Approximation of a volume-preserving homeomorphism by a volume-preserving diffeomorphism

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The purpose of this note is to prove the following statement, which is "well known" but for which there does not seem to exist a published proof.

Proposition. *Let M be a smooth n-manifold equipped with a smooth volume element. Let φ be a volume-preserving homeomorphism. Assume that φ can be C^0-approximated by smooth diffeomorphisms [this is always true if $n \leq 3$]. Then φ can be C^0-approximated by smooth volume-preserving diffeomorphisms.*

Proof. To simplify and get at the heart of the matter, let us assume that M is compact and oriented, and that φ preserves the smooth oriented volume element $\omega \in \Omega^n(M)$.

Fix any Riemannian metric, and let $\varepsilon > 0$. Let K be a smooth triangulation such that $\mathrm{diam}(\varphi(\sigma)) < \varepsilon$ for every n-simplex $\sigma \in K^{(n)}$. By hypothesis, for any $\eta > 0$ there exists $\psi_1 \in \mathrm{Diff}(M)$ such that $d(\psi_1, \varphi) < \eta$. This implies

$$(*) \qquad \max_{\sigma \in K^{(n)}} |\mathrm{vol}(\psi_1(\sigma)) - \mathrm{vol}(\sigma)| = O(\eta)\ ,\ \eta \to 0.$$

Using an idea of E. Giroux, we choose some maximal tree T in the dual triangulation, and write T as a union of almost-disjoint simplicial paths $P_1, \cdots, P_k$. Thus the n-simplices in all these paths are distinct and are all the n-simplices in $K^{(n)}$. We modify ψ_1 successively on each n-simplex on $P_1, \cdots, P_k$ to obtain ψ_2 such that $\mathrm{vol}(\psi_2(\sigma)) = \mathrm{vol}(\sigma)$ for every σ. If η is sufficiently small, $(*)$ implies that this can be done with the property

$$(**) \qquad (\forall \sigma \in K^{(n)}) \quad d(\psi_2, \varphi) + \mathrm{diam}(\psi_2(\sigma)) < \varepsilon.$$

By construction, the n-form $\alpha := \psi_2^*\omega - \omega$ satisfies $\int_\sigma \alpha = 0$ for every n-simplex. Thus it admits a primitive β which vanishes on $K^{(n-1)}$. Indeed, by [ST], Lemma 2 p.148 [p.165 in the Springer edition] it has a primitive β_1 such that $\int_\tau \beta_1 = 0$ for every $\tau \in K^{(n-1)}$; thus β_1 is exact on $K^{(n-1)}$, thus there exists $\gamma \in \Omega^{n-2}(M)$ such that $d\gamma = \beta_1$ on $K^{(n-1)}$, and one can take $\beta = \beta_1 - d\gamma$.

Applying Moser's method to the path $\omega_t = \omega + td\beta$ provides a time-dependent vector field X_t such that $\omega_t(X_t, .) = -\beta$, whose integration gives an isotopy φ_t with $\varphi_0 = \mathrm{Id}$, $\varphi_t^*\omega_t = \omega$ so that $\psi = \psi_2 \circ \varphi_1$ preserves ω. Since $\beta | K^{(n-1)} = 0$, we have $\varphi_t | K^{(n-1)} = \mathrm{Id}$, thus $\psi | K^{(n-1)} = \psi_2$. Finally, $(**)$ implies $d(\psi, \varphi) < \varepsilon$, which finishes the proof.

[ST] I.M. Singer, J.A. Thorpe, *Lecture notes on elementary topology and geometry*, Scott Foresman 1967. Republished in Springer Undergraduate Texts.

Remark (added on October 31, 2007). In fact, there exists a proof of this approximation result in a 2006 preprint of Yong-Geun Oh: *"C^0-coerciveness of Moser's problem and smoothing area preserving homeomorphims"*, http://xxx.lanl.gov/abs/math/0601183. I thank Stefan Mueller for making me aware of this reference.

Oh's proof is more sophisticated, using in particular a key analytical estimate due to Dacorogna and Moser. Thus I believe that our proof is of independent interest.