# Pseudo-Parallel submanifolds

by

MOHAMED BELKHELFA

ABSTRACT. Pseudo-parallel (shortly PP) submanifolds are defined as a generalization of semi-parallel (shortly SP ) submanifolds and as extrinsic analogue of pseudo-symmetric (shortly PS) manifolds (in the sense of R. Deszcz) [1], [26]. Asperti et al [1] obtained a description of PP hypersurfaces in space form as quasi-umbilic hypersurfaces or cyclids of Dupin and studied PP surfaces with maximal first normal bundle in space form and classified complete simply connected PP submanifolds in space. In the last years many results obtained about a class of surfaces in Euclidean space, pseudo-parallel hypersurfaces in pseudo-Riemannian space forms and pseudo-parallel real hypersurfaces in complex space forms, appeared in [19], [15], [16] respectively. In this paper we study the PP hypersurfaces in Riemannian manifold not necessarily a space form; Moreover we give a description of semi-parallel surfaces in BCV spaces.

*Mathematics Subject classification:* 53C40 53C35.

*Keywords:* Submanifold, symmetry, semi-symmetry, paseudo-symmetry, parallel, semi-parallel, pseudo-parallel.

*Acknowledgment* The author thanks the organizing committee for the invitation, hospitality and generosity.

## 1 Introduction

Let $(M, g)$ be a n-dimensional, $n \geq 3$, semi-Riemannian connected manifold of class $\mathcal{C}^\infty$. We denote by $\nabla$, $S$, and $\kappa$ the Levi-Civita connection, the Ricci tensor, and the scalar curvature of $(M, g)$, respectively. A $(0, k)$-tensor field $T$ on $(M, g)$ is called parallel when it is invariant under parallel translation, i.e. when

$$\nabla T = 0,$$

in particular, if the $(0,4)$-Riemann-Christoffel curvature tensor $R$ is parallel, i.e.

$$\nabla R = 0,$$

then $M$ is said to be locally symmetric.

In the following, by $R$ we will also denote the curvature operator such that:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

As is well known, every $(1,1)$-tensor field $A$ on a differential manifold determines a derivation $A\cdot$ of the tensor algebra on this manifold, commuting with contractions. In particular, the anti-symmetric $(1,1)$-tensor field $R(X,Y)$ induces derivations $R(X,Y)\cdot$, thus associating with a $(0,k)$-tensor field $T$, the $(0,2+k)$-tensor $R \cdot T$ defined by:

$$\begin{aligned} (R \cdot T)(X_1, X_2, \ldots, X_k; X, Y) &= (R(X,Y) \cdot T)(X_1, X_2, \ldots, X_k) \\ &= -T(R(X,Y)X_1, X_2, \ldots, X_k) - \cdots \\ &\quad -T(X_1, X_2, \ldots, R(X,Y)X_k). \end{aligned}$$

One has

$$R(X,Y) \cdot T = \nabla_X(\nabla_Y T) - \nabla_Y(\nabla_X T) - \nabla_{[X,Y]} T.$$

By $R \cdot R$ we denote in this way the $(0,6)$ -tensor field obtained by the derivation of the second $R$, the $(0,4)$- curvature tensor, using the curvature operator, the first $R$. If

$$R \cdot R = 0, \tag{1}$$

then $M$ is called a semi-symmetric space [20], [22]. It was already recognized by E. Cartan that all locally symmetric spaces and all 2-dimensional Riemannian space fulfill the condition (1). When talking about semi-symmetry, we will always assume $n \geq 3$. The condition of semi-symmetry for hypersurfaces $M$ of Euclidean spaces $\mathbb{E}^{n+1}$ was first studied by K. Nomizu in [18]. He asked the question if there exist complete irreducible and simply connected Riemannian manifolds of dimension $n \geq 3$ satisfying $R \cdot R = 0$ and which are not locally symmetric. In 1972, R. Takagi [24] and later also K. Sekigawa produced such examples, as hypersurfaces in Euclidean spaces. In 1982-84 Z. Szabó obtained a full intrinsic classification of semi-symmetric Riemannian manifolds. And soon after that Z. Szabó and J. Deprez, independently, obtained the classification of all semi-symmetric hypersurfaces in Euclidean spaces [9], [21].

Locally, each semi-symmetric space belongs to one of the following three classes:

(1) the trivial class consisting of all locally symmetric spaces and all 2-dimensional Riemannian manifolds,

(2) the exceptional class of all elliptic, hyperbolic, Euclidean and Kahlerian cones,

(3) the typical class of all Riemannian manifolds foliated by Euclidean leaves of codimension two [20], [22].

## 2 Pseudo-symmetry

To state the definition of pseudo-symmetry in Deszcz sense, we first determine a $(0, k+2)$-tensor $Q(g,T)$ associated with any $(0,k)$- tensor field $T$ on a semi-Riemannian manifold $M$ by:

$$\begin{aligned} Q(g,T)(X_1, X_2, \dots, X_k; X, Y) &= ((X \wedge_g Y) \cdot T)(X_1, X_2, .., X_k) \\ &= -T((X \wedge_g Y)X_1, X_2, .., X_k) - \\ &\quad -T(X_1, X_2, \dots, X_{k-1}, (X \wedge_g Y)X_k)), \end{aligned}$$

where $X \wedge_g Y$ denotes the endomorphism given by:

$$X \wedge_g Y = g(Y,Z)X - g(X,Z)Y.$$

A semi-Riemannian manifold $M^n = (M^n, g), n \geq 3$, is said to be pseudo-symmetric in the sense of R. Deszcz if the $(0,6)$-tensor fields $R \cdot R$ and $Q(g,R)$ on $M$ are linearly dependent, i.e., if there exists a function

$$L_R : M \longrightarrow \mathbb{R}$$

such that

$$R \cdot R = L_R \, Q(g,R) \tag{2}$$

holds on $U_R = \{x \in M \,|\, R - \frac{\kappa}{n(n-1)} G \neq 0 \text{ at } x\}$, where $G$ is the $(0,4)$-tensor of $(M,g)$ defined by

$$G(X_1, X_2, X_3, X_4) = g((X_1 \wedge_g X_2)X_3, X_4)$$

for all $X_1, X_2, X_3, X_4$ vector fields of $M$ [3], [10], [26].

Clearly, every semi-symmetric manifolds is pseudo-symmetric. The converse is not true; there are pseudo-symmetric manifolds which are not semi-symmetric. For instance, the warped product $S^p \times_F S^{n-p}, p \geq 2, n-p \geq 1$, of the standard spheres $S^p$ and $S^{n-p}$ with some warping function $F$ and Schwarzschild space time are a non semi-symmetric pseudo-symmetric manifold [10]. We mention also that certain spacetimes are pseudo-symmetric, among others: the Robertson-Walker spacetimes and generalized Robertson-Walker spacetimes, the Kottler spacetimes as well as the Riessner-Nordstrom spacetimes [7], [11]. The condition of pseudo-symmetry arose during the study of totally umbilical submanifolds of semi-symmetric manifolds, and also arose during the consideration of geodesic mapping onto semi-symmetric manifolds [3], [10], [26].

It is known that every totally umbilical submanifold, with parallel mean curvature vector field is also pseudo-symmetric (or more generally, in a pseudo-symmetric) manifold is also pseudo-symmetric. In particulair, extrinsic spheres in pseudo-symmetricRiemannian manifold are also pseudo-symmetric. Further, if a semi-Riemannian manifold $(M,g)$admits a geodesic mapping (A diffeomorphism which maps geodesic lines into geodesic lines) onto a semi-symmetric (or more generally, in a pseudo-symmetric) manifold then $(M,g)$ is pseudo-symmetric.

*Remark* 2.1. [10]

$$Q(g,R) = 0 \Longleftrightarrow R = \frac{\kappa}{n(n-1)} G$$

If at a point $x \in M$, we have $R = \frac{\kappa}{n(n-1)} G$ then $R \cdot R = 0$ holds at $x$. Thus the pseudosymmetry condition is trivially fulfilled on $M - U_R$

A hypersurface $M$ of $\mathbb{E}^{n+1}$ is pseudo-symmetric if and only if $M$ is semi-symmetric or has at most two distinct principal curvatures [10], [26].

# 3 Extrinsic notions of pseudo-symmetry

Let $M$ be a submanifold of Riemannian manifold $N$, when $N$ is a space form (i.e. With constant sectional curvature c) will be denote by $N(c)$ and let $h$ be the second fundamental form of $M$.

The extrinsic analogous classes of symmetric, semi-symmetric and pseudo-symmetric respectively are: symmetric or parallel submanifold ($\nabla h = 0$), semi-parallel submanifold ($R \cdot h = 0$) and pseudo-parallel submanifold ($R \cdot h = L_h \, Q(g, h)$). For parallel and semi-parallel submanifolds and others generalizations see [17].

The basic Gauss, Codazzi-Mainardi and Ricci equations give that the extrinsic conditions Parallel, semi-parallel and pseudo-parallel imply the correspondent intrinsic conditions symmetry, semi-symmetry and pseudo-symmetry respectively see [1]

**Proposition 3.1** *Let $M^n$ be a hypersurface of Riemannian manifold $(N^{n+1}, g)$ and let $\xi$ a unit normal vector and $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis of the tangent space of $M^n$ which diagonalizes the Weingarten operator $A_\xi$. If $M^n$ is pseudo-parallel then:*

$$[K(e_i, e_j) + \lambda_i \lambda_j - L_h](\lambda_j - \lambda_i) = 0, \tag{3}$$

*for all $i \neq j$*

*Where $\lambda_i = g(A_\xi e_i, e_i)$, $K(e_i, e_j) = g(R(e_i, e_j)e_j, e_i)$ and $R$ is the Riemannian curvature of $N^{n+1}$*

*Proof.* Let $\lambda_i = g(A_\xi e_i, e_i)$ be the principal curvature in the direction of $\xi$, following the proof in [1] and using the Gauss equation we get the equation 3 ☐

**Corollary 3.2** *If $(N^{n+1}, g)$ is a space form with constant curvature c and $M^n$ is PP we get*

$$[c + \lambda_i \lambda_j - L_h](\lambda_j - \lambda_i) = 0, \tag{4}$$

*for all $i \neq j$*

Which is the proposition 3.1 [1] up sign in the second term, because they used the wedge product $(X \wedge_g Y)Z = g(X, Z)Y - g(Y, Z)X$.

*Remark* 3.1. The work is in progress for the hypersurface in Sasakian space form.

The condition 3 is also sufficient for PP.

*Remark* 3.2. For $n = 2$. $L_h$ is the Gauss curvature; So every surface is PP. Which is not the case for SP surfaces.

In [4] We classified parallel surfaces in BCV spaces. The following 2-parameter family of Riemannian metrics.

$$g_{l,m} = \frac{dx^2+dy^2}{\{1+m(x^2+y^2)\}^2} + \left(dz + \frac{l}{2}\frac{ydx-xdy}{1+m(x^2+y^2)}\right)^2, \ \ l,\ m \in \mathbb{R}. \tag{5}$$

is called the *Bianchi-Cartan-Vranceanu metrics*. The metrics as above are defined over the whole 3-space $\mathbb{R}^3$ for $m \geq 0$ and over the region $x^2+y^2 < -1/m$ for $m < 0$. We shall call the homogeneous Riemannian 3-manifolds $M^3 = (M^3, g_{l,m})$ defined by

$$M^3 = \left(\{(x,y,z) \in \mathbb{R}^3 \mid 1+m(x^2+y^2) > 0\}, g_{l,m}\right)$$

the *Bianchi-Cartan-Vranceanu spaces* (*BCV-spaces* in short)[4].

**Theorem 3.3** *[4] Let $(M^3, g_{l,m})$ be the BCV-space with $4m - l^2 \neq 0$.*

*(1) If $l \neq 0$ : then the only parallel surfaces in $(M^3, g_{l,m})$ are Hopf cylinders over Riemannian circles in $\tilde{M}^2$.*

*(2) If $l = 0$ : then the only parallel surfaces in Riemannian symmetric space $M^3_{0,m}$ with $m \neq 0$ are totally geodesic leaves and Hopf cylinders over Riemannian circles in $\tilde{M}^2$.*

*Here Riemannian circles in $\tilde{M}^2$ are curves in $\tilde{M}^2$ with constant geodesic curvature, where*

$$\tilde{M}^2 = \left(\{(x,y) \in \mathbb{R}^2 \mid 1+m(x^2+y^2) > 0\ \}, \frac{dx^2+dy^2}{\{1+m(x^2+y^2)\}^2}\right).$$

Let $\tilde{\gamma}$ be a curve parametrised by arclength in $M^3/\xi$ with curvature $\tilde{\kappa}$. Then taking the inverse image $S := \pi^{-1}(\gamma)$ of $\tilde{\gamma}$ in $M^3$. We shall call this surface $S$ the *Hopf cylinder* (or *Boothby-Wang cylinder*) over $\tilde{\gamma}$, where $\xi = \frac{\partial}{\partial z}$.

Take an orthonormal frame field $\mathscr{E} = (e_1, e_2, e_3)$:

$$e_1 = \{1+m(x^2+y^2)\}\frac{\partial}{\partial x} - \frac{ly}{2}\frac{\partial}{\partial z}, \ \ e_2 = \{1+m(x^2+y^2)\}\frac{\partial}{\partial y} + \frac{lx}{2}\frac{\partial}{\partial z},$$

$$e_3 = \frac{\partial}{\partial z}.$$

Then the dual coframe field $\vartheta = (\theta^1, \theta^2, \theta^3)$ is given by

$$\theta^1 = \frac{dx}{1+m(x^2+y^2)}, \ \ \theta^2 = \frac{dy}{1+m(x^2+y^2)}, \ \ \theta^3 = dz + \frac{\lambda}{2}\frac{ydx-xdy}{1+m(x^2+y^2)}.$$

Note that the one-form $\eta := \theta^3$ is a contact form on $\mathfrak{M}^3$ if and only if $l \neq 0$. The Levi-Civita connection $\bar{\nabla}$ of $\mathfrak{M}$ is described by the formulae:

$$\bar{\nabla}_{e_1}e_1 = 2mye_2, \ \ \bar{\nabla}_{e_1}e_2 = -2mye_1 + \frac{l}{2}e_3, \ \ \bar{\nabla}_{e_1}e_3 = -\frac{l}{2}e_2,$$

$$\bar{\nabla}_{e_2}e_1 = -2mxe_2 - \frac{l}{2}e_3, \ \ \bar{\nabla}_{e_2}e_2 = 2mxe_1, \ \ \bar{\nabla}_{e_2}e_3 = \frac{l}{2}e_1, \tag{6}$$

$$\bar{\nabla}_{e_3}e_1 = -\frac{l}{2}e_2,\ \ \bar{\nabla}_{e_3}e_2 = \frac{l}{2}e_1,\ \ \bar{\nabla}_{e_3}e_3 = 0.$$

$$[e_1, e_2] = -2mye_1 + 2mxe_2 + le_3,\ \ [e_2, e_3] = [e_3, e_1] = 0. \tag{7}$$

the curvature tensor $\bar{R}$ is described by the formula:

$$\bar{R}_{1212} = 4m - \frac{3}{4}l^2,\ \ \bar{R}_{1313} = \bar{R}_{2323} = \frac{l^2}{4}. \tag{8}$$

The Ricci tensor $\overline{Ric}$ of $\mathscr{M}$ is given by

$$\bar{R}_{11} = \bar{R}_{22} = 4m - l^2,\ \ \bar{R}_{33} = \frac{l^2}{2}.$$

Hence the scalar curvature $\bar{s}$ is $\bar{s} = 8m - l^2/2$.

*Remark* 3.3. The geodesics of $\mathfrak{M}^3$ are calculated by Sitzia [23], see also [5], [2], for geometric properties of this metric.

*Proof.* Let $\{X_1, X_2, X_3\}$ be an orthonormal frame field of $S$ such that $X_3 = N$. Denote by $\{\omega^1, \omega^2, \omega^3\}$ the dual coframe field to $\{X_i\}$. We can write $\omega^3 = p\theta^1 + q\theta^2 + r\theta^3$. Since $\{\omega^i\}$ is orthonormal, $p^2 + q^2 + r^2 = 1$. The vector fields $u_1 = re_1 - pe_3,\ u_2 = re_2 - qe_3$ are tangent to $S$. In particular if $r \neq 0$, $\{u_1, u_2\}$ is a (local) frame field on $S$. By direct computations using (3.4) we obtain the following:

**Lemma 3.4** *The normal components of curvature tensor $\bar{R}$ are described by*

$$(\bar{R}(u_1, u_2)u_1)^{\perp} = qr^3(l^2 - 4m)N,\ \ (\bar{R}(u_1, u_2)u_1)^{\perp} = -pr^3(l^2 - 4m)N.$$

Assume that the second fundamental form $h$ of $S$ is parallel. Then the Codazzi equation implies:

$$(\bar{R}(u_1, u_2)u_1)^{\perp} = (\nabla_{u_1}h)(u_2, u_1) - (\nabla_{u_1}h)(u_1, u_1) = 0. \tag{9}$$

Similarly we have

$$(\bar{R}(u_2, u_1)u_2)^{\perp} = 0. \tag{10}$$

First we consider the case $r \neq 0$.

**Case I** $r \neq 0$.

The preceding lemma implies $p = q = 0$, since we assumed $\lambda^2 - 4\mu \neq 0$. Hence $\omega^3 = \theta^3$. Namely $S$ is an integral surface of the distribution $D$ defined by $\eta = 0$.

**Subcase (i)** $l = 0$: in this case $S$ is an integral surface of the distribution $D$ defined by $dz = 0$. Hence $S$ is a leaf $\tilde{\mathfrak{M}}^2 \times \{z_0\}$ for some $z_0$. Note that this leaf is totally geodesic.

**Subcase (ii)** $l \neq 0$: since $\eta$ in this case is a contact form, there are no integral surfaces of $D$. This implies that this case cannot occur.

**Case II** $r = 0$.

In this case $\omega^3$ has the form $\omega^3 = p\theta^1 + q\theta^2$. Since $p^2 + q^2 = 1$, we may write $p = \cos\varphi,\ q = \sin\varphi$. In addition, the orthonormal vector fields $v_1 = qe_1 - pe_2,\ v_2 = e_3$ give an orthonormal frame field tangent to $S$.

Next by using $(3.4)$, we have

$$\bar{\nabla}_{v_1} v_1 = v_1(q)e_1 - v_1(p)e_2 + 2m(px + qy)N,\ \ \bar{\nabla}_{v_1} v_2 = -\frac{l}{2}N,$$

$$\bar{\nabla}_{v_2} v_1 = -\frac{l}{2}N + (v_2(q)e_1 - v_2(p)e_2),\ \ \bar{\nabla}_{v_2} v_2 = 0.$$

By the symmetry of second fundamental form, we get $pv_2(q) - qv_2(p) = 0$. Equivalently we have $v_2(\varphi) = 0$.

By the Gauss formula we obtain the induced connection and the second fundamental form:

$$\nabla_{v_1} v_1 = 0,\ \ \nabla_{v_1} v_2 = 0,\ \ \nabla_{v_2} v_2 = 0, \tag{11}$$

$$h(v_1, v_1) = \{v_1(\varphi) + 2m(px + qy)\}N,\ \ h(v_1, v_2) = -\frac{l}{2}N,\ \ h(v_2, v_2) = 0. \tag{12}$$

The mean curvature vector field $\mathbb{H}$ of $S$ defined by

$$\mathbb{H} = \frac{1}{2}\mathrm{tr}\, h \tag{13}$$

is computed as

$$\mathbb{H} = HN,\ \ 2H = v_1(\varphi) + 2m(px + qy).$$

Since $h$ is parallel, the mean curvature $H$ is constant. Furthermore $(3.9)$ implies that $S$ is flat. Moreover $(3.9)$ implies that we can take $z$ as a (local) coordinate of $S$ and hence there exists a local coordinate system $(t, z)$ such that

$$\frac{\partial}{\partial t} = v_1.$$

Thus the equation $v_2(\varphi) = 0$ implies that $\varphi$ depends only on $t$. With respect to this coordinate system $(t, z)$, the constancy of $H$ is rewritten as

$$\frac{d\varphi}{dt} = 2\{H - m(\cos\varphi(t)x(t) + \sin\varphi(t)y(t))\},\ \ H \in \mathbb{R}. \tag{14}$$

It is easy to see that $S$ is generated by two coordinate curves. The $z$-coordinate curves are geodesics and integral curves of the vector field $e_3$. The $t$-coordinate curves are horizontal curves of curvature $2H$. Let us denote by $\gamma(t)$ the $t$-coordinate curve in $\mathfrak{M}^3$. Then it is easy to see that $\gamma$ is a Frenet curve of osculating order at most 3. The principal normal vector field of $\gamma$ is the restriction of $N$ on $\gamma$. The binormal vector field $B$ of $\gamma$ is $\pm e_3$. We can assume $B = e_3$. The Frenet-Serret formulas of $\gamma$ are given by

$$\bar{\nabla}_{\gamma'}(T, N, B) = (T, N, B)\begin{pmatrix} 0 & -2H & 0 \\ 2H & 0 & -\frac{l}{2} \\ 0 & \frac{l}{2} & 0 \end{pmatrix} \tag{15}$$

This implies that $t$-coordinate curves are of constant torsion $l/2$. Hence $t$-coordinate curves are horizontal curves of curvature $2H$. Let us denote by $\tilde{\gamma}(t) = (x(t), y(t))$ the projection of $\gamma$ to $\tilde{M}^2$, namely $\tilde{\gamma} = \pi \circ \gamma$. One can check that $\tilde{\gamma}(t)$ is of constant curvature $2H$. Equivalently $t$-coordinate curves are horizontal lifts of Riemannian circles of curvature $2H$ in $\tilde{M}^2$. (In particular the case $H = 0$, $\tilde{\gamma}$ is a geodesic in $\tilde{M}^2$.)

Recall the surface $S$ is parameterized by $t$ and $z$. And the $t$-coordinate curves are horizontal lifts of Riemannian circles in the base space with curvature $2H$. Thus we conclude that $S$ is a Hopf cylinder over a Riemannian circle of curvature $2H$. In particular, $S$ is parameterized by $(x(t), y(t), z)$. We finish this paper by considering these Riemannian circles more in detail.

From the definition of $v_1$, we notice that

$$\frac{dx}{dt} = \sin\varphi(t) f(t), \quad \frac{dy}{dt} = -\cos\varphi(t) f(t), \quad f(t) := 1 + \mu(x(t)^2 + y(t)^2). \tag{16}$$

Here $\varphi(t)$ is the solution to (3.12). The equation (3.14) says $t$ is the arc length parameter of $\tilde{\gamma}$.

The function $f(t)$ satisfies the following ordinary differential equation:

$$\frac{d}{dt} \log |f(t)| = 2\mu\{x(t)\sin\varphi(t) - y(t)\cos\varphi(t)\}. \tag{17}$$

On the other hand, differentiating (3.12),

$$\frac{d^2\varphi}{dt^2}(t) = \frac{d\varphi}{dt}(t)\frac{d}{dt}\log|f(t)|. \tag{18}$$

If $\varphi' = 0$, then from (3.14) we obtain that $\tilde{\gamma}$ is a straight line. So we assume that $\varphi' \neq 0$. Comparing (3.15) and (3.16), we have

$$f(t) = \alpha\varphi'(t), \quad \alpha \in \mathbb{R}.$$

Hence we obtain the following explicit expression for $\tilde{\gamma}$:

$$(x(t), y(t)) = (-\alpha\cos\varphi(t) + x_0, -\alpha\sin\varphi(t) + y_0). \tag{19}$$

Therefore the curve is a circle centered at $(x_0, y_0)$. ☐

Therefore it is interesting to study SP in some special manifolds, namely BCV spaces we got

**Theorem 3.5** *Let $M^2$ be a surface of Riemannian manifold $(N^3, g_{l,m})$ (BCV spaces) with $m \neq 0$ and $4m - l^2 \neq 0$.*

*Then $M^2$ is properly SP (non parallel) if and only if it is flat.*

*Proof.* The Corollary 8.4 [4] implies the nonexistence of totally geodesic surfaces and *extrinsic spheres–i.e.,* totally umbilical surfaces with parallel mean curvature vector fields, in contact 3-space forms with $c \neq 1$. Therefore using the corollary and suppose that the surface is semi-parallel we show that the surface is flat. ☐

*Example* 4. [14] Let $f$ be a linear function of $x$ and $y$. Then it is easy to see that the graph $r$ of $f$ is a plane and is minimal in $(N^3, g_{1,0})$ . However planes are not flat. In fact the Gaussian curvature is given by

$$K = \frac{3 + 2(b - x/2)^2 + 2(a + y/2)^2}{4\{1 + 2(b - x/2)^2 + (a + y/2)^2\}^2} > 0$$

for $f(x, y) = ax + by + c,\ a, b, c \in \mathbb{R}$. Which is not semi-parallel. We know that every surface is semi-symmetric, this shows that semi-symmetric do not imply semi-parallel.

*Example* 5. Let $(f(u), g(u))$ be a regular curve parametrised by arclength in $xz$-plane such that
$f(u) > 0$. The surface of revolution with profile curve $(f(u), g(u)) = (u, \frac{1}{6}\sqrt{(u^2+3)^3}$ is a flat surface in $(N^3, g_{1,0})$ [6] then it is SP surface.

*Example* 6. Let $(x(u), y(u))$ be a curve in the $xy-$plane parametrised by arclength $u$. The cylinder over a curve $(x(u), y(u))$ is $M(u, v) = (x(u), y(u), v)$, its mean curvature is $H = \frac{1}{2}\{\ddot{x}(u)\dot{y}(u) - \dot{x}(u)\ddot{y}(u)\} = -\frac{\kappa(u)}{2}$, where $\kappa(u)$ is the curvature of the curve $(x(u), y(u))$, and $\ddot{x}(u), \dot{x}(u)$ denote the second and the first derivative respectively [4]. We remark that if $\kappa(u)$ is non constant then the cylinder in $(N^3, g_{1,0})$ is SP non parallel.

*Remark* 3.4. (1) For the geometric interpretation of Pseudo symmetry see [13]

(2) J.Van der veken proved in his PhD thesis that ”A surface immersed in an arbitrary three-dimensional Riemannian manifold is semi-parallel if and only if it is flat or totally umbilical.” [25].

# References

[1] A. C. Asperti,G. A. Lobos and F. Mercuri, Pseuo-parallel submanifols of a space form, Adv. Geom. 2 (2002),57-71.

[2] M. Belkhelfa, R. Deszcz, and L. Verstraelen, Pseudosymmetry of 3-dimensional D’Atri space, Dept. Math., Agricultural Univ. Wrocław, Ser. A, Theory and Methods, Report No. **86** (2000).

[3] M. Belkhelfa, R. Deszcz, M. Gl ogowska, M. Hotlos, D.Kowalczyk and L. Verstraelen, On some type of curvature conditions, Banach Center Publications 57 (2002) 179-194.

[4] M. Belkhelfa, F. Dillen and J. Inoguchi, Surfaces with parallel second fundamental form in Bianchi-Cartan- Vrănceanu-spaces, to Banach Center Publications 57 (2002) 57-87.

[5] F. Borghero and R. Caddeo, Une structure de Séparabilité et Géodésiques dans les Huit Géométries Tridimensionalles de Thurston, Rend. Mat. Appl. (7) **9**, (1989), 607-624.

[6] R. Caddeo, P. Piu and A. Ratto Rotational surfaces in $H_3$ with constant Gauss curvature, Boll. Unione Mat. Ital., VII. Ser., B 10, **2**, (1996), 341-357.

[7] F. Defever, R. Deszcz, L. Verstraelen and L. Vrancken, On pseudo-symmetric space-times, J. Math. Phys. **35** (1994), 5908–5921.

[8] J. T. Cho and J. Inoguchi Pseudo-symmetric contact 3-manifolds, J. Korean Math. Soc. **42** (2005), 913–932.

[9] J. Deprez, Semi-parallel hypersurfaces, Rend. Semin. Mat., Torino **44** (1986), 303-316.

[10] R. Deszcz, On pseudo-symmetric spaces, Bull. Soc. Math. Belg, **44** (1992), 1–34.

[11] R. Deszcz, L. Verstraelen and L. Vrancken, On the symmetry of the warped product, Gen. Relativity Gravitation, **23** (1991), 671–681.

[12] D. Ferus, Symmetric submanifolds of Euclidean space, Math. Ann, **247** (1980), 81-93.

[13] S. Haesen and L. Verstraelen, Properties of a scalar curvature invariant de- pending on two planes, Manuscripta Math. 122 (2007), 59-72.

[14] J. Inoguchi, T. Kumamoto, N. Ohsugi and Y. Suyama, Differential Geometry of curves and surfaces in 3-dimensional homogeneous spaces II, Fukuoka Univ. Sci. Rep., **30** (2000), 17-47.

[15] G. A. Lobos, Pseudo-parallel Hypersurfaces in pseuod-Riemannian space forms, Publ. R. Soc.Mat. Esp. **6**,(2004), 229-234 . space forms, , (2004).

[16] G. A. Lobos and M. Ortegae, Pseudo-parallel real Hypersurfaces in complex space forms, Bull. Korean. Math. Soc, 41, No.4, (2004), 609-618.

[17] Ülo. Lumiste, Submanifolds with parallel fundamental form, in : Handbook of Differential Geometry, Amsterdam: North-Holland, **I** ( 2000), 779-906.

[18] K. Nomizu, On hypersurfaces satisfying a certain condition on the curvature tensor, Tôhoku Math. J, **20** (1968), 46-59.

[19] C. Özgür, K. Arslan and C. Murathan, On Class of surfaces in the Euclidean space, Commun. Fac. Univ.Ank. Series A1, Vol. **51**. No 1 (2002), 47-54

[20] Z.I. Szabó, Structure theorems on Riemannian spaces satisfying $R(X,Y)\cdot R=0$, I. The local version, J. Diff. Geom. **17** (1982), 531-582.

[21] Z.I. Szabó, Classification and construction of complete hypersurfaces satisfying $R(X,Y)R=0$. Acta Sci. Math. **47** (1984), 321-348

[22] Z.I. Szabó, Structure theorems on Riemannian spaces satisfying $R(X,Y)\cdot R=0$, II. Global versions, Geom. Dedicata **19** (1985), 65-108.

[23] P. Sitzia, Explicit formulas for geodesics of three-dimensional homogeneous manifolds with isometry group of dimension 4 or 6, preprint, Univ. Cagliari, (1989).

[24] H. Takagi, An example of a Riemannian manifold satisfying $R(X,Y)\cdot R=0$ but not $\nabla R=0$, Tôhoku Math. J. **24** (1972), 105-108.

[25] J. Van der veken, Submanifolds of homogeneous spaces, PhD thesis, KULeuven 2007.

[26] L. Verstraelen, Comments on pseudo-symmetry in the sense of Ryszard Deszcz, Geometry and Topology of Submanifolds **VI**, World Sci. Publ. (1994), 199-209.

Mohamed Belkhalfa
Laboratoire de Physique Quantique de la Matière et Modélisations Mathématiques
Centre Universitaire de Mascara
Route de Mamounia, Mascara 29000, Algérie
e-mail: Belkhelfa@univ-mascara.dz, Mohamed.Belkhelfa@gmail.com