{\alpha\beta}(u)\partial_\beta u)\,dx = 0. \quad (1.2)$$

We then say that the entropy is *strongly dissipated* if the quadratic term is non-negative and actually controls the dissipation flux. By this, we mean that

$$\sum_{\alpha,\beta} \mathrm{D}^2\eta(u)(X_\alpha, B^{\alpha\beta}(u)X_\beta) \geq \omega \sum_\alpha \Big|\sum_\beta B^{\alpha\beta}(u)X_\beta\Big|^2 =: \omega |B(u)\mathbf{X}|^2, \quad \forall u \in \mathcal{U},\ \forall \mathbf{X} = X_1, \dots, X_d \in \mathbf{M}_{n\times d}(\mathbb{R}). \quad (1.3)$$

Hereabove, $\omega = \omega(u)$ is positive and may be chosen continuous. Applying (1.3) to (1.2), we have a differential inequality

$$\frac{d}{dt} \int_{\mathbb{R}^d} \eta(u)\,dx + \int_{\mathbb{R}^d} \omega(u)|B(u)\nabla_x u|^2 dx \leq 0,$$

from which we obtain an *a priori* estimate of $u(t,\cdot)$ in some Lebesgue–Orlicz space (associated to η) and another one of $\omega(u)^{1/2}B(u)\nabla_x u$ in $L^2_{t,x}$.

3. On a third leg, we observe that in physical systems, some components of the unknown field u obey first-order PDEs. This is the case for instance when one writes the conservation of mass

$$\partial_t \rho + \operatorname{div}(\rho v) = 0,$$

in which there cannot be any second-order derivative. This means that in the Navier-Stokes-Fourier equations, the first line of the matrices $B^{\alpha\beta}$ vanishes identically. For more general systems, we make the assumption

(**A**) The matrices $B^{\alpha\beta}$ have the block form

$$B^{\alpha\beta}(u) = \begin{pmatrix} 0_{p\times n} \\ b^{\alpha\beta} \end{pmatrix}.$$

In addition, we assume that this property is sharp, in the following sense. Defining the symbol

$$B(\xi;u) := \sum_{\alpha,\beta} \xi_\alpha \xi_\beta B^{\alpha\beta}(u), \qquad \forall \xi \in \mathbb{R}^d,$$

we ask that $B(\xi;u)$ have rank $n-p$ for every $\xi \neq 0$.

4. On the last leg, the dissipation tensor enjoys, in suitable coordinates, a symmetry property, which we discuss in Section 2. It is reminiscent to Onsager's reciprocity relations, and was considered by Kawashima [4] in his analysis of the Cauchy problem.

We recall the results obtained in [7]. For this, we split u into two blocks v and w of respective sizes p and $n-p$.

THEOREM 1.1. *Let the variables $z_{p+1},\dots,z_n$ (dual to w) be defined by*

$$z_j = \frac{\partial \eta}{\partial u_j}\,.$$

Then the map

$$u = \begin{pmatrix} v \\ w \end{pmatrix} \quad \mapsto \quad U := \begin{pmatrix} v \\ z \end{pmatrix}$$

is a global diffeomorphism onto its image $\mathcal{V}$. The viscous flux $b(u)\nabla_x u$ can be written as $Z(u)\nabla_x z$. The tensor Z is uniquely defined and satisfies an inequality

$$\sum_{\alpha,\beta=1}^{d} \sum_{i,j\geq p+1} F_{i\alpha} F_{j\beta} Z_{ij}^{\alpha\beta}(u) \geq c_0(u)\|Z(u)F\|^2, \qquad \forall F \in \mathbf{M}_{(n-p)\times d}(\mathbb{R}). \tag{1.4}$$

In addition, the operator $Z(u)\nabla_x$ is strongly elliptic:

$$\sum_{\alpha,\beta=1}^{d} \sum_{i,j\geq p+1} \xi_\alpha \lambda_i \xi_\beta \lambda_j Z_{ij}^{\alpha\beta}(u) \geq c_1(u)\|\xi\|^2\|\lambda\|^2, \qquad \forall \xi \in \mathbb{R}^d,\ \forall \lambda \in \mathbb{R}^{n-p}. \tag{1.5}$$

Hereabove, c_0 and $c_1(u)$ are positive continuous functions. **The kernel of** $Z(u)$. The properties stated above do not imply that the linear map $F \mapsto Z(u)F$ be one-to-one over $\mathbf{M}_{(n-p)\times d}(\mathbb{R})$. At least, ellipticity tells us that its kernel does not intersect the cone of rank-one matrices. But for NSF, $\ker Z(u)$ consists of matrices $\begin{pmatrix} G \\ 0 \end{pmatrix}$ where G is skew-symmetric. We are thus led to Assumption

> (**B**) The kernel of $Z(u)$ does not depend upon $u \in \mathcal{U}$, and the derivatives of Z with respect to u (at suitable order) are bounded in terms of $Z(u)$ itself: for every derivative ∂ of order k, and every compact $K \subset \mathcal{U}$, there exists a finite $c_{k,K}$ such that

$$|\partial Z(u)F| \leq c_{k,K}|Z(u)F|. \tag{1.6}$$

This assumption is obviously met by NSF and by other systems of physical importance.

Another example: the MHD system. In MHD, the fluid is conducting but the electric field is stuck to the magnetic field by the constraint $q\vec{E}+v\wedge \vec{B} = 0$, with q the (constant) density of charge. The fluid is thus described by $(\rho, v, e, \vec{B})$, where (ρ, v, e) are as in Navier-Stokes-Fourier. Actually, if $\vec{B} \equiv 0$ at initial time, this identity remains true forever and the system reduces to Navier-Stokes-Fourier.

Let ν be the electrical resistivity. Define the stress tensor Π and the total energy E by the formulæ

$$\Pi := \left(p + \frac{1}{2}|\vec{B}|^2\right) I_3 - \vec{B} \otimes \vec{B}, \qquad E := \rho\left(\frac{1}{2}|v|^2 + e\right) + \frac{1}{2}|\vec{B}|^2.$$

The MHD system is

$$\begin{aligned} \rho_t + \mathrm{div}(\rho v) &= 0, \\ (\rho v)_t + \mathrm{Div}(\rho v \otimes v + \Pi) &= \mathrm{Div}\mathcal{T}, \\ \vec{B}_t + \mathrm{Div}(\vec{B} \otimes v - v \otimes \vec{B}) &= \mathrm{Div}(\nu(\nabla\vec{B} - \nabla\vec{B}^T)), \\ E_t + \mathrm{div}(Ev + \Pi v) &= \mathrm{div}(\mathcal{T}v + \kappa\nabla\theta), \\ \mathrm{div}\vec{B} &= 0, \end{aligned}$$

The MHD system contains a differential constraint ($\mathrm{div}\vec{B} = 0$) and thus looks a bit different from the abstract form (1.1). However, this constraint is compatible with the evolution equations and thus can be ignored in a first instance. If $\vec{B}|_{t=0}$ satifies it, and $(\rho, v, e, \vec{B})$ is a solution of the evolutionary part, even in the weaker sense of distributions, then $\vec{B}$ is solenoidal forever.

Entropy. Linear combinations of the conservation laws, together with the chain rule imply

$$\rho(v_t + (v \cdot \nabla)v) + \mathrm{Div}\Pi = 0, \qquad \rho(e_t + (v \cdot \nabla)e) + p\,\mathrm{div}v = 0.$$

Notice that the last equation is the same as in the Euler equations. Then the chain rule yields $s_t + v \cdot \nabla s = 0$, where s is the usual entropy, satisfying $\theta \mathrm{d}s = \mathrm{d}e + p\mathrm{d}\frac{1}{\rho}$, where $\theta > 0$ is the temperature.

Dissipation. Instead, we have

$$\rho(v_t + (v\cdot\nabla)v) + \mathrm{Div}\Pi = \mathrm{Div}\mathcal{T},$$
$$\rho(e_t + (v\cdot\nabla)e) + p\,\mathrm{div}v = \mathrm{div}(\kappa\nabla\theta) + \frac{\mu}{2}\left|\nabla v + \nabla v^T\right|^2 + (\zeta - 2\mu)(\mathrm{div}v)^2 + \nu|\mathrm{curl}\vec{B}|^2.$$

Whence

$$\theta\rho(s_t + (v\cdot\nabla)s) = \mathrm{div}(\kappa\nabla\theta) + \frac{\mu}{2}\left|\nabla v + \nabla v^T\right|^2 + (\zeta - 2\mu)(\mathrm{div}v)^2 + \nu|\mathrm{curl}\vec{B}|^2.$$

Denoting again $\eta := -\rho s$, we have

$$\eta_t + \mathrm{div}(\eta v) + \mathrm{div}\frac{\kappa\nabla\theta}{\theta} + \kappa\left|\frac{\nabla\theta}{\theta}\right|^2 + \frac{\mu}{2\theta}\left|\nabla v + \nabla v^T\right|^2 + \frac{\zeta - 2\mu}{\theta}(\mathrm{div}v)^2 + \frac{\nu}{\theta}|\mathrm{curl}\vec{B}|^2 = 0.$$

Strong dissipation requires $\kappa, \mu \geq 0$ and $\zeta \geq \frac{4}{3}\mu$.

1.2. Normal form and the Cauchy problem. In the new coordinates, the system (1.1) rewrites in the following quasilinear form

$$\partial_t U + \sum_\alpha \tilde{A}^\alpha(U)\partial_\alpha U = \begin{pmatrix} 0 \\ \mathrm{D}^2_{ww}\eta \sum_{\alpha,\beta}\partial_\alpha(Z^{\alpha\beta}\partial_\beta z) \end{pmatrix}, \tag{1.7}$$

where $\tilde{A}^\alpha = (\mathrm{d}U)A^\alpha(\mathrm{dU})^{-1}$. Multiplying by the block diagonal, positive definite matrix

$$S_0(U) := \begin{pmatrix} \mathrm{D}^2_{vv}\eta - \mathrm{D}^2_{vw}\eta\left(\mathrm{D}^2_{ww}\eta\right)^{-1}\mathrm{D}^2_{wv}\eta & 0 \\ 0 & \left(\mathrm{D}^2_{ww}\eta\right)^{-1} \end{pmatrix},$$

the system (1.7) is equivalent to

$$S_0(U)\partial_t U + \sum_\alpha S_\alpha(U)\partial_\alpha U = \begin{pmatrix} 0 \\ \sum_{\alpha,\beta}\partial_\alpha(Z^{\alpha\beta}\partial_\beta z) \end{pmatrix}, \tag{1.8}$$

where S_α is symmetric. This symmetrization is discussed in details in [8].

We remark that the normal form (1.8) depends upon the range of the dissipation tensor B. This might be a source of difficulty in the analysis of singular limits, when this range drops.

Local existence. The main result of [8] is

Theorem 1.2. *Consider a viscous system of conservation laws (1.1)*

$$\partial_t u + \sum_\alpha \partial_\alpha f^\alpha(u) = \sum_{\alpha,\beta}\partial_\alpha(B^{\alpha\beta}(u)\partial_\beta u).$$

Assume the following:

- *The maps* $u \mapsto f^\alpha(u)$ *and* $u \mapsto B^{\alpha\beta}(u)$ *are smooth over a convex open set* $\mathcal{U}$ *containing the origin,*
- *System (1.1) is strongly entropy-dissipative for some smooth strongly convex entropy* η*,*
- *(**A**) the range of the symbol matrix* $B(\xi; u)$ *does not depend neither on* $\xi \neq 0$ *in* $\mathbb{R}^d$*, nor on the state* u*.*
- *(**B**) the kernel of* $Z(u)$ *is independent of* u *and* $Z(u)$ *dominates its* u*-derivatives up to the order* $[s] + 1$*.*

Then, given an initial data u_0 *in* $H^s(\mathbb{R}^d)$ *with* $s > 1 + d/2$*, there exists* $T > 0$ *and a unique solution in the class*

$$u \in C(0, T; H^s), \quad \partial_t u \in L^2(0, T; H^{s-1}).$$

In addition, the component v *belongs to* $C^1(0, T; H^{s-1})$ *and* $Z(u)\nabla z$ *is in* $L^2(0, T; H^s)$*.*

The local existence is actually proved for the more general class of system of the form (1.8).

2. Singular limits *vs* Onsager's relations.

2.1. Onsager's reciprocity relations for viscous conservation laws. In practice, the tensor Z in the normal form of the equations displays a symmetry property. For instance, in the case of the Navier-Stokes-Fourier system, we have

$$Z^{\alpha\beta}(u) = \theta \begin{pmatrix} \mu(I_d + \mathbf{e}_\beta \mathbf{e}_\alpha^T) + (\zeta - 2\mu)\mathbf{e}_\alpha \mathbf{e}_\beta^T & \mu(v_\alpha \mathbf{e}_\beta + \delta_\alpha^\beta v) + (\zeta - 2\mu) v_\beta \mathbf{e}_\alpha \\ \mu(v_\beta \mathbf{e}_\alpha^T + \delta_\alpha^\beta v^T) + (\zeta - 2\mu) v_\alpha \mathbf{e}_\beta^T & \kappa\theta + \mu |v|^2 \delta_\alpha^\beta + (\zeta - \mu) v_\alpha v_\beta \end{pmatrix},$$

where the $\mathbf{e}_j$'s denote the vectors of the canonical basis of $\mathbb{R}^d$. We remark that these matrices satisfy

$$(Z^{\alpha\beta})^T = Z^{\beta\alpha}, \qquad \forall 1 \leq \alpha, \beta \leq d. \tag{2.1}$$

This symmetry is an example of Onsager's *reciprocity relations*, which originates in [5].

Onsager's relations (2.1) imply the symmetry of the symbol $Z(u; \xi)$ for every $\xi \in \mathbb{R}^d$. They are equivalent to saying that the linear map

$$F \mapsto G =: ZF, \qquad G_{\alpha i} := \sum_{\beta, j} Z_{ij}^{\alpha\beta} F_{\beta j}$$

is symmetric over $\mathbf{M}_{d \times (n-p)}(\mathbb{R})$, for the canonical scalar product $\langle F, G \rangle := \mathrm{Tr}(F^T G)$.

This property will be needed below. Let us remark that the optimal ellipticity constant ω_0 in

$$\langle F, ZF\rangle = \sum_{\alpha,\beta=1}^{d} \sum_{i,j\geq p+1} F_{\alpha i}F_{\beta j}Z_{ij}^{\alpha\beta} \geq \omega_0\|ZF\|^2 \tag{2.2}$$

can be taken equal to the inverse of the norm of Z:

$$\omega_0(u) = \frac{1}{\|Z(u)\|} \tag{2.3}$$

in the symmetric case. This is easily seen by diagonalizing Z in an orthonormal basis of $\mathbf{M}_{d\times(n-p)}(\mathbb{R})$.

2.2. Stable families of dissipation tensors. We focus in this paper in the situation where the tensor B depends upon a small parameter, $B = B_\epsilon(u)$, and we let $\epsilon \to 0+$. When the rank of $B_\epsilon(\xi; u)$ drops at $\epsilon = 0$, we face a *singular limit*. Exemples of this situations are encountered frequently:

- In the vanishing viscosity limit, when $B_\epsilon = \epsilon B(u)$,
- In the Navier-Stokes-Fourier system, when the viscosity coefficients have the form $\epsilon\mu$ and $\epsilon\zeta$, while the heat conductivity κ remains independent of ϵ,
- In MHD, when either the viscosity coefficients, or the electric resistivity, or both have a factor ϵ,

An important question is whether the solution u^ϵ converges towards the solution u of the Cauchy problem for the limit problem, on some non-trivial time interval. This will occur if the *a priori* estimates given in Theorem 1.2 are uniform as $\epsilon \to 0$. To formulate our main result, we make the following definition.

Definition 2.1. *The family* $(B_\epsilon)_{\epsilon\in[0,1]}$ *is* stable *if*

- *the range of $B_\epsilon(\xi; u)$ does not depend upon $\epsilon > 0$ (however it may, and does in general, be different when $\epsilon = 0$),*
- *$Z_\epsilon(u)$ is symmetric and the ellipticity parameter $\omega_0(u) > 0$ of Z_ϵ in (2.2) can be chosen independently of ϵ, and uniformly for u in compact subsets of $\mathcal{U}$,*
- *The kernel of $Z_\epsilon(u)$ does not depend upon $\epsilon > 0$ either,*
- *the partial derivatives of Z_ϵ are uniformly bounded in terms of Z_ϵ itself: For every multi-index ℓ of length less than s (s the regularity considered in Theorem 1.2)*

$$\|\partial_u^\ell Z_\epsilon(u)F\| \leq c_\ell(u)\|Z_\epsilon(u)F\|, \tag{2.4}$$

with c_ℓ independent of ϵ and bounded over compact sets of $\mathcal{U}$.

Discussion.

- In other words, we ask that Assumption ($\mathbf{B}$) be satisfied uniformly in $\epsilon > 0$.

- Because of (2.3), the uniform ellipticity is equivalent to the uniform boundedness of $Z_\epsilon(u)$ for u in a compact and $\epsilon > 0$.
  This is satisfied in the case of the Navier-Stokes-Fourier system provided that the coefficients μ_ϵ, ζ_ϵ and κ_ϵ remain bounded. This is also true in the singular limit towards the Euler-Fourier system and in the vanishing viscosity limit, when $Z_\epsilon \equiv \epsilon Z$.
- Condition (2.4) amounts to saying that $\partial_u^\ell Z_\epsilon(u) Z_\epsilon(u)^\dagger$ remains bounded, where $Z^\dagger$ denotes the Moore-Penrose inverse. Since $Z(u)$ is symmetric and positive semi-definite, $Z(u)^\dagger$ coincides with the usual inverse on the range of $Z(u)$, and vanishes over $\ker Z(u)$. Because of the symmetry of $Z^\dagger$ and of $\partial_u^\ell Z_\epsilon(u)$, and the fact that the norm of operators is unchanged under transposition, this is equivalent to saying that

$$\|Z_\epsilon(u)^\dagger \partial_u^\ell Z_\epsilon(u)\| \leq c_\ell(u). \tag{2.5}$$

2.3. Uniformity of the existence time. With this in hand, we have

THEOREM 2.1. *Let us consider a system as in Theorem 1.2 with a viscous tensor parametrized by $\epsilon \in (0,1)$ and a flux f independent of ϵ. We make the block structure hypothesis (**A**) and we assume that the family $(B_\epsilon)_{\epsilon\in[0,1]}$ is stable in the sense defined above.*

Let u_0 in $H^s(\mathbb{R}^d)$ with $s > 1 + d/2$ be a given initial data, independent of ϵ. Let u^ϵ denote the solution obtained in Theorem 1.2. Then there exists $T > 0$ such that u^ϵ is defined over $(0,T)$ and the following sequences are bounded:

$$u^\epsilon \text{ in } C(0,T;H^s), \quad \partial_t u^\epsilon \text{ in } L^2(0,T;H^{s-1})$$

and

$$v^\epsilon \text{ in } C^1(0,T;H^{s-1}), \qquad Z_\epsilon \nabla z^\epsilon \text{ in } L^2(0,T;H^s).$$

If in addition B_ϵ converges uniformly towards B as $\epsilon \to 0$, then u^ϵ converges towards the unique strong solution of the Cauchy problem associated to the viscous tensor B.

Comments.

- The stability assumption allows Z_ϵ to become singular at $\epsilon = 0$. For instance, we may have $Z_\epsilon = \epsilon Z$ (vanishing viscosity limit) and thus $Z_{\epsilon=0} = 0$. Therefore we do not expect that the components z^ϵ remain bounded in $L^2(0,T;H^{s+1})$.
- We do not discuss the behaviour of the *existence time*, defined as the maximal time for which the Cauchy problem has a solution in $C(0,T_{\max};H^s)$.

2.4. Proof of Theorem 2.1. The existence of a strong solution u^ϵ, respectively u in the limit problem, was proved in [8], where we obtained a positive lower bound T_ϵ (resp. T) of the existence time. This T_ϵ was that one for which we were able to get *a priori* estimates in the class described in Theorem 1.2. What we have to do here is to show that these estimates are uniform in ϵ. This will give us the first part of Theorem 2.1. The convergence follows from a classical compactness argument, plus the uniqueness of the strong solution of the limit problem.

We thus follow carefully the estimates in [8], Paragraph 3.1, and we examine where does they depend upon the dissipative tensor. We point out that since the existence has been proved yet, and because of the principle of continuation, we may work directly on the solutions u^ϵ, instead of dealing with approximate solutions. This means that we set $V = U$ in the estimates, simplifying a little bit the analysis.

The first estimate is obtained directly from the entropy inequality:

$$\int_{\mathbb{R}^d} \eta(u(T,x))\,dx + \int_0^T \int_{\mathbb{R}^d} \omega_0(u^\epsilon)|Z_\epsilon \nabla z^\epsilon|^2 dx\,dt \le \int_{\mathbb{R}^d} \eta(u_0(x))\,dx.$$

It gives a uniform bound of $Z_\epsilon \nabla z^\epsilon$ in $L^2((0,+\infty)\times\mathbb{R}^d)$, provided u^ϵ stays in some compact subset K of $\mathcal{U}$. This pointwise control is a part of the proof of Theorem 1.2, and we have to prove its uniformity in ϵ.

We now treat the higher order estimates. As in [8], we assume that $s = m > 1 + d/2$ is an integer. We point out that the change of variable $u \mapsto U$ does not depend upon ϵ, since the range of $B_\epsilon(\xi;u)$ is fixed for $\epsilon > 0$. In particular, the initial data U_0 is independent of ϵ. In the sequel, we employ the positive definite quadratic form

$$[[X]]^2 := X^T S_0(U^\epsilon) X,$$

which depends upon (x,t,ϵ) through the solution itself.

In order to estimate $\|\nabla_x^k U\|_{L^2}$, we denote by ∂ any spatial derivative of order $k \le m$ and apply ∂ to the equation (1.8). From now on, we adopt the convention of summation over repeated indices. Dropping the indices ϵ, we have

$$\begin{aligned} &S_0(U)\partial_t\partial U + S_\alpha(U)\partial_\alpha\partial U = \\ &\qquad \partial_\alpha(Y^{\alpha\beta}(U)\partial_\beta\partial U) + \partial_\alpha[\partial, Y^{\alpha\beta}(U)]\partial_\beta U + [S_\alpha(U),\partial]\partial_\alpha U \\ &\qquad +[S_0(U),\partial]\left(S_0(U)^{-1}\left\{\partial_\alpha(Y^{\alpha\beta}(U)\partial_\beta U) - S_\alpha(U)\partial_\alpha U\right\}\right). \end{aligned}$$

Multiplying scalarly by ∂U and using the symmetry of $S_0(U)$ and $S_\alpha(U)$, we derive

$$\begin{aligned} &\partial_t \frac{1}{2}[[\partial U]]^2 + \partial_\alpha\partial U \cdot Y^{\alpha\beta}\partial_\beta\partial U = \\ &\qquad \mathrm{div}_x(\cdots) + \frac{1}{2}\partial U(\partial_t S_0 + \partial_\alpha S_\alpha)\partial U - \partial_\alpha\partial U\cdot[\partial, Y^{\alpha\beta}]\partial_\beta U \qquad (2.6) \\ &\qquad +\partial U\cdot\left([S_\alpha,\partial]\partial_\alpha U + [S_0,\partial]\{S_0^{-1}(\partial_\alpha(Y^{\alpha\beta}\partial_\beta U) - S_\alpha\partial_\alpha U)\}\right), \end{aligned}$$

where we have pulled the dissipation rate in the left-hand side. We therefore deduce an inequality

$$\partial_t \frac{1}{2}[[\partial U]]^2 + \omega(K)|Z\nabla\partial z|^2 \leq \mathrm{div}_x(\cdots) + Q^{t\partial} + Q^{1\partial} + Q^{2\partial}, \tag{2.7}$$

with

$$\begin{aligned} Q^{t\partial} &:= \frac{1}{2}\partial U(\partial_t S_0)\partial U, \\ Q^{1\partial} &:= \frac{1}{2}\partial U(\partial_\alpha S_\alpha)\partial U + \partial U \cdot [S_\alpha, \partial]\partial_\alpha U - \partial U \cdot [S_0, \partial]\left(S_0^{-1} S_\alpha \partial_\alpha U\right) \end{aligned}$$

and

$$Q^{2\partial} := -\partial_\alpha \partial U \cdot [\partial, Y^{\alpha\beta}]\partial_\beta U + \partial U \cdot [S_0, \partial]\left(S_0^{-1}\partial_\alpha(Y^{\alpha\beta}\partial_\beta U)\right).$$

Only $Q^{2\partial}$ involves the tensor $Y = Y_\epsilon$ and thus could be sensitive to ϵ. We estimate its integral in terms of the dissipation rate $\|Z\nabla^{k+1}z\|_{L^2}$ and of $\|u\|_{H^m}$. Because of the block form of S_0 and Y, it rewrites in terms of derivatives of z only. In compact form, we have

$$Q^{2\partial} := -\nabla\partial z \cdot [\partial, Z]\nabla z + \partial z \cdot [s_0, \partial]\left(s_0^{-1}\mathrm{Div}(Z\nabla z)\right),$$

where $s_0 := (\mathrm{D}^2_{ww}\eta)^{-1}$ is the second diagonal block in S_0.

To begin with, we isolate one factor $\nabla^{k+1}z$ when possible, since it can be absorbed by the left-hand side of (2.7), *via* the Young inequality. On the one hand, we have

$$|\nabla\partial z \cdot [\partial, Z]\nabla z| = |Z\nabla\partial z \cdot Z^\dagger[\partial, Z]\nabla z| \leq \frac{\omega}{4}|Z\nabla\partial z|^2 + \frac{1}{\omega}\left|Z^\dagger[\partial, Z]\nabla z\right|^2.$$

According to Faà di Bruno's Formula, the expression $Z^\dagger[\partial, Z]\nabla z$ is a polynomial in the differentials of ∇z up to order $k-1$, whose coefficients are of the form $Z^\dagger\partial_u^\ell Z$ with $1 \leq |\ell| \leq k$. Because of (2.5), these coefficients are bounded as long as u^ϵ remains in a compact set K, *independently* of ϵ. Using Moser's inequalities (see [1], Chapter 10 and Appendix C for a description), and the fact that $1/\omega$ remains bounded over K, we therefore obtain

$$\int_{\mathbb{R}^d} \frac{1}{\omega}\left|Z^\dagger[\partial, Z]\nabla z\right|^2 \leq c(K)\|U(t)\|_{H^m}^4,$$

where $c(K)$ is independent from ϵ.

The remaining term in the integral of $Q^{2\partial}$ is treating mainly as in [8]. This again involves Moser's type inequalities:

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \partial z \cdot [s_0,\partial]\left(s_0^{-1}\mathrm{Div}(Z\nabla z)\right)\,dx \\
&\qquad \le \|\partial z\|_{L^2}\|[s_0,\partial]\left(s_0^{-1}\mathrm{Div}(Z\nabla z)\right)\|_{L^2} \\
&\qquad \le c(K)\|\nabla s_0\|_{H^{m-1}}\|s_0^{-1}\|_{H^{m-1}}\|Z\nabla z\|_{H^m}[[\partial U]] \\
&\qquad \le \frac{\omega}{4m}\|Z\nabla z\|_{H^m}^2 + \frac{mc(K)^2}{\omega}\|\nabla s_0\|_{H^{m-1}}^2\|s_0^{-1}\|_{H^{m-1}}^2[[\partial U]]^2 \\
&\qquad \le \frac{\omega}{4m}\|Z\nabla z\|_{H^m}^2 + \frac{c'(K)}{\omega}\|U\|_{H^m}^6[[\partial U]]^2.
\end{aligned}
$$

This is completed with

$$
\begin{aligned}
\frac{1}{2}\|\nabla^m(Z\nabla z)\|_{L^2}^2 &\le \|Z\nabla\nabla^m z\|_{L^2}^2 + \|[\nabla^m, Z]\nabla z\|_{L^2}^2 \\
&\le \|Z\nabla\nabla^m z\|_{L^2}^2 + \|U\|_{H^m}^4,
\end{aligned}
$$

where the first term will be absorbed by the dissipation after summing over k.

Finally, we obtained the same inequality as (15) in [8]:

$$
\frac{dY}{dt} + \frac{\omega(K)}{2m}\|\nabla z\|_{H^m}^2 \le c(K)\,\left(\|\partial_t U(t)\|_{H^{m-1}} + \|U(t)\|_{H^m}^2 + \|U(t)\|_{H^m}^6\right)Y,
$$

where the coefficients *do not depend* upon ϵ, and

$$
Y(t) := \int_{\mathbb{R}^d}\sum_{|\ell|\le m}[[\partial^\ell U]]^2 dx.
$$

We now close the loop as in Paragraph 3.2 of [8]. The only place where some care is needed is where we estimate $\partial_t U$ in $L^2(0,T;H^{m-1})$, since it involves explicitly $Z\nabla z$. However, the latter is controlled by (25), which writes here

$$
\|Z\nabla z\|_{L^2(0,T;H^m)} \le C(K)R_0\exp\frac{c_\rho}{2}\left\{(R_m^2+R_m^6)T + R_{m-1}\sqrt{T}\right\}. \quad (2.8)
$$

This, together with

$$
\partial_t U = S_0(V)^{-1}\left(\partial_\alpha(Y^{\alpha\beta}(V)\partial_\beta U) - S_\alpha(V)\partial_\alpha U\right)
$$

yields (26). Once again, both estimates above are ϵ-free. The rest is as in [8] and yields the uniform estimates mentionned in our Theorem 2.1. This ends the proof of the theorem.

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