

# KILLING FIELDS OF HOLOMORPHIC CARTAN GEOMETRIES

SORIN DUMITRESCU\*

ABSTRACT. We study local automorphisms of holomorphic Cartan geometries. This leads to classification results for compact complex manifolds admitting holomorphic Cartan geometries. We prove that a compact Kähler Calabi-Yau manifold bearing a holomorphic Cartan geometry of algebraic type admits a finite unramified cover which is a complex torus.

## 1. INTRODUCTION

We study here holomorphic Cartan geometries on compact complex manifolds  $M$ .

Let  $G$  be a complex connected Lie group and  $P \subset G$  a closed complex Lie subgroup. The Lie algebras of  $G$  and  $P$  will be denoted by  $\mathfrak{g}$  and  $\mathfrak{p}$ .

**Definition 1.1.** A holomorphic Cartan geometry  $(B, \omega)$  on  $M$  modeled on  $G/P$  is a holomorphic principal right  $P$ -bundle  $B$  over  $M$  endowed with a holomorphic  $\mathfrak{g}$ -valued one form  $\omega$  satisfying:

- (i)  $\omega_b : T_b M \rightarrow \mathfrak{g}$  is a linear complex isomorphism for all  $b \in B$ .
- (ii) If  $X \in \mathfrak{p}$  and  $X^*$  is the corresponding fundamental vector field on  $B$ , then  $\omega_b(X^*) = X$ , for all  $b \in B$ .
- (iii)  $(R_g)^* \omega = Ad(g^{-1})\omega$ , for all  $g \in P$ , where  $R_g$  is the right action on  $B$  of  $g \in P$ .

If the image of  $P$  through the adjoint representation is an algebraic subgroup of  $Aut(\mathfrak{g})$ , the Cartan geometry is said to be of *algebraic type*.

Recall that a local Killing field of the Cartan geometry is a local holomorphic vector field on  $M$  which lifts to a vector field on  $B$  acting by bundle automorphisms and preserving  $\omega$ . Denote by  $Kill^{loc}$  the Lie algebra of local Killing fields (recall that the sheaf of Killing fields on  $M$  is locally constant [8] (section 3.5)). If  $Kill^{loc}$  admits an open orbit  $U$  in  $M$ , we say that  $(B, \omega)$  is *locally homogeneous* on  $U$ .

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We show that, in many situations, a Cartan geometry of algebraic type admits a non trivial algebra  $Kill^{loc}$  of local Killing fields. This is inspired by the celebrated stratification theorem proved by Gromov in the context of *rigid geometric structures of algebraic type* [6, 8] (see also the nice Cartan geometries-adapted proof given by Karin Melnick [14]). Our result is:

**Theorem 1.2.** *Let  $M$  be a compact connected complex manifold of dimension  $n$  endowed with a holomorphic Cartan geometry  $(B, \omega)$  of algebraic type. Then:*

(i) *There exists a compact analytic subset  $S$  of  $M$  of positive codimension, such that  $M \setminus S$  is  $Kill^{loc}$ -invariant and the orbits of  $Kill^{loc}$  in  $M \setminus S$  are the fibers of a holomorphic fibration of constant rank.*

(ii) *For any distinct fibers of the previous fibration there exists a fibration-invariant meromorphic function on  $M$  taking distinct values on them. Consequently, the dimension of the fibers is  $\geq n - a(M)$ , where  $a(M)$  is the algebraic dimension of  $M$ .*

**Corollary 1.3.** *If  $a(M) = 0$ , then  $(B, \omega)$  is locally homogeneous on an open dense set in  $M$ .*

**Corollary 1.4.** *Let  $M$  be a compact simply connected complex manifold of positive dimension with trivial canonical bundle which doesn't admit nonconstant meromorphic functions. Then  $M$  doesn't admit holomorphic Cartan geometries of algebraic type.*

This enables us to prove the following result which was our main motivation:

**Theorem 1.5.** *A compact Kähler Calabi-Yau manifold  $M$  bearing a holomorphic Cartan geometry of algebraic type admits a finite unramified cover which is a complex torus.*

Benjamin McKay conjectured in [13] that compact Kähler Calabi-Yau manifolds bearing holomorphic Cartan geometries are holomorphically covered by complex tori. Theorem 1.5 answers positively this conjecture in the case of Cartan geometries of algebraic type. In a recent paper [4], Indranil Biswas and Benjamin McKay proved the conjecture in the case where  $M$  is a *projective* Calabi-Yau manifold.

Note that the particular case where  $P$  is the complex linear group  $GL(n, \mathbb{C})$  sitting in the complex affine group  $G = GL(n, \mathbb{C}) \ltimes \mathbb{C}^n$  corresponds to a holomorphic affine connexion on the holomorphic tangent bundle  $TM$ . This special case was proved in [10]. The conjecture was also solved in [13] for the particular case where  $P$  is parabolic or reductive.

A similar result was proved in [7] for holomorphic rigid geometric structures of algebraic affine type in Gromov's sense [6, 8], but the context here is different since the principal bundle  $B$  of a Cartan geometry is not supposed to be a frame bundle of the manifold  $M$ .

We give now the main steps in the proof of theorem 1.5 in the case where  $M$  is a nonprojective Calabi-Yau manifold. If  $M$  is Kähler but nonprojective, a result of Moishezon [15] implies that the algebraic dimension of  $M$  is not maximal and theorem 1.2 implies that any Cartan geometry on  $M$  admits nontrivial local Killing fields. We use then a structure theorem which asserts

that, up to a finite cover,  $M$  is biholomorphic to a direct product of a simply connected Calabi-Yau manifold with a complex torus [2] and a result of Amores-Nomizu [1, 16] about the extendibility of local Killing fields on simply connected manifolds (see also [6, 8, 14]).

In the case where  $M$  is projective our proof works also for Cartan geometries which are not of algebraic type. It is a very simplified version of that given in [4]. Actually, we obtain the more general:

**Theorem 1.6.** *Let  $N$  be a complex manifold.*

(i) *If  $M$  is a compact simply connected Kähler Calabi-Yau manifold of positive dimension, then  $M \times N$  doesn't admit holomorphic Cartan geometries of algebraic type.*

(ii) *If  $M$  is a compact simply connected projective Calabi-Yau manifold of positive dimension, then  $M \times N$  doesn't admit holomorphic Cartan geometries.*

(iii) *If  $M$  is a compact simply connected complex manifold of positive dimension with trivial canonical bundle  $K_M$  (or such that  $K_M^{-1}$  doesn't admit nontrivial holomorphic sections), then  $M \times N$  doesn't admit flat holomorphic Cartan geometries.*

*Moreover, the previous three points stand not only for  $M \times N$ , but also for complex manifolds  $W$  containing  $M$  as a submanifold such that there exists a holomorphic vector bundle morphism from  $TW|_M$  onto  $TM$ .*

After finishing this paper, we learnt from Benjamin McKay that he and Indranil Biswas just succeeded to adapt the proof of [4] for compact Kähler Calabi-Yau manifolds. Their new unpublished result uses Bogomolov's  $T$ -stability theory for coherent sheaves.

## 2. KILLING FIELDS OF CARTAN GEOMETRIES

Let  $M$  be a complex manifold endowed with a Cartan geometry of algebraic type modeled on  $G/P$ .

We can assume without loss of generality that  $P$  contains no nontrivial normal subgroups of  $G$ . Indeed, if a nontrivial normal subgroup  $N$  of  $G$  lies in  $P$ , then  $M$  also admits a Cartan geometry  $(B/N, \omega')$  locally modeled on  $G'/P'$ , where  $G' = G/N$  and  $P' = P/N$  (see [18], chapter 4).

Remark that  $\omega$  defines a holomorphic isomorphism  $TB \simeq B \times \mathfrak{g}$ , where  $TB$  is the holomorphic tangent bundle to  $B$ .

The curvature of the Cartan geometry  $(B, \omega)$  is a  $\mathfrak{g}$ -valued (holomorphic) 2-form on  $B$  defined by  $\Omega(X, Y) = d\omega(X, Y) + [\omega(X), \omega(Y)]$ , for all tangent vector fields  $X, Y$  to  $B$ . It is well known that  $\Omega(X, Y)$  vanishes if  $X \in \mathfrak{p}$  (see [18], chapter 5, corollary 3.10).

The Cartan geometry  $(B, \omega)$  is said to be *flat*, if the curvature vanishes on all of  $B$ .

Since  $TB \simeq B \times \mathfrak{g}$ , the curvature  $\Omega$  is completely determined by a  $P$ -equivariant function  $K : B \rightarrow V$ , where  $V = \wedge^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$  and  $P$  acts linearly on  $V$  by

$$p \cdot l(u, v) = (Ad(p) \circ l)(\bar{A}d(p^{-1})u, \bar{A}d(p^{-1})v),$$

for all  $p \in P$ , with  $\bar{Ad}$  being the induced  $P$ -action on  $\mathfrak{g}/\mathfrak{p}$  coming from the adjoint action  $Ad(P)$  (see [18], chapter 5, lemma 3.23).

Following [14] we define for all  $m \in \mathbb{N}$ , the  $m$ -jet of  $K$  with respect to  $\omega$ :

$$J^m K : B \rightarrow Hom(\otimes^m \mathfrak{g}, V)$$

$$(J^m K)(b) : X_1 \otimes X_2 \otimes \dots \otimes X_m \rightarrow (\tilde{X}_1 \cdot \tilde{X}_2 \cdot \dots \cdot \tilde{X}_m \cdot K)(b),$$

where  $X_i \in T_b B$  and  $\tilde{X}_i$  is the unique  $\omega$ -constant (holomorphic) vector field on  $B$  which extends  $X_i$ .

The  $m$ -jet of  $K$  is  $P$ -equivariant:

$$J^m K(bp^{-1}) = p \cdot (J^m K(b) \circ Ad^m p^{-1}),$$

where  $Ad^m$  is the tensor representation  $\otimes^m \mathfrak{g}$  of  $AdP$  (see proposition 3.1 in [14]).

**Definition 2.1.** A (local) automorphism of a Cartan geometry  $(B, \omega)$  on  $M$  is a (local) biholomorphism  $f$  of  $M$  which lifts to a (local) bundle automorphism of  $B$  preserving  $\omega$ .

Conversely, a local bundle automorphism of  $B$  preserving  $\omega$  is the lift of a unique local biholomorphism of  $M$  [14] (proposition 3.6).

The pseudogroup of local automorphisms of a Cartan geometry is a Lie pseudogroup  $Is^{loc}$  (of finite dimension) generated by the Killing Lie algebra of the Cartan geometry  $Kill^{loc}$ . We will say that  $m, n \in M$  are in the same  $Kill^{loc}$ -orbit if  $n$  can be reached from  $m$  by flowing along a finite sequence of local Killing fields. Locally the orbits of  $Is^{loc}$  and the orbits of  $Kill^{loc}$  are the same.

The following well known lemma will be useful in the sequel (see also [18], Appendix A).

**Lemma 2.2.** *Let  $M$  be a complex manifold endowed with a holomorphic Cartan geometry  $(B, \omega)$ . The holomorphic  $\mathfrak{g}$ -bundle  $B_{\mathfrak{g}}$  over  $M$  corresponding to  $B$  by the adjoint action of the structure group  $P$  on  $\mathfrak{g}$  admits a holomorphic connection.*

*Proof.* It is enough to prove that the holomorphic principal  $G$ -bundle  $B_G = B \times_P G$  obtained by extending the structure group of  $B$  using the inclusion map of  $P$  in  $G$  admits a holomorphic connection [11].

Consider  $\omega_{MC} : TG \rightarrow G \times \mathfrak{g}$  be the  $\mathfrak{g}$ -valued Maurer-Cartan one form on  $G$  constructed using the left invariant vector fields. Consider the  $\mathfrak{g}$ -valued holomorphic one form

$$\tilde{\omega}(b, g) = Ad(g^{-1})\pi_1^* \omega + \pi_2^* \omega_{MC},$$

on  $B \times G$ , where the  $\pi_i$  are the canonical projections on the two factors. The one form  $\tilde{\omega}$  descends on  $B_G$  to a  $\mathfrak{g}$ -valued holomorphic one form which defines a holomorphic connection.  $\square$

**Remark 2.3.** (i) *The proof of Lemma 2.2 only requires points (ii) and (iii) in the definition of a Cartan geometry.*

(ii) *If the Cartan geometry  $(B, \omega)$  is flat, then the holomorphic connection constructed in Lemma 2.2 is easily seen to be flat.*

## 3. CARTAN GEOMETRIES AND ALGEBRAIC DIMENSION

The maximal number of algebraically independent meromorphic functions on a complex manifold  $M$  is called *the algebraic dimension*  $a(M)$  of  $M$ .

Recall that a theorem of Siegel proves that a complex  $n$ -manifold  $M$  admits at most  $n$  algebraically independent meromorphic functions [19]. Then  $a(M) \in \{0, 1, \dots, n\}$  and for algebraic manifolds  $a(M) = n$ .

We will say that two points in  $M$  are in the same *fiber of the algebraic reduction* of  $M$  if any meromorphic function on  $M$  takes the same value at the two points. There exists some open dense set in  $M$  where the fibers of the algebraic reduction are the fibers of a holomorphic fibration on an algebraic manifold of dimension  $a(M)$  and any meromorphic function on  $M$  is the pull-back of a meromorphic function on the basis [19].

Theorem 1.2 shows that each fiber of the algebraic reduction lies in an orbit of the pseudogroup of local isometries for any holomorphic Cartan geometry of algebraic type on  $M$ . Let's give the proof.

*Proof.* (i) For each positive integer  $m$  we consider the  $m$ -jet  $J^m K$  of the curvature of  $(B, \omega)$ . This is a  $P$ -equivariant holomorphic map

$$J^m K : P \rightarrow W = \text{Hom}(\otimes^m \mathfrak{g}, V).$$

The proof of theorem 4.1 in [14] shows that two points in  $M$  are in the same  $\text{Kill}^{loc}$ -orbit if and only if the corresponding fibers of  $B$  are sent on the same  $P$ -orbit in  $W = \text{Hom}(\otimes^m \mathfrak{g}, V)$ , for a certain  $m$  large enough.

Since the  $\text{Ad}(P)$ -action on  $W$  is supposed to be algebraic, Rosenlicht's theorem (see [17]) shows that there exists a  $P$ -invariant stratification

$$W = Z_0 \supset \dots \supset Z_l,$$

such that  $Z_{i+1}$  is Zariski closed in  $Z_i$ , the quotient of  $Z_i \setminus Z_{i+1}$  by  $P$  is a complex manifold and rational  $P$ -invariant functions on  $Z_i$  separate orbits in  $Z_i \setminus Z_{i+1}$ .

Consider the open dense  $\text{Kill}^{loc}$ -invariant subset  $U$  of  $M$ , where  $J^m K$  is of constant rank and the image of  $B|_U$  through  $J^m K$  is contained in the maximal subset  $Z_i \setminus Z_{i+1}$  of the stratification which intersects the image of  $J^m K$ . Then  $U = M \setminus S$ , with  $S$  a compact analytic subset in  $M$ , and the orbits of  $\text{Kill}^{loc}$  in  $U$  are the fibers of a fibration of constant rank.

(ii) If  $n$  and  $n'$  are two points in  $U$  which are not in the same  $\text{Kill}^{loc}$ -orbit, then the corresponding fibers of  $B|_U$  are sent by  $J^m K$  on two distinct  $P$ -orbits in  $Z_i \setminus Z_{i+1}$ . By Rosenlicht's theorem there exists a  $P$ -invariant rational function  $F : Z_i \setminus Z_{i+1} \rightarrow \mathbb{C}$ , which takes distinct values at these two orbits.

The meromorphic function  $F \circ J^m K : B \rightarrow \mathbb{C}$  is  $P$ -invariant and descends in a  $\text{Kill}^{loc}$ -invariant meromorphic function on  $M$  which takes distinct values at  $n$  and at  $n'$ .

Consequently, the complex codimension in  $U$  of the  $\text{Kill}^{loc}$ -orbits is  $\leq a(M)$ , which finishes the proof.  $\square$

## 4. CARTAN GEOMETRIES AND SIMPLY-CONNECTED MANIFOLDS

We prove first the corollary 1.4.

*Proof.* Assume, by contradiction, that the complex manifold  $M$  bearing the Cartan geometry  $(B, \omega)$  verifies the hypothesis. Since  $a(M) = 0$ , theorem 1.2 implies  $(B, \omega)$  is locally homogeneous on an open dense set  $U$  in  $M$ . As  $M$  is simply connected, elements in the Killing Lie algebra  $\mathcal{G}$  extend to all of  $M$  [1, 8, 14, 16]: the unique connected simply connected complex Lie group  $G'$  associated to  $\mathcal{G}$  acts isometrically on  $M$  with an open dense orbit. The open dense orbit  $U$  identifies with a homogeneous space  $G'/H$ , where  $H$  is a closed subgroup of  $G'$ .

Consider  $X_1, X_2, \dots, X_n$  global Killing fields on  $M$  which are linearly independent at some point of the open orbit  $U$ . Consider the function  $\text{vol}(X_1, X_2, \dots, X_n)$ , where  $\text{vol}$  is the holomorphic volume form associated to a nontrivial section of the canonical bundle. Since  $\text{vol}(X_1, X_2, \dots, X_n)$  is a holomorphic function on  $M$ , it is a nonzero constant (by maximum principle) and, consequently,  $X_1, X_2, \dots, X_n$  are linearly independent on  $M$ . Hence Wang's theorem [20] implies that  $M$  is a quotient of a  $n$ -dimensional connected simply connected complex Lie group  $G_1$  by a discrete subgroup. Since  $M$  is simply connected, this discrete subgroup has to be trivial and  $M$  identifies with  $G_1$ . But there is no compact simply connected complex Lie group: a contradiction.  $\square$

**Theorem 4.1.** *Let  $M$  be a compact connected simply connected complex  $n$ -manifold without nonconstant meromorphic functions and admitting a holomorphic Cartan geometry  $(B, \omega)$  of algebraic type. Then  $M$  is biholomorphic to an equivariant compactification of  $\Gamma \backslash G'$ , where  $\Gamma$  is a discrete noncocompact subgroup in a complex Lie group  $G'$ .*

*Proof.* Since  $a(M) = 0$ , theorem 1.2 implies  $(B, \omega)$  is locally homogeneous on an open dense set  $U$ . As before, the extension property of local Killing fields implies  $U$  is a complex homogeneous space  $G'/H$ , where  $G'$  is a connected simply connected complex Lie group and  $H$  is a closed subgroup in  $G'$ .

We show now that  $H$  is a discrete subgroup of  $G'$ . Assume by contradiction the Lie algebra of  $H$  is nontrivial. Take at any point  $u \in U$ , the isotropy subalgebra  $\mathcal{H}_u$  (i.e. the Lie subalgebra of Killing fields vanishing at  $u$ ). Remark that  $\mathcal{H}_{gu} = \text{Ad}(g)\mathcal{H}_u$ , for any  $g \in G'$  and  $u \in U$ , where  $\text{Ad}$  is the adjoint representation. Let  $d$  be the complex dimension of  $\mathcal{H}_u$ .

The map  $u \rightarrow \mathcal{H}_u$  is a meromorphic map from  $M$  to the Grassmannian of  $d$ -dimensional vector spaces in  $\mathcal{G}$  (see [19], chapter 2). Since  $M$  doesn't admit nontrivial meromorphic function, this map has to be constant. It follows that  $\mathcal{H}_u$  is  $\text{Ad}(G')$ -invariant and  $H$  is a normal subgroup of  $G'$ : a contradiction, since the  $G'$ -action on  $M$  is faithful. Thus  $G'$  is of dimension  $n$  and  $H$  is a discrete subgroup  $\Gamma$  in  $G'$ .

As  $M$  is simply connected,  $U$  has to be strictly contained in  $M$  and  $M$  is an equivariant compactification of  $\Gamma \backslash G'$ .  $\square$

We don't know if such compactifications of  $\Gamma \backslash G'$  admit holomorphic Cartan geometries, but the previous result has the following application.

Recall that an open question asks whether the 6-dimensional real sphere  $S^6$  admits complex structures. In this context, we have the following:

**Corollary 4.2.** *If  $S^6$  admits a complex structure  $M$ , then  $M$  doesn't admit holomorphic Cartan geometries of algebraic type.*

*Proof.* The starting point of the proof is a result of [5] where it is proved that  $M$  doesn't admit nonconstant meromorphic functions. If  $M$  admits a holomorphic Cartan geometry, then the previous proof shows that  $M$  supports a holomorphic action of a three-dimensional complex Lie group  $G'$  with an open orbit. This is in contradiction with the main theorem of [9].  $\square$

## 5. CARTAN GEOMETRIES AND CALABI-YAU MANIFOLDS

Recall that Kähler Calabi-Yau manifolds are Kähler manifolds with vanishing first (real) Chern class [2].

The aim of this section is to prove theorem 1.5. We settle first the case where  $M$  is simply connected.

**Lemma 5.1.** *A simply connected Kähler Calabi-Yau manifold of positive dimension  $n$  doesn't admit holomorphic Cartan geometries of algebraic type.*

*Proof.* Assume first  $M$  is nonprojective. A theorem of Moishezon [15] shows that the algebraic dimension of a Kähler nonprojective complex manifold of dimension  $n$  is  $\leq n-1$ . Theorem 1.2 implies then that the Killing Lie algebra of a Cartan geometry on  $M$  is nontrivial. Since  $M$  is simply-connected, a nontrivial element of the Killing Lie algebra extends to a global holomorphic (Killing) vector field on  $M$  [1, 8, 14, 16].

But a simply connected compact Calabi-Yau manifold doesn't admit nontrivial holomorphic vector fields [12]: a contradiction.

Consider now the case where  $M$  is projective. Assume by contradiction that  $M$  admits a Cartan geometry  $(B, \omega)$  locally modelled on  $G/P$ .

Let  $B_{\mathfrak{g}}$  (respectively  $B_{\mathfrak{p}}$ ) be the holomorphic vector bundle over  $M$  with fiber  $\mathfrak{g}$  (respectively  $\mathfrak{p}$ ) associated to  $B$  and corresponding to the action of the structure group  $P$  on  $\mathfrak{g}$  (respectively on  $\mathfrak{p}$ ) by the adjoint representation.

The point (i) in the definition of a Cartan geometry implies that the holomorphic tangent bundle  $TM$  is isomorphic to  $B_{\mathfrak{g}}/B_{\mathfrak{p}}$  which is also the holomorphic vector bundle  $B_{\mathfrak{g}/\mathfrak{p}}$  corresponding to the adjoint  $P$ -action on the quotient  $\mathfrak{g}/\mathfrak{p}$ .

Lemma 2.2 shows that  $B_{\mathfrak{g}}$  admits a holomorphic connection. By theorem A(1) in [3],  $B_{\mathfrak{g}}$  also admits a *flat* holomorphic connection. Since  $M$  is simply connected,  $B_{\mathfrak{g}}$  is holomorphically trivial.

Let  $p$  be the canonical projection of  $B_{\mathfrak{g}}$  onto  $TM$ . Choose  $s_1, \dots, s_n$  global holomorphic sections of  $B_{\mathfrak{g}}$  such that  $p(s_1), \dots, p(s_n)$  span  $TM$  at a chosen point in  $M$ . Then  $p(s_1) \wedge \dots \wedge p(s_n)$  is a nontrivial holomorphic section of  $K_M^{-1}$ , where  $K_M$  is the canonical bundle of  $M$ . Since  $K_M$  is trivial, the section  $p(s_1) \wedge \dots \wedge p(s_n)$  doesn't vanish on  $M$ . Consequently,  $p(s_1), \dots, p(s_n)$  trivialize  $TM$ . Wang's theorem [20] implies that  $M$  is a complex torus: a contradiction (for  $M$  is supposed to be simply connected).  $\square$

We give now the proof of theorem 1.5.

*Proof.* Let  $M$  be a Kähler Calabi-Yau manifold bearing a holomorphic Cartan geometry  $(B, \omega)$ . It is known that, up to a finite unramified cover,  $M$  is biholomorphic to a product of nontrivial simply connected Kähler Calabi-Yau manifolds  $M_1, M_2, \dots, M_l$  and a complex torus  $\mathbb{C}^p/\Lambda$ , with  $\Lambda$  being a lattice in  $\mathbb{C}^p$  [2].

Remark that the proof of lemma 5.1 still works if at least one of the simply connected factors  $M_i$  is nonprojective. Indeed, in this case meromorphic functions on  $M$  don't separate points in the fibers  $M_1 \times M_2 \times \dots \times M_l \times \{t\}$ , with  $t \in \mathbb{C}^p/\Lambda$ , and the proof of theorem 1.2 shows that the foliation given by the  $Kill^{loc}$ -orbits intersects a generic fiber, say  $M_1 \times M_2 \times \dots \times M_l \times \{t\}$ , on a foliation with positive dimensional leaves. Hence we can choose a local Killing field  $X$  defined on a connected simply connected open set  $U$  in  $M$  which is tangent to  $M_1 \times M_2 \times \dots \times M_l \times \{t\}$  at a given point  $m \in U \cap (M_1 \times M_2 \times \dots \times M_l \times \{t\})$  and  $X(m) \neq 0$ .

By Amores-Nomizu's extendibility result,  $X$  extends to a holomorphic Killing vector field  $\tilde{X}$  on  $M_1 \times M_2 \times \dots \times M_l \times U'$ , with  $U'$  being a simply connected open set in  $\mathbb{C}^p/\Lambda$  containing  $t$ . Consider the image of  $\tilde{X}$  through the projection on the first factor of the canonical decomposition

$$T(M_1 \times M_2 \times \dots \times M_l \times U') \simeq \pi_1^*T(M_1 \times M_2 \times \dots \times M_l) \oplus \pi_2^*TU'$$

where the  $\pi_i$  are the canonical projections on the simply connected factor and on the torus.

We constructed a holomorphic vector field on the simply connected Calabi-Yau manifold  $M_1 \times M_2 \times \dots \times M_l \times \{t\}$  which doesn't vanish at  $m$ . This is in contradiction with [12] as before.

Consider now the remaining case where all simply connected factors  $M_i$  are projective. Assume, by contradiction, that  $M_1 \times M_2 \times \dots \times M_l$  is non-trivial.

Let  $B_{\mathfrak{g}}$  (respectively  $B_{\mathfrak{p}}$ ) be the holomorphic vector bundle over  $M$  with fiber  $\mathfrak{g}$  (respectively  $\mathfrak{p}$ ) associated to  $B$  and corresponding to the action of the structure group  $P$  on  $\mathfrak{g}$  (respectively on  $\mathfrak{p}$ ) by the adjoint representation. As before  $TM$  is isomorphic to  $B_{\mathfrak{g}}/B_{\mathfrak{p}}$ . Let  $p$  be the canonical projection of  $B_{\mathfrak{g}}$  onto  $TM$ .

Let  $p_1$  (respectively  $p_2$ ) be the projections on the first factor (respectively second factor) of the decomposition

$$TM = \pi_1^*T(M_1 \times M_2 \times \dots \times M_l) \oplus \pi_2^*T(\mathbb{C}^p/\Lambda).$$

Then  $p_1 \circ p : B_{\mathfrak{g}} \rightarrow \pi_1^*T(M_1 \times M_2 \times \dots \times M_l)$  is a surjective morphism of vector bundles. When restricted to a fiber  $M_1 \times M_2 \times \dots \times M_l \times \{t\}$ , this induces a surjective morphism  $\bar{B}_{\mathfrak{g}} \rightarrow T(M_1 \times M_2 \times \dots \times M_l)$ , where  $\bar{B}_{\mathfrak{g}}$  denotes the restriction of  $B_{\mathfrak{g}}$  to  $M_1 \times M_2 \times \dots \times M_l \times \{t\}$ .

On the other hand, Lemma 2.2 shows that  $B_{\mathfrak{g}}$  admits a holomorphic connection. In particular,  $\bar{B}_{\mathfrak{g}}$  admits a holomorphic connection. By theorem A(1) in [3],  $\bar{B}_{\mathfrak{g}}$  also admits a *flat* holomorphic connection. Since  $M_1 \times M_2 \times \dots \times M_l$  is simply connected,  $\bar{B}_{\mathfrak{g}}$  is holomorphically trivial. We obtained that  $T(M_1 \times M_2 \times \dots \times M_l)$  is a quotient of a holomorphically trivial vector bundle. We conclude as in the proof of Lemma 5.1.  $\square$

Remark that, in the case where the simply connected factor  $M_1 \times M_2 \times \dots \times M_l$  is projective, the previous proof also works for Cartan geometries which are not necessarily of algebraic type.

Actually the previous proof doesn't use neither that the second factor of the holomorphic decomposition of a Calabi-Yau manifold as a product is a complex torus. In our proof this second factor might be any complex

manifold  $N$ . This proves also points (i) and (ii) in theorem 1.6. Whereas for point (iii) of theorem 1.6 we give the proof here:

*Proof.* Consider a flat Cartan geometry  $(B, \omega)$  on  $M \times N$  and the associated vector bundle  $B_{\mathfrak{g}}$  over  $M \times N$ . As before  $TM$  is a quotient of the vector bundle  $\bar{B}_{\mathfrak{g}}$ , where  $\bar{B}_{\mathfrak{g}}$  denotes the restriction to the factor  $M$  of  $B_{\mathfrak{g}}$ .

Since the Cartan geometry  $(B, \omega)$  is supposed to be flat, Lemma 2.2 and Remark 2.3 (point (ii)) show that  $B_{\mathfrak{g}}$  and hence also  $\bar{B}_{\mathfrak{g}}$  admit a flat holomorphic connection. As  $M$  is simply connected, the vector bundle  $\bar{B}_{\mathfrak{g}}$  over  $M$  is holomorphically trivial.

As in the proof of Lemma 5.1, let  $p$  be the canonical projection of  $\bar{B}_{\mathfrak{g}}$  onto  $TM$ . Let  $n$  be the complex dimension of  $M$  and choose  $s_1, \dots, s_n$  global holomorphic sections of  $\bar{B}_{\mathfrak{g}}$  such that  $p(s_1), \dots, p(s_n)$  span  $TM$  at a chosen point in  $M$ . Then  $p(s_1) \wedge \dots \wedge p(s_n)$  is a nontrivial holomorphic section of  $K_M^{-1}$ , where  $K_M$  is the canonical bundle of  $M$ . We get a contradiction in the case where  $K_M^{-1}$  doesn't admit nontrivial holomorphic sections.

Consider now the case where  $K_M$  is trivial. Then the section  $p(s_1) \wedge \dots \wedge p(s_n)$  doesn't vanish on  $M$ . Consequently,  $p(s_1), \dots, p(s_n)$  trivialize  $TM$ . Wang's theorem [20] implies that  $M$  is a quotient of a nontrivial connected simply connected complex Lie group  $G_1$  by a discrete subgroup. Since  $M$  is simply connected, this discrete subgroup has to be trivial and  $M$  identifies with  $G_1$ . But there is no nontrivial compact simply connected complex Lie group: a contradiction.

Remark that all over the proof we didn't use the product structure of the manifold  $W = M \times N$ , but only the existence of a holomorphic vector bundle morphism from  $TW|_M$  onto  $TM$ .  $\square$

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\* DÉPARTEMENT DE MATHÉMATIQUES D'ORSAY, ÉQUIPE DE TOPOLOGIE ET DYNAMIQUE,  
BAT. 425, U.M.R. 8628 C.N.R.S., UNIV. PARIS-SUD (11), 91405 ORSAY CEDEX,  
FRANCE

*E-mail address:* Sorin.Dumitrescu@math.u-psud.fr