

LOCALLY HOMOGENEOUS RIGID GEOMETRIC STRUCTURES ON SURFACES

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ABSTRACT. We study locally homogeneous rigid geometric structures on surfaces. We show that a locally homogeneous projective connection on a compact surface is flat. We also show that a locally homogeneous unimodular affine connection ∇ on a two dimensional torus is complete and, up to a finite cover, homogeneous.

Let ∇ be a unimodular real analytic affine connection on a real analytic compact connected surface M . If ∇ is locally homogeneous on a nontrivial open set in M , we prove that ∇ is locally homogeneous on all of M .

1. INTRODUCTION

Riemannian metrics are the most common (rigid) geometric structures. A locally homogeneous Riemannian metric on a surface has constant sectional curvature and it is locally isometric either to the standard metric on the two-sphere, or to the flat metric on \mathbb{R}^2 , or to the hyperbolic metric on the Poincaré's upper-half plane (hyperbolic plane). Obviously the flat metric is translation invariant on \mathbb{R}^2 . Recall also that the isometry group of the hyperbolic plane is isomorphic to $PSL(2, \mathbb{R})$ and contains copies of the affine group of the real line (preserving orientation) $Aff(\mathbb{R})$, which act locally free (they are the stabilizers of points on the boundary). Consequently, the hyperbolic plane is locally isometric to a translation invariant Riemannian metric on $Aff(\mathbb{R})$.

We generalize here this phenomena to all rigid geometric structures in Gromov's sense (see the definition in the following section).

Theorem 1.1. *Let ϕ be a locally homogeneous rigid geometric structure on a surface. Then ϕ is locally isomorphic to a rigid geometric structure which is either rotation invariant on the two-sphere, or translation invariant on \mathbb{R}^2 , or translation invariant on the affine group of the real line preserving orientation $Aff(\mathbb{R})$.*

If ϕ is an affine connection this was first proved by B. Opozda [32] for the case torsion-free (see also the group-theoretical approach in [22]), and then by T. Arias-Marco and O. Kowalski in the case of arbitrary torsion [3].

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Theorem 1.1 stands, in particular, for projective connections. In this case the theorem can also be deduced from the results obtained in [9]. Recall that a well-known class of locally homogeneous projective connections ϕ on surfaces are those which are *flat*, i.e. locally isomorphic to the standard projective connection of the projective plane $P^2(\mathbb{R})$. The automorphism group of $P^2(\mathbb{R})$ is the projective group $PGL(3, \mathbb{R})$ which contains \mathbb{R}^2 acting freely (by translation) on the affine plane: it is the subgroup which fixes each point in the line at the infinity. Consequently, ϕ is locally isomorphic to a translation invariant projective connection on \mathbb{R}^2 .

We prove also the following global results dealing with projective and affine connections on surfaces:

Theorem 1.2. *A locally homogeneous projective connection on a compact surface is flat.*

Theorem 1.3. *Let M be a compact connected real analytic surface endowed with a unimodular real analytic affine connection ∇ . If ∇ is locally homogeneous on a nontrivial open set in M , then ∇ is locally homogeneous on M .*

The main result of the article is Theorem 1.3. It is motivated by the celebrated open-dense orbit theorem of M. Gromov [13, 18] (see also [5, 11, 16]). Gromov's result asserts that a rigid geometric structure admitting an automorphism group which acts with a dense orbit is locally homogeneous on an open dense set. This maximal locally homogeneous open (dense) set appears to be mysterious and it might very well happen that it coincides with all of the (connected) manifold in many interesting geometric backgrounds. This was proved, for instance, for Anosov flows with differentiable stable and instable foliations and transverse contact structure [6] and for three dimensional compact Lorentz manifolds admitting a nonproper one parameter group acting by automorphisms [35]. In [7], the authors deal with this question and their results indicate ways in which the rigid geometric structure cannot degenerate off the open dense set.

Surprisingly, the extension of a locally homogeneous open dense subset to all of the (connected) manifold might stand *even without assuming the existence of a big automorphism group*. This is known to be true in the Riemannian setting [17], as a consequence of the fact that all scalar invariants are constant (see also Corollary 4.2 in our section 4). This was also recently proved in the frame of three dimensional real analytic Lorentz metrics [14] and for complete real analytic pseudo-Riemannian metrics [25].

Theorem 1.3 proves the extension phenomenon in the setting of affine connections on surfaces. We don't know if the result is still true when ∇ is not unimodular analytic, or in higher dimension.

As a by-product of the proof we get the following:

Theorem 1.4. *A locally homogeneous unimodular affine connection on a two dimensional torus is complete and, up to a finite cover, homogeneous.*

The composition of the article is the following. In section 2 we introduce the basic facts about rigid geometric structures and prove theorem 1.1. Section 3 deals with global phenomena and proves theorem 1.2. Theorems 1.3 and 1.4 will be proved in sections 4 and 5 respectively.

2. LOCALLY HOMOGENEOUS RIGID GEOMETRIC STRUCTURES

In the sequel all manifolds will be supposed to be smooth and connected. The geometric structures will be also assumed to be smooth.

Consider a n -manifold M and, for all integers $r \geq 1$, consider the associated bundle $R^r(M)$ of r -frames, which is a $D^r(\mathbb{R}^n)$ -principal bundle over M , with $D^r(\mathbb{R}^n)$ the real algebraic group of r -jets at the origin of local diffeomorphisms of \mathbb{R}^n fixing 0 (see [1]).

Let us consider, as in [13, 18], the following

Definition 2.1. A *geometric structure* (of order r) ϕ on a M is a $D^r(\mathbb{R}^n)$ -equivariant smooth map from $R^r(M)$ to a real algebraic variety Z endowed with an algebraic action of $D^r(\mathbb{R}^n)$.

Riemannian and pseudo-Riemannian metrics, affine and projective connections and the most encountered geometric objects in differential geometry are known to verify the previous definition [13, 18, 5, 11, 16]. For instance, if the image of ϕ in Z is exactly one orbit, this orbit identifies with a homogeneous space $D^r(\mathbb{R}^n)/G$, where G is the stabilizer of a chosen point in the image of ϕ . We get then a reduction of the structure group of $R^r(M)$ to the subgroup G . This is exactly the classical definition of a G -structure (of order r): the case $r = 1$ and $G = O(n, \mathbb{R})$ corresponds to a Riemannian metric and that of $r = 2$ and $G = GL(n, \mathbb{R})$ gives a torsion free affine connection [20, 1].

Definition 2.2. A (local) Killing field of ϕ is a (local) vector field on M whose canonical lift to $R^r(M)$ preserves ϕ .

Following Gromov [18, 13] we define rigidity as:

Definition 2.3. A geometric structure ϕ is rigid at order $k \in \mathbb{N}$, if local Killing fields are determined by their k -order jet at any chosen point in M .

Consequently, in the neighborhood of any point of M , the algebra of Killing fields of a rigid geometric structure is finite dimensional.

Recall that (pseudo)-Riemannian metrics, as well as affine and projective connections, or conformal structures in dimension ≥ 3 are known to be rigid [13, 18, 5, 11, 16, 20].

Definition 2.4. The geometric structure ϕ is said to be locally homogeneous on the open subset $U \subset M$ if for any tangent vector $V \in T_u U$ there exists a local Killing field X of ϕ such that $X(u) = V$.

The Lie algebra of Killing fields is the same at the neighborhood of any point of a locally homogeneous geometric structure ϕ . In this case it will be simply called *the Killing algebra of ϕ* .

Let G be a connected Lie group and I a closed subgroup of G . Recall that M is said to be *locally modelled* on the homogeneous space G/I if it admits an atlas with open sets diffeomorphic to open sets in G/I such that the transition maps are given by restrictions of elements in G .

In this situation any G -invariant geometric structure $\tilde{\phi}$ on G/I uniquely defines a locally homogeneous geometric structure ϕ on M which is locally isomorphic to $\tilde{\phi}$.

We recall that there exists locally homogeneous Riemannian metrics on 5-dimensional manifolds which are not locally isometric to an invariant Riemannian metric on a homogeneous space [21, 23]. However this phenomenon cannot happen in lower dimension:

Theorem 2.5. *Let M be a manifold of dimension ≤ 4 bearing a locally homogeneous rigid geometric structure ϕ with Killing algebra \mathfrak{g} . Then M is locally modelled on a homogeneous space G/I , where G is a connected Lie group with Lie algebra \mathfrak{g} and I is a closed subgroup of G . Moreover, (M, ϕ) is locally isomorphic to a G -invariant geometric structure on G/I .*

Proof. Let \mathfrak{g} be the Killing algebra of ϕ . Denote by \mathfrak{J} the (isotropy) subalgebra of \mathfrak{g} composed by Killing fields vanishing at a given point in M . Let G be the unique connected simply connected Lie group with Lie algebra \mathfrak{g} . Since \mathfrak{J} is of codimension ≤ 4 in \mathfrak{g} , a result of Mostow [28] (chapter 5, page 614) shows that the Lie subgroup I in G associated to \mathfrak{J} is *closed*. Then ϕ induces a G -invariant geometric structure $\tilde{\phi}$ on G/I locally isomorphic to it. Moreover, M is locally modelled on G/I . \square

Remark 2.6. *By the previous construction, G is simply connected and I is connected (and closed), which implies that G/I is simply connected (see [28], page 617, Corollary 1). In general, the G -action on G/I admits a nontrivial discrete kernel. We can assume that this action is effective considering the quotient of G and I by the maximal normal subgroup of G contained in I (see proposition 3.1 in [33]).*

Theorem 1.1 is a direct consequence of theorem 2.5 and of the following:

Proposition 2.7. *A two dimensional homogeneous space G/I of a connected simply connected Lie group G is either the two sphere with G being S^3 , or it bears an action of a two dimensional subgroup of G (isomorphic either to \mathbb{R}^2 , or to the affine group of the real line preserving orientation) which admits an open orbit.*

Here the 3-sphere S^3 is endowed with its standard structure of Lie group [19, 31].

Proof. This follows directly from Lie's classification of the two dimensional homogeneous space (see the list in [31] or [28]). More precisely, any (finite dimensional) Lie algebra acting transitively on a surface either admits a two dimensional subalgebra acting simply transitively (or equivalently, which trivially intersects the isotropy subalgebra), or it is isomorphic to the Lie algebra of S^3 acting on the real sphere S^2 by the standard action. \square

3. GLOBAL RIGIDITY RESULTS

Recall that a manifold M locally modelled on a homogeneous space G/I gives rise to a *developing map* defined on its universal cover \tilde{M} with values in G/I and to a *holonomy morphism* $\rho : \pi_1(M) \rightarrow G$ (well defined up to conjugacy in G) [33]. The developing map is a local diffeomorphism which is equivariant with respect to the action of the fundamental group $\pi_1(M)$ on \tilde{M} (by deck transformations) and on G (by its image through ρ).

The manifold M is said to be *complete* if the developing map is a global diffeomorphism. In this case M is diffeomorphic to a quotient of G/I by a discrete subgroup of G acting properly and without fixed points.

We give now a last definition:

Definition 3.1. A geometric structure ϕ on M is said to be of *Riemannian type* if there exists a Riemannian metric on M preserved by all Killing fields of ϕ .

Roughly speaking a locally homogeneous geometric structure is of Riemannian type if it is constructed by putting together a Riemannian metric and any other geometric structure (e.g. a vector field). Since Riemannian metrics are rigid, a geometric structure of Riemannian type it is automatically rigid.

With this terminology we have the following corollary of theorem 2.5.

Theorem 3.2. *Let M be a compact manifold of dimension ≤ 4 equipped with a locally homogeneous geometric structure ϕ of Riemannian type. Then M is isomorphic to a quotient of a homogeneous space G/I , endowed with a G -invariant geometric structure, by a lattice in G .*

Proof. By theorem 2.5, M is locally modelled on a homogeneous space G/I . Since ϕ is of Riemannian type, G/I admits a G -invariant Riemannian metric. This implies that the isotropy I is compact.

On the other hand, compact manifolds locally modelled on homogeneous space G/I with compact isotropy group I are classically known to be complete: this is a consequence of the Hopf-Rinow's geodesical completeness [33]. \square

Remark 3.3. *A G -invariant geometric structure on G/I is of Riemannian type if and only if I is compact.*

Recall that a homogeneous space G/I is said to be *imprimitive* if the canonical G -action preserves a non trivial foliation.

Proposition 3.4. *If M is a compact surface locally modelled on an imprimitive homogeneous space, then M is a torus.*

Proof. The G -invariant one dimensional foliation on G/H descends on M to a non singular foliation. Hopf-Poincaré's theorem implies then that the genus of M equals one: M is a torus. \square

Note that the results of [22, 3] imply in particular:

Theorem 3.5. *A locally homogeneous affine connection on a surface which is neither torsion free and flat, nor of Riemannian type, is locally modelled on an imprimitive homogeneous space.*

Indeed, T. Arias-Marco and O. Kowalski study in [3] all possible local normal forms for locally homogeneous affine connections on surfaces with the corresponding Killing algebra. Their results are summarized in a nice table (see [3], pages 3-5). In all cases, except for the Killing algebra of the (standard) torsion free affine connection and for Levi-Civita connections of Riemannian metrics of constant sectional curvature, there exists at least one

Killing field non contained in the isotropy algebra which is normalized by the Killing algebra. Its direction defines then a G -invariant line field on G/I .

The previous result combined with proposition 3.4 imply the main result in [32]:

Theorem 3.6. (*Opozda*) *A compact surface M bearing a locally homogeneous affine connection of non Riemannian type is a torus.*

Recall first that a well known result of J. Milnor shows that a compact surface bearing a flat affine connection is a torus [26] (see also [8]).

Proof. In the case of a non flat connection, theorem 3.5 shows that M is locally modelled on an imprimitive homogeneous space. Then proposition 3.4 finishes the proof. \square

We give now the proof of theorem 1.2.

Proof. The starting point of the proof is the classification obtained in [9] of all possible Killing algebras of a two dimensional locally homogeneous projective connection. Indeed, Lemma 3 and Lemma 4 in [9] prove that either ϕ is flat, or the Killing algebra of ϕ is the Lie algebra of one of the following Lie groups: $Aff(\mathbb{R})$ or $SL(2, \mathbb{R})$. Moreover, in the last case the isotropy is generated by a one parameter unipotent subgroup.

Assume, by contradiction, that the Killing algebra of ϕ is that of $Aff(\mathbb{R})$. Then, by theorem 2.5, M is locally modelled on $Aff(\mathbb{R})$ and, by theorem 3.2, M has to be a quotient of $Aff(\mathbb{R})$ by a uniform lattice. Or, $Aff(\mathbb{R})$ is not unimodular and, consequently, doesn't admit lattices: a contradiction.

Assume, by contradiction, that the Killing algebra is that of $SL(2, \mathbb{R})$. Then, by theorem 2.5, M is locally modelled on $SL(2, \mathbb{R})/I$, with I a one parameter unipotent subgroup in $SL(2, \mathbb{R})$.

Equivalently, I is conjugated to $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, with $b \in \mathbb{R}$. The homogeneous space $SL(2, \mathbb{R})/I$ is diffeomorphic to $\mathbb{R}^2 \setminus \{0\}$ endowed with the linear action of $SL(2, \mathbb{R})$.

Notice that the action of $SL(2, \mathbb{R})$ on $SL(2, \mathbb{R})/I$ preserves a nontrivial vector field. The expression of this vector field in linear coordinates (x_1, x_2) on $\mathbb{R}^2 \setminus \{0\}$ is $x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}$, which is the fundamental generator of the one parameter group of homotheties. The flow of this vector field doesn't preserve the standard volume form and has a nonzero constant divergence: $\lambda = 2$.

Let X be the corresponding vector field induced on M and $div(X)$ the divergence of X with respect to the volume form vol induced on M by the standard $SL(2, \mathbb{R})$ -invariant volume form of $\mathbb{R}^2 \setminus \{0\}$. Recall that, by definition, $L_X vol = div(X) \cdot vol$, where L_X is the Lie derivative. Here $div(X)$ is the constant function λ .

Denote by Ψ^t the time t of the flow generated by X . We get $(\Psi^t)^* vol = exp(\lambda t) \cdot vol$, for all $t \in \mathbb{R}$. But the flow of X has to preserve the global volume $\int_M vol$. This implies $\lambda = 0$: a contradiction.

It remains that ϕ is flat. \square

We terminate the section with the following.

Proposition 3.7. *If M is a compact surface endowed with a locally homogeneous rigid geometric structure admitting a semi-simple Killing algebra of dimension 3, then either M is globally isomorphic to a rotation invariant geometric structure on the two-sphere (up to a double cover), or the Killing algebra of ϕ preserves a hyperbolic metric on M (and M is of genus $g \geq 2$).*

Proof. By theorem 2.5, M is locally modelled on G/I , with G a 3-dimensional connected simply connected semi-simple Lie group and I a closed one parameter subgroup in G .

Up to isogeny, there are only two such G : S^3 and $SL(2, \mathbb{R})$ [19]. If $G = S^3$ then I is compact and coincides with the stabilizer of a point under the standard S^3 -action on S^2 . Consequently, G/I identifies with S^2 and the G -action on G/I preserves the canonical metric of the two-sphere.

The developing map from \tilde{M} to G/I has to be a diffeomorphism (see theorem 3.2). Consequently, M is a quotient of the sphere S^2 by a discrete subgroup of G acting by deck transformations. Since a nontrivial isometry of S^2 which is not $-Id$ always admits fixed points, this discrete subgroup has to be of order two. Up to a double cover, (M, ϕ) is isomorphic to S^2 endowed with a rotation invariant geometric structure.

Consider now the case where $G = SL(2, \mathbb{R})$. Then M is locally modelled on G/I , where I is a closed one parameter subgroup in $SL(2, \mathbb{R})$. We showed in the proof of theorem 1.2 that I is not conjugated to a unipotent subgroup.

Assume now that I is conjugated to a one parameter semi-simple subgroup in $SL(2, \mathbb{R})$. We prove that this assumption yields a contradiction. In order to describe the geometry of $SL(2, \mathbb{R})/I$, consider the adjoint representation of $SL(2, \mathbb{R})$ into its Lie algebra $sl(2, \mathbb{R})$. This $SL(2, \mathbb{R})$ -action preserves the Killing quadratic form q , which is a non degenerate Lorentz quadratic form. Choose $x \in sl(2, \mathbb{R})$ a vector of unitary q -norm and consider its orbit under the adjoint representation. This orbit identifies with our homogeneous space $SL(2, \mathbb{R})/I$, on which the restriction of the Killing form induces a two dimensional $SL(2, \mathbb{R})$ -invariant complete Lorentz metric g of constant nonzero sectional curvature [34].

We prove now that there is no *compact* surface locally modelled on the previous homogeneous space $SL(2, \mathbb{R})/I$.

Observe that x induces on $SL(2, \mathbb{R})/I$ a $SL(2, \mathbb{R})$ -invariant vector field X and $g(X, \cdot)$ induces on $SL(2, \mathbb{R})/I$ an invariant one form ω . Remark that $d\omega$ is a volume form. Indeed, $d\omega(Y, Z) = -g(X, [Y, Z])$, for all $SL(2, \mathbb{R})$ -invariant vector fields Y, Z tangents to $SL(2, \mathbb{R})/I$.

Assume, by contradiction, that M is locally modelled on $SL(2, \mathbb{R})/I$. Then M inherits the one form ω whose differential is a volume form. This is in contradiction with Stokes' theorem. Indeed $\int_M d\omega = \int vol \neq 0$, but $d\omega$ is exact.

It remains that I is conjugated to a one parameter subgroup of rotations in $SL(2, \mathbb{R})$ and then the $SL(2, \mathbb{R})$ -action on $SL(2, \mathbb{R})/I$ identifies with the action by homographies on the Poincaré's upper-half plane. This action preserves the hyperbolic metric. Therefore M inherits a hyperbolic metric and, consequently, its genus is ≥ 2 . \square

4. DYNAMICS OF LOCAL KILLING ALGEBRA

A manifold M bearing a geometric structure ϕ admits a natural partition given by the orbits of the action of the Killing algebra of ϕ . Precisely, two points $m_1, m_2 \in M$ are in the same subset of the partition if m_1 can be reached from m_2 by flowing along a finite sequence of local Killing fields. A connected open set in M where ϕ is locally homogeneous lies in the same subset of this partition.

The Gromov's celebrated stratification theorem [13, 18] which was used and studied by many authors [13, 18, 5, 11, 16] roughly states that, if ϕ is rigid, the subsets of this partition are locally closed in M . We adapt here Gromov's proof and get a more precise result in the particular case, where ϕ is of Riemannian type.

Theorem 4.1. *Let M be a connected manifold endowed with a geometric structure of Riemannian kind ϕ . Then the orbits of the Killing algebra of ϕ are closed.*

Corollary 4.2. *If ϕ is locally homogeneous on an open dense set, then ϕ is locally homogeneous on M .*

Proof. Let g be a Riemannian metric preserved by all Killing fields of ϕ . Consider also the g -orthonormal frame bundle $\pi : B \rightarrow M$. Then B is a principal sub-bundle of the frame bundle $R^1(M)$ with structure group $O(n, \mathbb{R})$.

Gromov's proof shows that (for any rigid geometric structure) there exists an integer $s \in \mathbb{N}$ such that two points of M where the s -jet of ϕ is the same are in the same orbit of the Killing algebra [13, 18, 5, 11, 16].

We consider exponential local coordinates with respect to g . For each element of $(m, b) \in B$ we get local exponential coordinates around the point $m \in M$ in which we take the k -jet ϕ^k of ϕ . This gives a map

$$\phi^k : B \rightarrow Z^k$$

with values in the variety Z^k of k -jets of ϕ .

By Gromov's proof, orbits of the Killing algebra are the connected components of the projections on M (through π) of the pull-back through ϕ^k of $O(n, \mathbb{R})$ -orbits of Z^k (see the arguments in [13], section 3.5). The $O(n, \mathbb{R})$ -orbits of Z^k being compact, their pull-back through ϕ^k in B are saturated closed sets. By compactness of the fibers, the projection to M of a saturated closed subset in B is a closed set. Since connected components in M of closed sets are also closed, we get that the orbits of the Killing algebra are closed. \square

The corollary was known for Riemannian metrics. Indeed, in [17] the authors proved that Riemannian metrics whose all scalar invariants are constant are locally homogeneous (this is known to fail in the pseudo-Riemannian setting [10]).

In the real analytic realm this implies the following more precise:

Corollary 4.3. *If M and ϕ are real analytic and ϕ is locally homogeneous on a nontrivial open set, then ϕ is locally homogeneous on M .*

Proof. In the real analytic setting, Gromov's proof shows that away from a nowhere dense analytic subset S in M , the orbits of the Killing algebra are connected components of fibers of an analytic map of constant rank [18] (section 3.2). With our hypothesis, the orbits of the Killing algebra are exactly connected components of $M \setminus S$. Consequently, they are open sets. One apply now corollary 4.2 and get that the orbits are also closed. Since M is connected, the Killing algebra admits exactly one orbit. \square

In the previous results the compactness of the orthogonal group was essential.

We prove now the following result for affine unimodular connections which are not (necessarily) of Riemannian kind:

Theorem 4.4. *Let M be a compact connected real analytic surface endowed with a real analytic unimodular affine connection ∇ . If ∇ is locally homogeneous on a nontrivial open set in M , then ∇ is locally homogeneous on M .*

Recall that ∇ is said *unimodular* if there exists a volume form on M which is invariant by the parallel transport [1]. This volume form is automatically preserved by any local Killing field of ∇ . We prove first the following useful:

Lemma 4.5. *Let ∇ be a unimodular analytic affine connection on an analytic surface M . Then the dimension of the isotropy algebra at a point of M is $\neq 2$.*

Proof. Assume by contradiction that the isotropy algebra \mathcal{I} at a point $m \in M$ has dimension two. Consider a system of local exponential coordinates at m with respect to ∇ and, for all $k \in \mathbb{N}$ take the k -jet of ∇ in these coordinates. Any volume preserving linear isomorphism of $T_m M$ gives another system of local exponential coordinates at m , with respect to which we consider the k -jet of the connection. This gives an algebraic $SL(2, \mathbb{R})$ -action on the vector space Z^k of k -jets of affine connections on \mathbb{R}^2 admitting a trivial underlying 0-jet [13, 18].

Elements of \mathcal{I} linearize in exponential coordinates at m . Since they preserve ∇ , they preserve in particular the k -jet of ∇ at m , for all $k \in \mathbb{N}$. This gives an embedding of \mathcal{I} in the Lie algebra of $SL(2, \mathbb{R})$ such that the corresponding (two dimensional) connected subgroup of $SL(2, \mathbb{R})$ preserves the k -jet of ∇ at m for all $k \in \mathbb{N}$.

Now we use the fact that *the stabilizers of a linear algebraic $SL(2, \mathbb{R})$ -action are of dimension $\neq 2$* . Indeed, it suffices to check this statement for irreducible linear representations of $SL(2, \mathbb{R})$ for which it is well-known that the stabilizer in $SL(2, \mathbb{R})$ of a nonzero element is one dimensional [19].

It follows that the stabilizer of the k -jet of ∇ at m is of dimension three and contains the connected component of identity in $SL(2, \mathbb{R})$. Consequently, in exponential coordinates at m , each element of the connected component of the identity in $SL(2, \mathbb{R})$ gives rise to a local linear vector field which preserves ∇ (for it preserves all k -jets of ∇). The isotropy algebra \mathcal{I} contains a copy of the Lie algebra of $SL(2, \mathbb{R})$: a contradiction, since \mathcal{I} is of dimension two. \square

Our proof of theorem 4.4 will need analyticity in another essential way. We will make use of an extendibility result for local Killing fields proved first

for Nomizu in the Riemannian setting [30] and generalized then for rigid geometric structures by Amores et Gromov [2, 18] (see also [11, 13, 16]). This phenomena states roughly that a local Killing field of a *rigid analytic* geometric structure can be extended along any curve in M . We then get a multivalued Killing field defined on all of M or, equivalently, a global Killing field defined on the universal cover. In particular, the Killing algebra in the neighborhood of any point is the same (as long as M is connected).

As an application of the Nomizu's phenomena we give (compare with theorem 2.5):

Theorem 4.6. *Let M be a compact simply connected real analytic manifold admitting a real analytic locally homogeneous rigid geometric structure. Then M is isomorphic to a homogeneous space G/I endowed with a G -invariant geometric structure.*

Proof. Since ϕ is locally homogeneous and M is simply connected and compact, the local transitive action of the Killing algebra extends to a global action of the associated simply connected Lie group G (we need compactness to insure that vector fields on M are complete). All orbits have to be open, so there is only one orbit: the action is transitive and M is a homogeneous space. \square

Let's go back now to the proof of theorem 4.4. As before, in the real analytic setting Gromov's stratification theorem shows that the locally homogeneous open dense set has to be dense [18, 13]. Note also that Nomizu's extension phenomena doesn't imply that the extension of a family of linearly independent Killing fields, stays linearly independent. In general, the extension of a locally transitive Killing algebra, fails to be transitive on a nowhere dense analytic subset S in M . The unimodular affine connection is locally homogeneous on each connected component of $M \setminus S$.

We prove now that S is empty.

Assume by contradiction that S is not empty. Then we have the following crucial:

Lemma 4.7. *(i) The Killing algebra \mathfrak{g} of ∇ has dimension two and the isotropy algebra at a point of S is one dimensional.*

(ii) \mathfrak{g} is isomorphic to the Lie algebra of the affine group of the line.

Proof. (i) Since the Killing algebra admits a nontrivial open orbit in M , its dimension is ≥ 2 . Pick up a point $s \in S$ and consider the linear morphism $ev(s) : \mathfrak{g} \rightarrow T_s M$ which associates to an element $K \in \mathfrak{g}$ its value $K(s)$. The kernel of this morphism is the isotropy \mathcal{I} at s . Since the \mathfrak{g} -action is nontransitive in the neighborhood of s , the range of $ev(s)$ is ≤ 1 . This implies that the isotropy at s is of dimension at least $\dim \mathfrak{g} - 1$.

Assume, by contradiction, that the Killing algebra has dimension at least three. Then the isotropy at s is of dimension at least two. By lemma 4.5, this dimension never equals two. Consequently, the isotropy algebra at $s \in S$ is three dimensional. The isotropy algebra contains then a copy of the Lie algebra of $SL(2, \mathbb{R})$ (see the proof of lemma 4.5).

The local action of $SL(2, \mathbb{R})$ in the neighborhood of s is conjugated to the standard linear action of $SL(2, \mathbb{R})$ on \mathbb{R}^2 . This action has two orbits: the point s and $\mathbb{R}^2 \setminus \{s\}$. The open orbit $\mathbb{R}^2 \setminus \{s\}$ identifies with a homogeneous space $SL(2, \mathbb{R})/I$. Precisely, the stabilizer I in $G = SL(2, \mathbb{R})$ of a nonzero vector $x \in T_s M$ is conjugated to the following one parameter unipotent subgroup of $SL(2, \mathbb{R})$: $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, with $b \in \mathbb{R}$. The action of $SL(2, \mathbb{R})$ on $SL(2, \mathbb{R})/I$ preserves the induced flat torsion free affine connection coming from \mathbb{R}^2 .

By proposition 8 in [3], the only $SL(2, \mathbb{R})$ -invariant affine connection on $SL(2, \mathbb{R})/I$ is the previous flat torsion free connection. Another way to prove this result is to consider the difference of a $SL(2, \mathbb{R})$ -invariant connection with the standard one. We get a $(2, 1)$ -tensor on $SL(2, \mathbb{R})/I$ which is $SL(2, \mathbb{R})$ -invariant. Equivalently, we get a $ad(I)$ -invariant $(2, 1)$ -tensor on the quotient of the Lie algebra $sl(2, \mathbb{R})$ by the infinitesimal generator of I [1]. A straightforward computation shows that the tensor has to be trivial. This gives a different proof of the unicity of a $SL(2, \mathbb{R})$ -invariant connection.

By analyticity, ∇ is torsion free and flat on all of M . In particular, ∇ is locally homogeneous on all of M . This is in contradiction with our assumption.

(ii) The Killing algebra is two dimensional. Thus it coincides with \mathbb{R}^2 or with the Lie algebra of the affine group. Consider K_1, K_2 a basis of the local Killing algebra and extend K_1, K_2 along a topological disk reaching a point s in S .

Recall that ∇ is unimodular and let vol be the volume form associated to ∇ .

In the case where the Lie algebra is \mathbb{R}^2 , $vol(K_1, K_2)$ is a nonzero constant (for being invariant by the Killing algebra and thus constant on the locally homogenous open set). Hence the Lie algebra acts transitively in the neighborhood of $s \in S$. This is a contradiction: S is then empty and ϕ is locally homogeneous on all of M . \square

For the sequel, let K_1, K_2 be two local Killing fields at $s \in S$ which span the Killing algebra. We assume, without loss of generality, that K_1 and K_2 verify the Lie bracket relation $[K_1, K_2] = K_1$.

Recall that K_1, K_2 don't vanish both at a point $s \in S$. Indeed, if not the isotropy at s has dimension 2, which is impossible by lemma 4.5.

Notice that $vol(K_1, K_2)$ is not invariant by the action of the Killing algebra (since the adjoint representation of $Aff(\mathbb{R})$ is nontrivial). But we still have:

Proposition 4.8. *The local function $vol(K_1, K_2)$ is constant on the orbits of the flow generated by K_1 .*

Proof. The adjoint action $ad(K_1)$ on \mathfrak{g} is nilpotent. In particular, the adjoint representation of the one parameter subgroup generated by K_1 is unimodular. Consequently, the local flow generated by K_1 preserves $vol(K_1, K_2)$. \square

Proposition 4.9. *S is a smooth 1-dimensional manifold (diffeomorphic to a finite union of circles).*

Proof. By lemma 4.7, the isotropy \mathcal{I} at a chosen point $s \in S$ is one dimensional and the range of the map $ev(s)$ equals one. In particular, the orbit of s under the action of \mathfrak{g} is one dimensional. The image of $ev(s)$ coincides with $T_s S$. Consequently, the \mathcal{I} -action on $T_s M$ preserves the line $T_s S$.

This shows that \mathfrak{g} acts transitively on each connected component of S .

In particular, each connected component of S is a one dimensional smooth submanifold in M (recall that S is nowhere dense in M).

Since M is compact, each connected component of S is diffeomorphic to a circle. □

We also have:

Proposition 4.10. (i) $M \setminus S$ is locally modelled on the affine group of the line endowed with a translation invariant connection.

(ii) $M \setminus S$ admits a (nonsingular) \mathfrak{g} -invariant foliation \mathcal{F}_1 by lines.

(iii) \mathcal{F}_1 coincides with the kernel of a (nonsingular) closed \mathfrak{g} -invariant one form ω .

(iv) The leafs of \mathcal{F}_1 are closed (in $M \setminus S$). They are endowed with a \mathfrak{g} -invariant translation structure.

(v) The space of leafs of \mathcal{F}_1 is Hausdorff.

Proof. (i) This comes from the fact that the \mathfrak{g} -action on $M \setminus S$ is simply transitive.

(ii) The Lie bracket relation $[K_1, K_2] = K_1$ implies that the flow of K_2 normalizes the flow of K_1 and thus the foliation spanned by K_1 is K_2 -invariant. Consequently, the foliation spanned by K_1 is \mathfrak{g} -invariant and it defines a foliation \mathcal{F}_1 well defined on $M \setminus S$.

(iii) We locally define the one form ω such that $\omega(K_1) = 0$ and $\omega(K_2) = 1$. Since the basis K_1, K_2 is well defined, up to a Lie algebra isomorphism (which necessarily preserves the derivative algebra $\mathbb{R}K_1$ and sends K_2 on $K_2 + \beta K_1$, for $\beta \in \mathbb{R}$), ω is globally defined on $M \setminus S$. Also by Lie-Cartan's formula [1] $d\omega(K_1, K_2) = -\omega([K_1, K_2]) = 0$.

Thus the foliation is transversaly Riemannian in the sense of [27]. Another way to see it, is to observe that the projection of the local Killing field $K_2 + \beta K_1$ on $TM/T\mathcal{F}_1$ is well defined (it doesn't depend on β). It defines a transverse vector field \tilde{K}_2 and, consequently, \mathcal{F}_1 is transversaly Riemannian.

(iv) By proposition 4.8 and by the previous point, $vol(K_1, K_2)$ is a well define (nonconstant) function on $M \setminus S$ which is constant on the leafs of \mathcal{F}_1 . Therefore the leafs are closed in $M \setminus S$.

Moreover, the action of the derivative algebra $\mathbb{R}K_1$ preserves each leaf of \mathcal{F}_1 . Hence each leaf inherits a \mathfrak{g} -invariant translation structure.

(v) This is a consequence of (iii) and (iv) (see [27], chapter 3, Proposition 3.7). The connected components of the space of leafs are parametrized by the flow of \tilde{K}_2 . □

Denote by $Y \in \mathfrak{g}$ a generator of \mathcal{I} and by $X \in \mathfrak{g}$ a Killing field such that $X(s)$ span $T_s S$. By applying an automorphism of the Lie algebra of the affine group we can assume that either $Y = K_1$ and $X = K_2$, or $Y = K_2$ and $X = K_1$.

Case I: unipotent isotropy

Assume first that $Y = K_1$ and $X = K_2$.

We study the local situation in the neighborhood of $s \in S$. For this local analysis we will also denote by S the connected component of s in S .

Proposition 4.11. *The isotropy at any point of S is unipotent.*

Proof. Because of the Lie bracket relation, the isotropy \mathcal{I} at s acts trivially on $T_s S$. This implies that its generator Y acts trivially on the unique geodesic passing through s and tangent to the direction $T_s S$. Consequently Y vanishes on this geodesic which has to coincide locally to S (for S is the subset of M where \mathfrak{g} doesn't act freely).

Since the Y -action on $T_s M$ is trivial on $T_s S$ and volume preserving, it is unipotent. In local exponential coordinates (x, y) at s , Y is linearized and so conjugated to the linear vector field $y \frac{\partial}{\partial x}$. In these coordinates S identifies locally with $y = 0$ and the time t of the flow generated by Y is $(x, y) \rightarrow (x + ty, y)$.

Notice that the flow of Y preserves a unique foliation in the neighborhood of s , which is given by $dy = 0$. We will see later that this rules out the case of a connected components of S with isotropy $Y = K_2$ (see Step 2 in the proof of Case II, below). \square

Proposition 4.12. \mathcal{F}_1 extends to a nonsingular foliation with closed leaves defined on all of M .

Proof. The foliation \mathcal{F}_1 spanned by Y extends on M by adding the leaves S . Moreover, all the leaves of the foliation \mathcal{F}_1 are closed on M . Indeed, the first integral $vol(K_1, K_2)$ on $M \setminus S$ vanishes exactly on the extra leaves S . Therefore, $vol(K_1, K_2)$ defines a first integral on all of M . \square

Proposition 4.13. (i) Each connected component of $M \setminus S$ is diffeomorphic to a cylinder $\mathbb{R} \times S^1$.

(ii) The elements of \mathfrak{g} extend to global Killing fields on each connected component of $M \setminus S$.

Proof. By proposition 4.12, the leaves of \mathcal{F}_1 are circles.

(i) The flow of the transverse vector field \tilde{K}_2 (see the proof of proposition 4.10) acts transitively on each connected component of the space of leaves of \mathcal{F}_1 . Let $]a, b[$ be the maximal domain of definition of a integral curve of \tilde{K}_2 . Then the corresponding connected component of $M \setminus S$ is diffeomorphic to $]a, b[\times S^1$ (we will see further that the maximal domain of definition is \mathbb{R}).

(ii) Choose a local determination of K_1 and extend it along a leaf of \mathcal{F}_1 . Our local determination changes into αK_1 , with $\alpha \in \mathbb{R}$. Also extending K_2 along the same leaf, we will find $K_2 + \beta K_1$, with $\beta \in \mathbb{R}$. We get then two multivalued vector fields defined on a small cylinder containing the chosen leaf.

But $vol(\alpha K_1, K_2 + \beta K_1) = vol(K_1, K_2)$ is a nonzero constant function on our leaf (for it is locally constant by proposition 4.8). This implies $\alpha = 1$, so K_1 is globally defined (and univalued) on the small cylinder containing the

leaf. Since the fundamental group of the corresponding connected component of $M \setminus S$ is generated by our leaf (as a simple closed curve), K_1 extends in a global Killing field on all of the corresponding connected component of $M \setminus S$.

This also proves that the projection of K_2 on $T\mathcal{F}_1$ extends in a global vector field (which is not Killing but preserves each leaf of \mathcal{F}_1 and its translation structure: its local expression is fK_1 , with f a function defined on the space of leaves). Since the kernel of this projection is also well defined globally, K_2 extends on the corresponding connected component of $M \setminus S$ as well. \square

We have now that X and Y extend to globally defined vector fields on any connected component of $M \setminus S$. We have seen that the flow of X parametrized the space of leaves of \mathcal{F}_1 . In particular, the positive and the negative orbit of any point in $M \setminus S$ accumulate on S .

A contradiction is obtained in the following;

Proposition 4.14. *The negative orbit of points in $M \setminus S$ under the flow of X (equivalently the positive orbit under the flow of $-X$) cannot accumulate on S .*

Proof. Recall that the Lie bracket relation implies that the flow of X contracts Y exponentially. Thus the action of the negative flow of X expands Y . Therefore, a negative orbit under the flow of X of a point m in $M \setminus S$ never accumulates on S (for Y vanishes on S). \square

Another way to get a contradiction is to prove the following:

Proposition 4.15. *X and Y are complete on $M \setminus S$.*

Proof. By proposition 4.14, the negative orbits of X are complete since they stay in a compact subset of $M \setminus S$. The positive orbits of X are complete because they stay in a compact subset of the maximal domain of definition of X (for X can be extended to an open neighborhood of S as in the proof of point (ii) in proposition 4.13).

As for Y , its orbits lie in the (compact) leaves of \mathcal{F}_1 , so they are complete. \square

Proposition 4.15 implies then that there is a locally free transitive action of $Aff(\mathbb{R})$ on each connected component of $M \setminus S$. Consequently, each connected component of $M \setminus S$ is diffeomorphic to a homogeneous space $Aff(\mathbb{R})/\Gamma$, where Γ is a discrete subgroup of $Aff(\mathbb{R})$.

Since the $Aff(\mathbb{R})$ -action preserves the finite volume of $M \setminus S$ given by vol , Γ has to be a lattice in $Aff(\mathbb{R})$: a contradiction.

Case II: semi-simple isotropy

We assume now that $Y = K_2$ and $X = K_1$.

We will show that the Killing algebra \mathfrak{g} preserves two transverse foliations by lines \mathcal{F}_1 and \mathcal{F}_2 on M , with \mathcal{F}_2 geodesic.

Step 1: Construction of \mathcal{F}_1 . We prove that the foliation \mathcal{F}_1 constructed in proposition 4.10 extends to all of M .

Assume first that S is connected. In the neighborhood of $s \in S$ the local foliation spanned by the nonsingular vector field K_1 agrees with \mathcal{F}_1 on $M \setminus S$. Since this stands in the neighborhood of each point of S , the foliation \mathcal{F}_1 extends to a nonsingular foliation on M by adding the leaf S .

If S admits several connected components, the previous argument applies in the neighborhood of each component of S .

Notice that Poincaré-Hopf's theorem shows that M is a torus.

Step 2: Construction of \mathcal{F}_2 . Since the action of \mathcal{I} on $T_s M$ is unimodular (for it preserves $vol(s)$) and contracts the direction $T_s S$ (for $[Y, X] = -X$), this action (preserves and) expands a unique line $\mathcal{F}_2(s) \in T_s M$. This constructs a smooth \mathfrak{g} -invariant line field along S : the unique line field (preserved and) expanded by the local isotropy.

In the neighborhood of s , consider the unique geodesic foliation \mathcal{F}_2 which extends the previous line field. It is the image of the line field through the exponential map, with respect to ∇ , along S . Since \mathcal{F}_2 is \mathfrak{g} -invariant, it extends to a foliation (which we still denote by \mathcal{F}_2) on M transverse to \mathcal{F}_1 . We obtain then:

Proposition 4.16. *M admits a flat \mathfrak{g} -invariant lorentz metric.*

Proof. Let $v \in T_m M$ and consider $q(v) = vol(v_1, v_2)$, where v_1, v_2 are the components of v under the splitting

$$T_m M = T\mathcal{F}_1(m) \oplus T\mathcal{F}_2(m).$$

This constructs a \mathfrak{g} -invariant lorentz metric on M . In particular, q is locally homogeneous and hence of (lorentzian) constant sectional curvature [34].

Since M is a torus, Gauss-Bonnet's theorem (see its Lorentzian version in [34]) implies that q is flat (locally isomorphic to \mathbb{R}^2 endowed with $dx dy$) [34]. \square

Step 3: Construction of a \mathfrak{g} -invariant vector field T on M which vanishes exactly on S .

Recall that the local vector field Y generates the isotropy in the neighborhood of $s \in S$. Let T be the projection of Y on the second factor of the decomposition

$$TM = T\mathcal{F}_1 \oplus T\mathcal{F}_2.$$

Then T is obviously Y -invariant. Since X spans \mathcal{F}_1 and X and Y commutes modulo X , it follows that T is also X -invariant. The vector field T is \mathfrak{g} -invariant and vanishes on S . As before, T is defined locally in the neighborhood of $s \in S$, but since T is \mathfrak{g} -invariant, it extends to a global vector field on M .

Note that \mathfrak{g} being transitive on $M \setminus S$, the vector field T doesn't vanish on $M \setminus S$. We prove now:

Proposition 4.17. *The flow of T preserves the volume form of the unimodular connection (and also the volume form of the flat lorentz metric q).*

Proof. Since both T and the volume form vol are \mathfrak{g} -invariant, the divergence $div(T)$ is also \mathfrak{g} -invariant. Consequently, $div(T)$ is constant on each connected component of $M \setminus S$. The time t of the T -flow acts then on vol by multiplication with $exp(\lambda t)$, where $div(T) = \lambda \in \mathbb{R}$.

But this T -action has to preserve the global (finite) volume of each connected component of $M \setminus S$. This implies $\lambda = 0$ and, consequently, vol is T -invariant.

Remark that the underlying volume form of the lorentz metric being also \mathfrak{g} -invariant, it is a constant multiple of vol . Consequently, it is also T -invariant. \square

We will get a contradiction by showing the following

Proposition 4.18. *The divergence of T with respect to the volume form of the flat lorentz metric equals 1.*

Proof. Choose a point s in S . In exponential coordinates (x, y) , at s , with respect to the flat lorentz metric g , the foliations \mathcal{F}_1 and \mathcal{F}_2 are generated by the isotropic directions $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ of the standard lorentz metric $dx dy$.

The isotropy \mathcal{I} at s is generated by the linear vector field $Y = -x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$.

It follows, by construction, that T equals $y \frac{\partial}{\partial y}$. The divergence of T with respect to $dx \wedge dy$ equals 1. \square

5. LOCALLY HOMOGENEOUS AFFINE CONNECTIONS

Recall that T. Nagano and K. Yagi completely classified torsion free flat affine connections (affine structures) on the real two torus [29]. They proved that, except a well described family of incomplete affine structures (see the nice description given in [4]), all the other are homogeneous constructed from faithful affine actions of \mathbb{R}^2 on the affine plane $GL(2, \mathbb{R}) \ltimes \mathbb{R}^2 / GL(2, \mathbb{R})$ admitting an open orbit. This induces a translation invariant affine structure on \mathbb{R}^2 . The quotient of \mathbb{R}^2 by a lattice is a homogeneous affine two dimensional tori.

Notice that the previous homogeneous affine structures are complete if and only if the corresponding \mathbb{R}^2 -actions on the affine plane are transitive.

In particular, Nagano-Yagi's result prove that a real two dimensional torus locally modelled on the affine space $GL(2, \mathbb{R}) \ltimes \mathbb{R}^2 / GL(2, \mathbb{R})$ such that the linear part of the holonomy morphism lies in $SL(2, \mathbb{R})$ is always complete and homogeneous (the corresponding torsion free flat affine connection on the torus is translation invariant).

Here we prove:

Theorem 5.1. *Let M be a compact surface locally modelled on a homogeneous space G/I such that G/I admits a G -invariant affine connection ∇ and a volume form. Then:*

(i) *the corresponding affine connection on M is homogeneous (up to a finite cover), except if ∇ is the Levi-Civita connection of a hyperbolic metric (and the genus of M is ≥ 2).*

(ii) *the corresponding affine connection on M is complete.*

Remark 5.2. *In particular, a unimodular affine connection on the two torus has constant Christoffel symbols with respect to global translation invariant coordinates.*

Proof. Note first that, in the case where I is compact, theorem 3.2 proves that M is complete. Moreover, if M is of genus 0 then the underlying locally homogeneous Riemannian metric has to be of positive sectional curvature (by Gauss-Bonnet's theorem) and G/I coincides with the standard sphere S^2 seen as a homogeneous space of S^3 . Since M is simply connected, the developing map gives a global isomorphism with S^2 .

Also if the genus of M is one, the underlying locally homogeneous metric on M has to be flat. It follows that G/I coincides with the only homogeneous flat Riemannian space $O(2, \mathbb{R}) \ltimes \mathbb{R}^2 / O(2, \mathbb{R})$. Bieberbach's theorem (see, for instance, [34]) implies then that M is homogeneous (up to a finite cover).

Surfaces of genus $g \geq 2$ cannot admit a homogeneous geometric structure locally modelled on G/I , with I compact. Indeed, here the Riemannian metric induced on M is of constant negative curvature. It is well known that the isometry group of a hyperbolic metric on M is finite (hence it cannot act transitively) [20].

Consider now the case where I is noncompact.

Denote also by ∇ the locally homogeneous affine connection induced on M . The Killing algebra \mathfrak{g} acts locally on M preserving ∇ and a volume form vol . The proof of theorem 4.4 implies that \mathfrak{g} is of dimension at most 3. Therefore G is of dimension 3 and I is a one parameter subgroup in G . Since I is supposed noncompact, its (faithful) isotropy action on T_oG/I , where o is the origin point of G/I , identifies I either with a semi-simple, or with a unipotent one parameter subgroup in $SL(2, \mathbb{R})$.

Case I: semi-simple isotropy

The isotropy action on T_oG/I preserves two line fields. Consequently, there exists on G/I two G -invariant line fields. Since G/I admits also a G -invariant volume form, there exists on G/I a G -invariant Lorentz metric q (see the construction in proposition 4.16).

By Poincaré-Hopf's theorem M is a torus and q has to be flat (by Gauss-Bonnet's theorem). It follows that G is the automorphism group SOL of the flat Lorentz metric $dxdy$ on \mathbb{R}^2 . By the Lorentzian version of Bieberbach's theorem, this structure is known to be complete and homogeneous (up to a finite cover of M) [12].

Case II: unipotent isotropy

Here the isotropy action preserves a vector field in T_oG/I . This yields a G -invariant vector field \tilde{X} on G/I and a G -invariant one form $\tilde{\omega} = vol(\tilde{X}, \cdot)$. We then have:

Proposition 5.3. *M admits a \mathfrak{g} -invariant vector field X and a \mathfrak{g} -invariant closed one form ω vanishing on X . Thus the foliation \mathcal{F} generated by X is transversally Riemannian.*

Proof. Since $d\tilde{\omega}$ and vol are G -invariant, there exists $\lambda \in \mathbb{R}$ such that $d\tilde{\omega} = \lambda vol$.

Denote by ω the one form on M associated to $\tilde{\omega}$ and by X the vector field induced on M by \tilde{X} . It follows that ω vanishes on X . Also $\int_M d\omega = \lambda vol(M)$, where the volume of M is calculated with respect to the volume form induced by vol . Stokes' theorem yield $\lambda = 0$ and, consequently, ω is closed. \square

Proposition 5.4. *The normal subgroup H of G which preserves each leaf of the foliation $\tilde{\mathcal{F}}$ generated by \tilde{X} on G/I is two dimensional and abelian.*

Proof. The group G preserves the foliation $\tilde{\mathcal{F}}$ and its transverse Riemannian structure. Therefore an element of G fixing one leaf of $\tilde{\mathcal{F}}$, will fixe all the leaves. The subgroup H is nontrivial, since it contains the isotropy. The action of G/H on the transversal of the foliation preserves a Riemannian structure, so it is of dimension at most one. Since this action has to be transitive, the dimension of G/H is exactly one and the dimension of H is two.

Since H preserves \tilde{X} , the elements of H commutes in restriction to each leaf of $\tilde{\mathcal{F}}$. Hence H is abelian. \square

We will make use of the following result proved in [14] (pages 17-19):

Lemma 5.5. *\tilde{X} is a central element in \mathfrak{g} . Consequently, X is a global Killing field on M preserved by \mathfrak{g} .*

In order to prove that M is homogeneous, we will construct a second global Killing field on M which is not tangent to the foliation \mathcal{F} generated by X . For this we will study the holonomy group.

Once again M is of genus one, since it admits a nonsingular vector field. Since the fundamental group of the torus is abelian, its image Γ by the holonomy morphism is an abelian subgroup of G .

We prove first that Γ is nontrivial. Assume by contradiction that Γ is trivial. Then the developing map is well defined on M and we get a local diffeomorphism $dev : M \rightarrow G/I$. The image has to be open and closed (for M is compact). Hence, dev is surjective. By Ehresmann's submersion theorem, dev is a covering map. Remark 2.6 shows that G/I can be considered simply connected, which implies dev is a diffeomorphism: a contradiction, since M is not simply connected.

Considering finite covers of M , the previous proof rules out the case where Γ is finite. Consequently, Γ is an infinite subgroup of G .

Consider its (real) Zariski closure $\bar{\Gamma}$ in G , which is an abelian subgroup of positive dimension. This is possible, since by Lie's classification [28, 31], G is locally isomorphic to a real algebraic group acting (transitively) on G/I . Therefore we can assume that G is algebraic.

Up to a finite cover of M , we can assume $\bar{\Gamma}$ connected (algebraic groups admit at most finitely many connected components).

Proposition 5.6. *Any element a of the Lie algebra of $\bar{\Gamma}$ defines a global Killing field on M .*

Moreover, if the one parameter subgroup of G generated by a intersects Γ nontrivially, then the orbits of the corresponding Killing field on M are closed.

Proof. The action of Γ on a (by adjoint representation) being trivial, a defines a Γ -invariant vector field on G/I which descends on M .

Assume now that Γ intersects nontrivially the one parameter subgroup generated by a . One orbit of the corresponding Killing field on M develops in the model G/I as one orbit of the Killing field a . Since vector fields on compact manifolds are complete, the image of the developing map contains all of the orbit of a . In particular, the corresponding orbit of a contains distinct points which are in the same Γ -orbit. Therefore, the corresponding orbit on M is closed. \square

We prove now:

Proposition 5.7. *$\bar{\Gamma}$ is not a subgroup of H .*

Proof. Consider a one parameter subgroup in G , generated by an element in the Lie algebra of $\bar{\Gamma}$, which nontrivially intersects Γ . By proposition 5.6, a defines a global Killing field K with closed orbits on M .

Assume by contradiction that $\bar{\Gamma}$ is a subgroup of H . It follows that the orbits of K coincide with those of X . Consequently, the orbits of X are closed and the space of leaves of \mathcal{F} is a one dimensional manifold (see proposition 4.10, point (v)). Since M is compact, the space of leaves is diffeomorphic to a circle S^1 .

Consider the developing map $dev : \tilde{M} \rightarrow G/I$ of the G/I -structure. In particular, this is also the developing map of the transverse structure of the foliation \mathcal{F} . Since the holonomy group acts trivially on the transversal \tilde{T} of $\tilde{\mathcal{F}}$, dev descends to a local diffeomorphism from the space of leaves of \mathcal{F} (parametrized by S^1) to \tilde{T} .

Since G/I is simply connected, the closed one form $\tilde{\omega}$ admits a primitive $\tilde{f} : G/I \rightarrow \mathbb{R}$. Consequently, \tilde{f} is a first integral for $\tilde{\mathcal{F}}$ and $\tilde{f} \circ dev$ descends to a local diffeomorphism from S^1 to \mathbb{R} . This map has to be onto since the image is open and closed. We get a topological contradiction. \square

By proposition 5.6, any one parameter subgroup in G generated by an element of the Lie algebra of $\bar{\Gamma}$ non contained in the Lie algebra of H provides a global Killing field K on M such that the abelian group generated by the flows of X and of K acts transitively on M . Therefore the G/I -structure on M is homogeneous.

(ii) Consider \tilde{X}, \tilde{K} the corresponding Killing vector fields on G/I . They generate a two dimensional abelian subgroup A of G acting with an open orbit on G/I . Recall that $vol(\tilde{X}, \tilde{K})$ is an A -invariant function on G/I . Hence, $vol(\tilde{X}, \tilde{K})$ is a nonzero constant in restriction to the open orbit. By continuity, $vol(\tilde{X}, \tilde{K})$ equals the same nonzero constant on the closure of the open orbit. This implies that \tilde{X}, \tilde{K} remain linearly independent on the closure of the open orbit, which implies that the open orbit is also closed. Since G/I is connected, the open orbit is all of G/I . \square

The previous proof combined with a result of [14] leads to the following classification result:

Theorem 5.8. *Let M be a compact surface locally modelled on a homogeneous space G/I such that G/I admits a G -invariant affine connection ∇ and a volume form.*

If I is noncompact, then G is three dimensional and:

(i) either G is the unimodular SOL group and its action preserves a flat (two dimensional) Lorentz metric;

(ii) or G is the Heisenberg group and its action preserves the standard flat torsion free affine connection on \mathbb{R}^2 together with a nonsingular closed one form $\tilde{\omega}$ and with a nontrivial parallel vector field \tilde{X} such that $\tilde{\omega}(\tilde{X}) = 0$;

(iii) or G is isomorphic to the product $\mathbb{R} \times \text{Aff}(\mathbb{R})$ and its action preserves the canonical bi-invariant torsion free and complete connection of $\text{Aff}(\mathbb{R})$ (for which the full automorphism group is $\text{Aff}(\mathbb{R}) \times \text{Aff}(\mathbb{R})$) together with a geodesic Killing field.

The model (i) was obtained in the previous proof in the case where the isotropy is semi-simple. The models (ii) and (iii) correspond to the case where the isotropy is unipotent. This classification follows from [14] (see Proposition 3.5 and pages 15-19).

Theorem 5.1 can be deduced from theorem 5.8. Indeed, in the case (ii) G preserves a flat torsion free affine connection and Nagano-Yagi's result applies. The completeness of compact surfaces locally modelled on the homogeneous space (iii) was proved in Proposition 9.3 of [35] (the homogeneity follows from the proof of Proposition 10.1 in [35]).

REFERENCES

- [1] *D. Alekseevskij, A. Vinogradov, V. Lychagin*, Basic Ideas and Concepts in Differential Geometry, E.M.S., Geometry I, Springer-Verlag (1991).
- [2] *A. M. Amores*, Vector fields of a finite type G -structure, J. Differential Geom., **14**(1), (1979), 1-6.
- [3] *T. Arias-Marco, O. Kowalski*, Classification of locally homogeneous affine connections with arbitrary torsion on 2-dimensional manifolds, Monatsh. Math., **153**, (2008), 1-18.
- [4] *Y. Benoist*, Tores affines, Contemp. Math., **262**, (2000), 1-32.
- [5] *Y. Benoist*, Orbites de structures rigides, Integable systems and foliations (Montpellier), Boston, Birkäuser, (1997).
- [6] *Y. Benoist, P. Foulon, F. Labourie*, Flots d'Anosov à distributions stables et instables différentiables, Jour. Amer. Math. Soc., **5**, (1992), 33-74.
- [7] *J. Benveniste, D. Fisher*, Nonexistence of invariant rigid structures and invariant almost rigid structures, Comm. Annal. Geom., **13**(1), (2005), 89-111.
- [8] *J. Benzecri*, Sur les variétés localement affines et localement projectives, Bull. Soc. Math. France, **88**, (1960), 229-332.
- [9] *R. Bryant, G. Manno, V. Matveev*, A solution of a problem of Sophus Lie: Normal forms of 2-dim metrics admitting two projective vector fields, Math. Ann., **340**(2), (2008), 437-463.
- [10] *P. Bueken, L. Vanhecke*, Examples of curvature homogeneous Lorentz metrics, Class. Quant. Grav., **5**, (1997), 93-96.
- [11] *A. Candel, R. Quiroga-Barranco*, Gromov's centralizer theorem, Geom. Dedicata **100**, (2003), 123-155.
- [12] *Y. Carrière, F. Dalbo*, Généralisations du premier théorème de Bieberbach sur les groupes cristallographiques, Enseign. Math., **35**(2), (1989), 245-263.

- [13] *G. D'Ambra, M. Gromov*, Lectures on transformations groups: geometry and dynamics, Surveys in Differential Geometry (Cambridge), (1990), 19-111.
- [14] *S. Dumitrescu*, Dynamique du pseudo-groupe des isométries locales sur une variété lorentzienne analytique de dimension 3, Ergodic Th. Dyn. Systems, **28(4)**, (2008), 1091-1116.
- [15] *I. Egorov*, On the order of the group of motions of spaces with affine connection, Dokl. Akad. Nauk. SSSR, **57**, (1947), 867-870.
- [16] *R. Feres*, Rigid geometric structures and actions of semisimple Lie groups, Rigidité, groupe fondamental et dynamique, Panorama et synthèses, **13**, Soc. Math. France, Paris, (2002).
- [17] *P. Friedbert, F. Tricerri, L. Vanhecke*, Curvature invariants, differential operators and local homogeneity, Trans. Amer. Math. Soc., **348**, (1996), 4643-4652.
- [18] *M. Gromov*, Rigid transformation groups, Géométrie Différentielle, (D. Bernard et Choquet-Bruhat Ed.), Travaux en cours, Hermann, Paris, **33**, (1988), 65-141.
- [19] *A. Kirilov*, Eléments de la théorie des représentations, M.I.R., (1974).
- [20] *S. Kobayashi*, Transformation groupes in differential geometry, Springer-Verlag, (1972).
- [21] *O. Kowalski*, Counter-example to the second Singer's theorem, Ann. Global Anal. Geom., **8(2)**, (1990), 211-214.
- [22] *O. Kowalski, B. Opozda, Z. Vlášek*, A classification of locally homogeneous connections on 2-dimensional manifolds via group-theoretical approach, CEJM, **2(1)**, (2004), 87-102.
- [23] *F. Lastaria & F. Tricceri*, Curvature-orbits and locally homogeneous Riemannian manifolds, Ann. Mat. Pura Appl., **165(4)**, (1993), 121-131.
- [24] *S. Lie*, Theorie der Transformationsgruppen, Math. Ann., **16**, (1880), 441-528.
- [25] *K. Melnick*, Compact Lorentz manifolds with local symmetry, Journal of Diff. Geom., **81(2)**, (2009), 355-390.
- [26] *J. Milnor*, On the existence of a connection of curvature zero, Comment. Math. Helv., **32**, (1958), 215-223.
- [27] *P. Molino*, Riemannian Foliations, Birkhauser, (1988).
- [28] *G. Mostow*, The extensibility of local Lie groups of transformations and groups on surfaces, Ann. of Math., **52(2)**, (1950), 606-636.
- [29] *T. Nagano, K. Yagi*, The affine structures on the real two torus, Osaka J. Math., **11**, (1974), 181-210.
- [30] *K. Nomizu*, On local and global existence of Killing vector fields, Ann. of Math. (2), **72**, (1960), 105-120.
- [31] *P. Olver*, Equivalence, invariants and symmetry, Cambridge Univ. Press., Cambridge, (1995).
- [32] *B. Opozda*, Locally homogeneous affine connections on compact surfaces, Proc. Amer. Math. Soc., **9(132)**, (2004), 2713-2721.
- [33] *R. Sharpe*, Differential Geometry, Cartan's Generalization of Klein's Erlangen Program, Springer, (2000).
- [34] *J. Wolf*, Spaces of constant curvature, McGraw-Hill Series in Higher Math., (1967).
- [35] *A. Zeghib*, Killing fields in compact Lorentz 3-manifolds, J. Differential Geom., **43**, (1996), 859-894.

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