

Simplicity of $\text{Homeo}(\mathbb{D}^2, \partial\mathbb{D}^2, \text{Area})$ and fragmentation of symplectic diffeomorphisms

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This paper is concerned with the algebraic study of the group

$$G = \text{Homeo}(\mathbb{D}^2, \partial\mathbb{D}^2, \text{Area})$$

of area-preserving homeomorphisms of the 2-disc that are the identity near the boundary. The central open question is the following.

Question 1 ([Fa80]). *Is G a simple group?*

Normal subgroups of G have been defined by Müller and Oh ([MO07]), and Ghys ([Gh07], see [Bo08]), but so far no one has been able to prove that these are proper subgroups: they might turn out to be equal to G . In this text, I propose to define still another family of normal subgroups $\{N_\varphi\}$ of G . I have not been able to prove that these subgroups are proper ; however, we can prove that they are good candidates:

Theorem 1. *If some normal subgroup of the family $\{N_\varphi\}$ is equal to G , then G is simple.*

The present work has its origin in Fathi's paper [Fa80] dealing with higher dimensions, and especially in the proof of the perfectness of the group of volume preserving homeomorphisms of the n -ball which are the identity near the boundary, when $n \geq 3$. Fathi's argument has two steps. The first step is a fragmentation result: any element of the group can be written as a product of two elements which are supported on a topological ball whose volume is $\frac{3}{4}$ of the total volume. The second step shows how this fragmentation result implies the perfectness of the group. While the second step is still valid in dimension 2, the first one fails (see section 6 below). The content of the present paper is to show, by defining the normal subgroups N_φ , that Fathi's second step essentially admits a converse in dimension 2: the perfectness (and simplicity) of our group is equivalent to some fragmentation property. Furthermore it is also equivalent to a similar fragmentation property on the subgroup $G^{\text{diff}} = \text{Diffeo}(\mathbb{D}^2, \partial\mathbb{D}^2, \text{Area})$ consisting of those elements of G that are C^∞ -diffeomorphisms (see theorems 2 and 3). Thus the problem is translated into a problem dealing with Hamiltonian diffeomorphisms. Our fragmentation problem is actually a family of problems depending on a parameter $\rho \in (0, 1]$. We will see in section 6 that Entov-Polterovich quasi-morphisms, coming from Floer homology, immediately give the solution for $\rho \in (\frac{1}{2}, 1]$. The problem remains unsolved for the other values of ρ .

1 A family of bi-invariant fragmentation metrics on G

The definition of the fragmentation metric is not new, see example 1.24 in [BIP07].

The group G is endowed with the topology of uniform convergence (also called C^0 topology) that turns it into a topological group. We recall that G is arcwise connected: an elementary proof is provided by the famous Alexander trick. We will use the term *topological disc* to denote any image of a euclidean closed disc under an element of the group G . As a consequence of the classical theorems by Schönflies and Oxtoby-Ulam, any Jordan curve of null area bounds a topological disc (see [OU41]). Remember that the *support* of some $g \in G$ is the closure of the set of non-fixed points. For any topological disc D we denote by G_D the subgroup of G consisting of the elements whose support is included in the interior of D . Note that each group G_D is isomorphic to G , as we see by “re-scaling”. More precisely by definition there exists some $\Phi \in G$ such that $D = \Phi^{-1}(D_0)$ where D_0 is a euclidean disc, and the map $g \mapsto \Phi g \Phi^{-1}$ provides an isomorphism between the groups G_{D_0} and G_D ; we furthermore choose an homothety Ψ that sends the whole disc \mathbb{D}^2 onto D_0 , and similarly get an isomorphism $g \mapsto \Psi g \Psi^{-1}$ between G and G_{D_0} .

For any parameter $\rho \in (0, 1]$ we consider the set

$$G_\rho = \bigcup \{G_D \text{ such that } D \text{ is a topological disc of area } \rho\}.$$

In other words, some $g \in G$ belongs to G_ρ if and only if the connected component of the interior of the fixed point set of g that contains $\partial\mathbb{D}^2$ has area greater than $1 - \rho$. Let us emphasise the importance of the word *disc*: an element g which is supported on an annulus of small area surrounding a disc of area bigger than ρ is *not* in G_ρ .

The following proposition says that any element can be fragmented into elements having arbitrarily small support, and that this fragmentation is “locally uniform”.

Proposition 1.1. *For any $\rho \in (0, 1]$ the set G_ρ generates G . Moreover, there exists a neighbourhood \mathcal{V}_ρ of the identity in G such that any g in \mathcal{V}_ρ can be written as a product of less than $\frac{2}{\rho}$ elements of G_ρ .*

The proof is essentially the proof of lemma 6.5 in [Fa80]. It is also similar to the proof of proposition 4.1 below (see section 4), which is the smooth analog.

When H is a subset of G we denote by $(H)^m$ the set of products of m elements of H . For any $g \in G$, let us define the number $\varphi_g(\rho)$ as the least number of factors in a fragmentation of g by elements of G_ρ :

$$\varphi_g(\rho) := \min\{m, g \in (G_\rho)^m\}.$$

The set G_ρ is clearly invariant under conjugacy in G . Thus the following properties are obvious.

Proposition 1.2.

$$\begin{aligned} \varphi_{hgh^{-1}}(\rho) &= \varphi_g(\rho), \\ \varphi_{g_1 g_2}(\rho) &\leq \varphi_{g_1}(\rho) + \varphi_{g_2}(\rho), \\ \varphi_{g^{-1}} &= \varphi_g. \end{aligned}$$

Corollary 1.3. *The formula*

$$d_\rho(g_1, g_2) = \varphi_{g_1 g_2^{-1}}(\rho)$$

defines a bi-invariant metric on G .

It is natural to ask whether the locally uniform bound in proposition 1.1 can be extended, namely if there exists a universal bound, depending on ρ , in the number of factors in the fragmentation by elements of G_ρ . Here is a more general (and a priori less natural) problem.

Question 2. *Does there exist some numbers ρ, ρ' with $0 < \rho' < \rho \leq 1$ and some positive integer m such that every element of G_ρ can be written as a product of m elements of $G_{\rho'}$?*

The following result shows that the simplicity problem given in question 1 is equivalent to question 2.

Theorem 2. *The group G is simple if and only if there exists some ρ, ρ' with $0 < \rho' < \rho \leq 1$ and some positive integer m such that*

$$G_\rho \subset (G_{\rho'})^m.$$

Here are some easy remarks. Let us denote by $P(\rho, \rho')$ the property that there exists some positive integer m with $G_\rho \subset (G_{\rho'})^m$. Assuming property $P(\rho, \rho')$, by re-scaling we get for any $\rho'' < \rho$

$$G_{\rho''} \subset (G_{k\rho''})^m$$

where k satisfies $\rho' = k\rho$. In particular we can iterate the fragmentation to get, for every positive n ,

$$G_\rho \subset (G_{k^n \rho})^{(m^n)}.$$

This shows that property $P(\rho, \rho'_0)$ implies property $P(\rho, \rho'_1)$ for every $\rho'_1 < \rho'_0$. The converse is clearly true, so that property $P(\rho, \rho')$ depends only on ρ and not on ρ' . In particular it is equivalent to the following property $P(\rho)$:

There exists some positive integer m such that $G_\rho \subset \left(G_{\frac{\rho}{2}}\right)^m$.

Also note that property $P(\rho_0)$ implies property $P(\rho_1)$ if $\rho_1 < \rho_0$ (again by re-scaling).

The theorem is proved in section 3. We will see in section 4 that this is still equivalent to the analog fragmentation problem on the diffeomorphisms group G^{diff} .

2 Normal subgroups

Here we define the family of normal subgroups N_φ . Given some element $g \in G$, we consider the function

$$\rho \mapsto d_\rho(\text{Id}, g) = \varphi_g(\rho).$$

We call it the *complexity profile* of g . Let $\varphi : (0, 1] \rightarrow \mathbb{R}^+$ be any non-increasing function. We define the subset N_φ containing those elements of G whose complexity profile is essentially bounded by φ :

$$N_\varphi := \{g \in G, \varphi_g = O(\varphi)\}$$

where the notation $\psi = O(\varphi)$ means that there exists some $K > 0$ such that $\psi(\rho) < K\varphi(\rho)$ for every small enough ρ . The following is an immediate consequence of proposition 1.2.

Proposition 2.1. *For any non-increasing function $\varphi : (0, 1] \rightarrow \mathbb{R}^+$, the set N_φ is a normal subgroup of G .*

3 Fragmentation and simplicity

In this section we prove theorem 2. The proof will also imply theorem 1.

We first assume that question 2 admits a negative answer:

- (\star) for every $\rho \in (0, 1]$ and every positive integer m there exists some element g in G_ρ such that $d_{\frac{\rho}{2}}(g, \text{Id}) > m$.

We fix any function $\varphi : (0, 1] \rightarrow \mathbb{R}^+$, and we will construct some element g in G that is not included in N_φ . Applying this for the function $\varphi = \varphi_f$ for any $f \in G$ shows that the normal subgroup N_{φ_f} , which contains f , is different from G . This will prove the “only if” part of theorem 2.

Let us explain the construction of g . We define $D_0 = \mathbb{D}^2$. We first pick two sequences of discs $(C_i)_{i \geq 1}$ and $(D_i)_{i \geq 1}$ converging to a point, such that for every i ,

- C_i and D_i are disjoint and included in D_{i-1} ,
- the area of D_i is less than half the area of C_i .

We denote the area of C_i by ρ_i . We will construct a sequence $(g_i)_{i \geq 1}$, with each g_i supported in the interior of C_i , and define g as the (infinite) product of the g_i 's. Note that since the discs C_i 's are pairwise disjoint this product has a meaning, and since the sequence (C_i) converges to a point it actually defines an element of G . Since all the g_j 's with $j > i$ will be supported in the interior of the disc D_i whose area is less than $\rho_i/2$ we will get

$$d_{\frac{\rho_i}{2}}(g, \text{Id}) \geq d_{\frac{\rho_i}{2}}(g_i \dots g_1, \text{Id}) - 1. \quad (1)$$

The sequence (g_i) is constructed by induction. Assume g_1, \dots, g_{i-1} have been constructed. Using hypothesis (\star), we may choose g_i such that $d_{\frac{\rho_i}{2}}(g_i, \text{Id})$ is arbitrarily high, more precisely we demand the following inequality:

$$d_{\frac{\rho_i}{2}}(g_i, \text{Id}) \geq \frac{1}{\rho_i} \varphi\left(\frac{\rho_i}{2}\right) + d_{\frac{\rho_i}{2}}(g_{i-1} \dots g_1, \text{Id}) + 1. \quad (2)$$

Using inequality (1), the triangular inequality and inequality (2) we get

$$\begin{aligned} \varphi_g\left(\frac{\rho_i}{2}\right) = d_{\frac{\rho_i}{2}}(g, \text{Id}) &\geq d_{\frac{\rho_i}{2}}(g_i \dots g_1, \text{Id}) - 1 \\ &\geq d_{\frac{\rho_i}{2}}(g_i, \text{Id}) - d_{\frac{\rho_i}{2}}(g_{i-1} \dots g_1, \text{Id}) - 1 \\ &\geq \frac{1}{\rho_i} \varphi\left(\frac{\rho_i}{2}\right). \end{aligned}$$

This proves that the complexity profile of g is not equal to $O(\varphi)$. In other words g does not belong to N_φ .

We now turn to the “if” part of theorem 2. This is just a slight generalisation of Fathi’s argument showing that a universal bound implies the perfectness of G (remember that due to “Thurston’s trick”, perfectness implies simplicity, for completeness the argument is written below).

We assume that there exists a number $\rho \in (0, 1]$ and a positive integer m such that

$$G_\rho \subset (G_{\frac{\rho}{2}})^m$$

and we call this assumption the “fragmentation hypothesis”. Let C_1 be a small disc. By usual fragmentation (proposition 1.1), any element of G is a product of elements supported in a disc of area less than that of C_1 , and any such element is conjugate to an element supported in the interior of C_1 . Thus to prove perfectness it is enough to consider some element g supported in the interior of C_1 and to prove that g is a product of commutators.

Let us first prove that such a g is a product of two commutators when considered in the group $\text{Homeo}(\mathbb{D}^2, \partial\mathbb{D}^2)$, that is, let us forget for a while about the area (this is a “pedagogical” step). Choose some sequence of disjoint discs $(C_i)_{i \geq 1}$ converging to a point, and another sequence $(D_i)_{i \geq 1}$ such that

- the interior of D_i contains both C_i and C_{i+1} ,
- the D_{2i} ’s (resp. the D_{2i+1} ’s) are pairwise disjoint.

For any $i \geq 1$ choose some $h_i \in \text{Homeo}(\mathbb{D}^2, \partial\mathbb{D}^2)$, supported on D_i , that sends C_i onto C_{i+1} . We let $g_1 := g$, thus g_1 is supported on C_1 , and define inductively $g_{i+1} := h_i g_i h_i^{-1}$; thus g_i is a copy of g supported on C_i and the g_i ’s are pairwise commuting. Let

$$K := g_2 g_3^{-1} g_4 g_5^{-1} \cdots, \quad K' := g_1 g_2^{-1} g_3 g_4^{-1} \cdots$$

so that $KK' = K'K = g$. The map $K = [g_2, h_2][g_4, h_4] \cdots$ may be seen as an infinite product of commutators, but we need a finite product. Now define

$$G := g_2 g_4 \cdots, \quad H := h_2 h_4 \cdots, \quad G' := g_1 g_3 \cdots, \quad H' := h_1 h_3 \cdots$$

and observe that $K = [G, H]$ and $K' = [G', H']$: indeed these equalities may be checked independently on each disc D_i . Thus $g = [G, H][G', H']$ is a product of two commutators in $\text{Homeo}(\mathbb{D}^2, \partial\mathbb{D}^2)$.

Now let us take care of the area. We will use sequences (C_i) and (D_i) as before, and we will get around the impossibility of shrinking C_i onto C_{i+1} inside the group G by using the fragmentation hypothesis. We may assume, for every i , the equality

$$\text{Area}(C_{i+1}) = \frac{1}{2} \text{Area}(C_i).$$

Moreover, by fragmentation, we may assume this time that g is supported in the interior of a disc $C'_1 \subset C_1$ of area $\rho \text{Area}(C_1)$. We use the fragmentation hypothesis re-scaled on C_1 to write

$$g = f_{1,1} \cdots f_{1,m}$$

with each $f_{1,j}$ supported in the interior of a disc included in C_1 and whose area is

$$\frac{\rho}{2} \text{Area}(C_1) = \rho \text{Area}(C_2).$$

We choose a disc $C'_2 \subset C_2$ whose area is $\rho \text{Area}(C_2)$ and, for each $j = 1, \dots, m$, some $h_{1,j} \in G$ supported on D_1 and sending the support of $f_{1,j}$ inside C'_2 . We define

$$g_2 := \prod_{j=1, \dots, m} h_{1,j} f_{1,j} h_{1,j}^{-1}$$

which is supported on C'_2 . We apply recursively the (re-scaled) fragmentation hypothesis to get a sequence $(g_i)_{i \geq 1}$ with each g_i supported in the interior of a disc $C'_i \subset C_i$ having area $\rho \text{Area}(C_i)$ and sequences $(f_{i,j})_{j=1, \dots, m, i \geq 1}$ and $(h_{i,j})_{i \geq 1, j=1, \dots, m}$ with $f_{i,j}$ supported on C_i and $h_{i,j}$ supported on D_i , such that

$$g_i = \prod_{j=1, \dots, m} f_{i,j} \text{ and } g_{i+1} = \prod_{j=1, \dots, m} h_{i,j} f_{i,j} h_{i,j}^{-1}.$$

Obviously g_i and g_{i+1} are equal up to a product of commutators. More precisely, we have

$$\begin{aligned} g_i g_{i+1}^{-1} &= \prod_{j=1, \dots, m} f_{i,j} \prod_{j=m, \dots, 1} h_{i,j} f_{i,j}^{-1} h_{i,j}^{-1} \\ &= [f_{i,1}, P] \left(\prod_{j=2, \dots, m} f_{i,j} \prod_{j=m, \dots, 2} h_{i,j} f_{i,j}^{-1} h_{i,j}^{-1} \right) [f_{i,1}, h_{i,1}] \end{aligned}$$

where P is equal to the term between parentheses, and we see recursively that $g_i g_{i+1}^{-1}$ is a product of $2m$ commutators of elements supported in D_i ; we write

$$g_i g_{i+1}^{-1} = \prod_{j=1, \dots, 2m} [s_{i,j}, t_{i,j}].$$

It remains to define the infinite commutative products

$$K := g_2 g_3^{-1} g_4 g_5^{-1} \dots, \quad K' := g_1 g_2^{-1} g_3 g_4^{-1} \dots$$

$$S_j := s_{2,j} s_{4,j} \dots, \quad T_j := t_{2,j} t_{4,j} \dots, \quad S'_j := s_{1,j} s_{3,j} \dots, \quad T'_j := t_{1,j} t_{3,j} \dots$$

and to check that

$$g = K K' = \prod_{j=1, \dots, 2m} [S_j, T_j] \prod_{j=1, \dots, 2m} [S'_j, T'_j]$$

is a product of $4m$ commutators. This proves that G is perfect.

Let us recall briefly, according to Thurston, how perfectness implies simplicity. Let $g, h \in G$ be such that the discs $D, g(D), h(D)$ are pairwise disjoint. Let $u, v \in G$ be supported in D . In this situation the identity

$$[u, v] = [[u, g], [v, h]]$$

may easily be checked, and shows that $[u, v]$ belongs to the normal subgroup generated by g . Now given any $g \neq \text{Id}$ in G , one can find an $h \in G$ and a disc D such that the above situation takes place. If G is perfect then so is the isomorphic group G_D , hence every f supported in D is a product of commutators supported in D , and by the above equality such an f belongs to the normal subgroup generated by g . By fragmentation this subgroup is thus equal to G . This proves that G is simple, and completes the proof of theorem 2.

Proof of theorem 1. If some normal subgroup N_φ is equal to G , then by the first part of the proof of theorem 2 we find some $\rho \in (0, 1]$ and some m such that $G_\rho \subset (G_\rho^{\text{diff}})^m$, and by the second part this in turn implies that G is simple. \square

The following is an attempt to express that if G is not simple, then it has to contain “lots of” normal subgroups.

Corollary 3.1. *Assume G is not simple. Then every compact subset K of G is included in a proper normal subgroup of G .*

Note that the situation is radically different for the diffeomorphisms group G^{diff} , since (by Banyaga’s theorem [Ba97], and since the centralizer of G^{diff} is trivial) any one-parameter subgroup of diffeomorphisms that is not included in the commutator subgroup $[G^{\text{diff}}, G^{\text{diff}}]$ normally generates G^{diff} . However these are not purely algebraic statements since they involve the topology of the groups G and G^{diff} .

Proof of the corollary. Consider some $\rho \in (0, 1]$. Let \mathcal{V}_ρ be the neighbourhood of the identity given by the fragmentation lemma (proposition 1.1): we have $d_\rho(g, \text{Id}) < \frac{2}{\rho}$ for every $g \in \mathcal{V}_\rho$. By compactness we may find a finite family g_1, \dots, g_k such that the sets $g_i \cdot \mathcal{V}_\rho$ cover K . Thus the fragmentation is also uniform on K , in other words the set K is bounded with respect to the distance d_ρ . Define

$$\varphi(\rho) := \sup \{d_\rho(g, \text{Id}), g \in K\}.$$

This defines a non-increasing function, and clearly K is included in the normal subgroup N_φ . According to the proof of theorem 2, if G is not simple then N_φ is a proper subgroup of G , which completes the proof of the corollary. \square

4 Fragmentation of diffeomorphisms

Here we further translate questions 1 and 2 into the smooth group G^{diff} .

Let $G^{\text{diff}} = \text{Diffeo}(\mathbb{D}^2, \partial\mathbb{D}^2, \text{Area})$ be the group of elements of G that are C^∞ -diffeomorphisms. For every $\rho \in (0, 1]$ we consider the set

$$G_\rho^{\text{diff}} = G^{\text{diff}} \cap G_\rho.$$

Note that for every topological disc D the group

$$G_D^{\text{diff}} := G^{\text{diff}} \cap G_D$$

is isomorphic to G^{diff} , even if D is not smooth: indeed we may use Riemann conformal mapping theorem and Moser’s lemma to find a smooth diffeomorphism Φ between the interiors of \mathbb{D}^2 and D with constant Jacobian, and the conjugacy by Φ provides an isomorphism.

We also define

$$d_\rho^{\text{diff}}(g_1, g_2) = \min \{m, g_1 g_2^{-1} \in (G_\rho^{\text{diff}})^m\}.$$

Again d_ρ^{diff} is a bi-invariant metric on the group G^{diff} . Clearly we have $d_\rho(g, \text{Id}) \leq d_\rho^{\text{diff}}(g, \text{Id})$ for every $g \in G^{\text{diff}}$. The following question is the smooth analog to question 2.

Question 3. *Does there exist some numbers ρ, ρ' with $0 < \rho' < \rho \leq 1$ and some positive integer m such that every element of G_ρ^{diff} can be written as a product of m elements of $G_{\rho'}^{\text{diff}}$?*

The following theorem says this question is still equivalent to question 2 (and thus also to question 1).

Theorem 3. *For any $\rho \in (0, 1]$, the following properties are equivalent:*

- *there exists some m such that $G_\rho \subset (G_{\frac{\rho}{2}})^m$,*
- *there exists some m such that $G_\rho^{\text{diff}} \subset (G_{\frac{\rho}{2}}^{\text{diff}})^m$.*

The proof of this equivalence requires two ingredients. The first one is the density of G^{diff} in G ; this is a classical result, see [Si]. The second one is the uniformity of the fragmentation in G^{diff} inside some C^0 -neighbourhood of the identity. This is provided by the following proposition which is similar to proposition 1.1 above.

Proposition 4.1. *For any $\rho \in (0, 1)$, there exists in G^{diff} a neighbourhood \mathcal{V}_ρ of the identity in the C^0 -topology such that any g in \mathcal{V}_ρ can be written as a product of less than $\frac{2}{\rho}$ elements of G_ρ^{diff} .*

The classical proof of fragmentation for diffeomorphisms relies on the inverse mapping theorem and would only give uniformity in a C^1 -neighbourhood of the identity (see [Ba97, Bo08]). Thus we will rather try to mimic the proof of the fragmentation for homeomorphisms.

Proof of proposition 4.1. We choose an integer m bigger than $\frac{2}{\rho}$, and we cut the disc into m strips of area less than $\frac{\rho}{2}$: more precisely, we choose m topological discs D_1, \dots, D_m such that

1. $\text{Area}(D_i) \leq \frac{\rho}{2}$,
2. $\mathbb{D}^2 = D_1 \cup \dots \cup D_m$,
3. $D_i \cap D_j = \emptyset$ if $|j - i| > 1$,
4. $D_i \cup D_{i+1} \cup \dots \cup D_j$ is a topological disc for every $i \leq j$, and the intersection of $D_1 \cup \dots \cup D_i$ and $D_i \cup \dots \cup D_m$ with the boundary of \mathbb{D}^2 is non-empty and connected for every i .

Now we define \mathcal{V}_ρ as the set of elements g of G^{diff} such that, for every i, j with $|j - i| > 1$, $g(D_i) \cap D_j = \emptyset$. Note that \mathcal{V}_ρ is a C^0 -neighbourhood of the identity in G^{diff} . We will prove that each element of \mathcal{V}_ρ can be written as a product of $m - 1$ elements of G_ρ^{diff} .

Let $g \in \mathcal{V}_\rho$. By hypothesis D_1 and $g(D_1)$ are disjoint from $D_3 \cup \dots \cup D_m$. By the classical lemma 4.2 below, we can find $\Psi_1 \in G$ such that

- $\Psi_1 = g$ on some neighbourhood of D_1 ,
- Ψ_1 is the identity on some neighbourhood of $D_3 \cup \dots \cup D_m$.

The diffeomorphism Ψ_1 is supported in the interior of the topological disc $D_1 \cup D_2$ whose area is less than or equal to ρ . Let $g_1 := \Psi_1^{-1}g$, thus g_1 is supported in the interior of $D_2 \cup \dots \cup D_m$, and we easily check that this diffeomorphism is still in \mathcal{V}_ρ . In particular $D_1 \cup D_2$ and its image under g_1 are disjoint from $D_4 \cup \dots \cup D_m$. We apply again the lemma to get some $\Psi_2 \in G$ such that

- $\Psi_2 = g_1$ on some neighbourhood of $D_1 \cup D_2$,
- Ψ_2 is the identity on some neighbourhood of $D_4 \cup \dots \cup D_m$.

Thus Ψ_2 is supported in the interior of $D_2 \cup D_3$. Let $g_2 := \Psi_2^{-1}g_1$; this diffeomorphism is in \mathcal{V}_ρ and is supported in the interior of $D_3 \cup \dots \cup D_m$. In the same way we construct diffeomorphisms $\Psi_1, \dots, \Psi_{m-1}$, such that each Ψ_i is supported in the interior of $D_i \cup D_{i+1}$, and such that $g = \Psi_1 \circ \dots \circ \Psi_{m-1}$. This completes the proof. \square

Lemma 4.2. *Let D'_1, D'_2 be two disjoint topological discs in \mathbb{D}^2 , and assume that the intersection of D'_1 (resp. D'_2) with the boundary of $\partial\mathbb{D}^2$ is non-empty and connected (and thus $\mathbb{D}^2 \setminus \text{Int}(D'_1 \cup D'_2)$ is again a topological disc). Let $\Phi \in G^{\text{diff}}$, and suppose that $\Phi(D'_1)$ is disjoint from D'_2 . Then there exists $\Psi \in G^{\text{diff}}$ whose support is disjoint from D'_2 and such that $\Psi = \Phi$ on some neighbourhood of D'_1 .*

Proof of the lemma. By Smale's theorem ([Sm59]) and Moser's lemma, the group G is arcwise connected. Let $(\Phi_t)_{t \in [0,1]}$ be a smooth isotopy from the identity to Φ in G . It is easy to find another smooth isotopy $(g_t)_{t \in [0,1]}$ (that does not preserve area), supported in the interior of \mathbb{D}^2 , such that

- for every t , $g_t(\Phi_t(D'_1))$ is disjoint from D'_2 ,
- $g_0 = g_1 = \text{Id}$.

The isotopy $(\Phi'_t) = (g_t\Phi_t)$ still goes from Id to Φ . Consider the vector field tangent to this isotopy, and multiply it by some smooth function that is equal to 1 on some neighbourhood of $\cup_t \Phi'_t(D'_1)$ and vanishes on D'_2 . By integrating this truncated vector field we get another isotopy (Ψ_t) such that

- on some neighbourhood of D'_1 we have $\Psi_t = g_t\Phi_t$ for every t , and in particular $\Psi_1 = \Phi$,
- the support of Ψ_1 is disjoint from D'_2 .

Thus Ψ_1 satisfies the conclusion of the lemma, except that it does not preserve the area. Let w_1 be the pre-image of the area form w_0 under Ψ_1 . Then $w_1 = w_0$ on some neighbourhood of $\partial\mathbb{D}^2 \cup D'_1 \cup D'_2$. By Moser's lemma we may find some $\Psi_2 \in G$, whose support is disjoint from D'_1 and D'_2 , and that sends w_1 to w_0 . The diffeomorphism $\Psi = \Psi_1\Psi_2^{-1}$ suits our needs. \square

Proof of theorem 3. Assume there exists some $\rho \in (0, 1]$ and some positive integer m such that

$$G_\rho^{\text{diff}} \subset \left(G_{\frac{\rho}{2}}^{\text{diff}}\right)^m.$$

Let $g \in G_\rho$. The set G^{diff} is dense in G , and thus G_ρ^{diff} is also dense in G_ρ . We choose some $g' \in G_\rho^{\text{diff}}$ close to g : according to proposition 1.1, we get

$$d_{\frac{\rho}{2}}(g, g') \leq \frac{2}{\rho}.$$

By hypothesis we have $d_{\frac{\rho}{2}}^{\text{diff}}(g', \text{Id}) \leq m$ and thus also

$$d_{\frac{\rho}{2}}(g', \text{Id}) \leq m.$$

We conclude that $d_{\frac{\rho}{2}}(g, \text{Id}) \leq m + \frac{2}{\rho}$.

Conversely, assume that there exists some $\rho \in (0, 1]$ and some positive integer m such that

$$G_\rho \subset \left(G_{\frac{\rho}{2}}\right)^m.$$

Let $g \in G_\rho^{\text{diff}}$. By hypothesis we can write

$$g = g_1 \cdots g_m$$

with $g_i \in G_{\frac{\rho}{2}}$. By density we choose for each i a diffeomorphism $g'_i \in G_{\frac{\rho}{2}}^{\text{diff}}$ which is C^0 -close to g_i , so that the map

$$g' := g'_1 \cdots g'_m$$

belongs to $\left(G_{\frac{\rho}{2}}^{\text{diff}}\right)^m$ and is C^0 -close to g . According to proposition 4.1 we have $d_{\frac{\rho}{2}}^{\text{diff}}(g', g) \leq \frac{2}{\rho}$, and thus $d_{\frac{\rho}{2}}^{\text{diff}}(g, \text{Id}) \leq m + \frac{2}{\rho}$. This completes the proof of theorem 3. \square

5 Profiles of diffeomorphisms

One can wonder what the complexity profile looks like for a diffeomorphism, both inside the group G^{diff} and inside the group G . The following proposition only partially solves this problem. For $g \in G^{\text{diff}}$ we define $\varphi_g^{\text{diff}}(\rho) = d_\rho^{\text{diff}}(g, \text{Id})$.

Proposition 5.1.

- For any $g \in G^{\text{diff}}$ we have $\varphi_g^{\text{diff}}(\rho) = O\left(\frac{1}{\rho^2}\right)$.
- For any g in the commutator subgroup $[G^{\text{diff}}, G^{\text{diff}}]$ we have $\varphi_g^{\text{diff}}(\rho) = O\left(\frac{1}{\rho}\right)$.

If the support of $g \in G$ has area a , then for any ρ we need at least $\frac{a}{\rho}$ elements of G_ρ or G_ρ^{diff} to get g . Thus, according to the second point, the profile of any g in $[G^{\text{diff}}, G^{\text{diff}}]$ is bounded from above and below by multiples of the function $\varphi_0 : \rho \mapsto \frac{1}{\rho}$; and this holds both in G and G^{diff} . In particular we see that N_{φ_0} is the smallest non-trivial subgroup of our family $\{N_\varphi\}$. I have not been able to decide whether the first point is optimal, and whether $G^{\text{diff}} \subset N_{\varphi_0}$ or not (if not, of course, then G is not simple).

Proof of proposition 5.1. For every $\rho > 0$ we can find some discs D_1, \dots, D_m as in the proof of proposition 4.1, such that for every i, j with $|i - j| > 1$,

$$d(D_i, D_j) > \frac{C}{m} \sim \frac{C}{2}\rho$$

where d is the euclidean metric of the unit disc and C is some constant (maybe $C = \frac{1}{\sqrt{\pi}}$). Let $g \in G^{\text{diff}}$, and fix some smooth isotopy $(g_t)_{t \in [0, 1]}$ from the identity to g in G^{diff} . Let $M > 0$ be such that every trajectory of the isotopy has speed bounded from above by M . Within any interval of time less than $\frac{C}{mM}$, no point moves a distance more than $\frac{C}{m}$. We deduce that for every $t, t' \in [0, 1]$ with

$$|t - t'| < \frac{C}{mM}$$

the disc $g_{t'} g_t^{-1}(D_i)$ remains disjoint from D_j for every $|i - j| > 1$; that is, the diffeomorphism $g_{t'} g_t^{-1}$ belongs to the neighbourhood \mathcal{V}_ρ defined in proposition 4.1.

Thus we can write g as the product of no more than $\frac{mM}{C} + 1$ elements of \mathcal{V}_ρ , each of them is the product of at most $m - 1$ elements of G_ρ^{diff} , thus we get

$$d_\rho^{\text{diff}}(g, \text{Id}) \leq (m - 1) \left(\frac{mM}{C} + 1 \right) \sim \frac{4M}{C} \frac{1}{\rho^2}$$

which proves the first point of the proposition.

We turn to the second point. We first prove the result for some special commutator. Let D be any displaceable disc (say of area $\frac{1}{3}$), and let Φ be any non trivial element of G^{diff} supported in the interior of D . Choose some $\Psi \in G^{\text{diff}}$ such that $\Psi(D)$ is disjoint from D . Let us define $g := [\Phi, \Psi]$. We claim that for any ρ we have

$$\varphi_g^{\text{diff}}(\rho) \leq \frac{4}{3\rho}.$$

To prove the claim fix some positive ρ . It is easy to find, almost explicitly, some $\Psi_\rho \in G^{\text{diff}}$ which is a product of less than $\frac{2}{3\rho}$ elements of G_ρ^{diff} and that moves D disjoint from itself. Then

$$d_\rho^{\text{diff}}([\Phi, \Psi_\rho], \text{Id}) \leq \frac{4}{3\rho}.$$

We now notice that the map $g = [\Phi, \Psi]$ is conjugated to $[\Phi, \Psi_\rho]$: indeed we may find some $\Theta \in G^{\text{diff}}$ that is the identity on D and equals $\Psi_\rho \Psi^{-1}$ on $\Psi(D)$ (use a variation on lemma 4.2), and such a Θ provides the conjugacy. Since the fragmentation metric is bi-invariant this proves the claim.

We end the proof of the proposition by using Banyaga's theorem. Since the commutator subgroup $[G^{\text{diff}}, G^{\text{diff}}]$ is simple, the normal subgroup generated by g is equal to $[G^{\text{diff}}, G^{\text{diff}}]$. As in the C^0 case the set of elements $g' \in G^{\text{diff}}$ satisfying $\varphi_{g'}(\rho) = O(\frac{1}{\rho})$ is a normal subgroup, therefore it contains $[G^{\text{diff}}, G^{\text{diff}}]$. This proves the second point of the proposition. \square

6 Solution of the fragmentation problem for $\rho > \frac{1}{2}$

In [EPP] the authors describe some quasi-morphisms on the group of symplectic diffeomorphisms on various surfaces that are continuous with respect to the C^0 -topology and, as a consequence, extend continuously to the group of area-preserving homeomorphisms. It turns out that their family of quasi-morphisms on the disc provides a solution to the easiest half of the fragmentation problem expressed in questions 2 or 3 above. We discuss this briefly.

Remember that a map $\phi : G^{\text{diff}} \rightarrow \mathbb{R}$ is a *quasi-morphism* if the function

$$| \phi(gh) - \phi(g) - \phi(h) |$$

is bounded on $G^{\text{diff}} \times G^{\text{diff}}$ by some quantity $\Delta(\phi)$ called the *defect* of ϕ . A quasi-morphism is called *homogeneous* if it satisfies $\phi(g^n) = n\phi(g)$ for every g and every integer n . The construction of the continuous quasi-morphisms uses the Calabi quasi-morphisms of Entov and Polterovich on the 2-sphere ([EP03]). By definition, this quasi-morphism coincides with Calabi morphism when restricted to those diffeomorphisms supported on any displaceable disc. By embedding \mathbb{D}^2 inside the two-sphere as a non-displaceable disc ([EP03], theorem 1.11 and section 5.6), and subtracting the Calabi morphism ([EPP]), one can get a family $(\phi_{\rho'})_{\rho' \in (\frac{1}{2}, 1]}$ of homogeneous quasi-morphisms of the two-disc satisfying

1. $\phi_{\rho'}(g) = 0$ for any $g \in G_{\rho'}$,
2. for every $\rho > \rho'$ there exists some $g \in G_{\rho}$ with $\phi_{\rho'}(g) \neq 0$.

This entails the non-existence of a universal bound for the fragmentation of elements of G_{ρ} into elements of $G_{\rho'}$, for every $\rho \in (\frac{1}{2}, 1]$ and every $\rho' < \rho$: in other words for every such ρ, ρ' , there is no integer m such that $G_{\rho} \subset G_{\rho'}^m$. To see this, first remember that it suffices to prove this property when ρ' is greater than $\frac{1}{2}$ (see the easy remarks at the end of section 1). The first property of $\phi_{\rho'}$ gives a bound on $\phi_{\rho'}$ in $G_{\rho'}^m$, namely

$$\phi_{\rho'}(g') \leq (m-1)\Delta(\phi_{\rho'}) \quad \text{if } g' \in G_{\rho'}^m.$$

On the other hand there exists some $g \in G_{\rho}$ with $\phi_{\rho'}(g) \neq 0$, and since $\phi_{\rho'}$ is homogeneous, the sequence of values of $\phi_{\rho'}$ on the iterates of g is unbounded. Thus some g^n do not belong to $G_{\rho'}^m$.

We are naturally led to the following question.

Question 4. *Do there exist, for every $\rho' \in (0, \frac{1}{2}]$, some homogeneous quasimorphism $\phi_{\rho'}$ satisfying properties 1 and 2 as expressed above?*

A positive answer would imply a negative answer to questions 1, 2, 3.

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