# On Lorentz Dynamics

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1. **The question:**

— A subquestion: Homogeneous (non-proper) spacetimes

— A super-question: $Diff(M)$-action on the space of Lorentz metrics

— Some motivations

— Conformal case

2.**Examples:**

— Warped products

- Non-compact cases: Constant curvature spacetimes
- Compact cases:

— Flat cases: tori, $SOL$

— $SL(2, R) = AdS_3$,

— Warped Heisenberg groups,

— Non-homogeneous examples

3. **Results: compact case** (answer to the subquestion...)

4. **Results: non-compact case** (partial answers)

5. **Results:: Super-question** (the 2-dimensionnal case

**The Question:**

$(M, g)$ a Lorentz manifold,

$G = \mathrm{Isom}(M, g)$

**Question** When is the action of $G$ on $M$ **essential**?

$\Longleftrightarrow$

When the $G$-action can not preserve a Riemanniann metric on $M$?

$\Longleftrightarrow$

When is the action of $G$ on $M$ **non proper** ?

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The $G$ action is proper if: $\forall\ K \subset M$ compact, the set

$$G_K = \{g \in G, gK \cap K \neq \emptyset\}$$

is compact

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The compact case:

*When is the isometry group of a* **compact** *Lorentz manifold is* **non-compact***?*

**The sub-question:**

**Homogeneous space:** $M = G/H$, $G$ a Lie group, and $H$ a closed subgroup of $G$.

– We suppose everywhere that $G$ acts faithfully on $M$, i.e. we can not simplify $G/H$ to a smaller $G'/H'$

– $G$ acts on the left on $M$: $(g, xH) \in G \times M \to (gx)H \in M$

– This action preserves some "rigid geometric structure".

– The homogeneous space is of Riemannian type (resp. Lorentzian...) if the $G$-action preserves a Riemannian (resp. Lorentz...) metric on $M$.

– Stabilizer $(1.H) = Ad(H) \subset Ad(G) \subset GL(\mathcal{G})$, $\mathcal{G} =$ Lie algebra of $G$

**Remark** The $G$-action is of Riemannian type $\iff$ the action is proper $\iff$ $\bar{(}Ad(H))$ is compact

(in general $\iff$ $H$ is compact)

**Sub-question** Classify $M = G/H$ of Lorentzian type (i.e. the $G$-action preserves some Lorentz metric on $G/H$), with $H$ non compact.

**The super-question:** $Diff(M)$-action on the space of Lorentz metrics

$Diff^k(M)$ acts on $Lor^{k-1}(M) =$ space of $C^{k-1}$ Lorentz metrics on $M$.

Endow them with the Banach or Frechet topology (for $k = \infty$)

— It is known that $Diff(M)$ acts properly on $Rie(M)$ (space of Riemannian metrics).

— The quotient $Riem(M)/Diff(M)$ is Hausdorff = modular space.

— A function on $M$ is a Riemannian invariant.

QUESTION: When is the $Diff(M)$-action on $Lor(M)$ proper?

— If $g \in Lor(M)$, Stabilizer($g$) = Isom($g$)

— The $Diff(M)$-action proper $\Longrightarrow \forall\ g \in Lor(M)$, Isom($g$) is proper. (i.e. the super-question $\Longrightarrow$ the question).

– Gromov: the difficulty in the global studying of Lorentz manifolds lies in the fact that $Lor(M)/Diff(M)$ does not exist (as a Hausdorff space).

Some motivations:

**1. For the sub-question:**

• The homogeneous Riemannian problem (trivial for our talk here):

• We know very few about non-Riemannian homogeneous space.

– The interest of the Lorentz case: it is the easiest Non-Riemannian problem.

• The homogeneous compact Lorentz problem: Find $G$ a Lie group, and $H$ a closed Lie subgroup of $G$, such:

C1. The action of $G$ on $G/H$ preserves a Lorentz metric.

C2. $M = G/H$ is compact.

**Fact**

1. If $H$ is **discrete**, then:

– C1 is equivalent to that the Lie algebra $\mathcal{G}$ has an $Ad(H)$- Lorentz scalar product.

– C2 means (by definition) that $H$ is a co-compact lattice in $G$.

Explanation: Left translate to $G$ a Lorentz scalar product on $\mathcal{G}$ which is $Ad(H)$-invariant. The Lorentz metric on $G$ is: $G$-left-invariant, and $H$-right invariant. Therefore, it passes to a $G$-invariant Lorentz metric on $G/H$.

(exercise: Where have we used discreetness of $H$?)

## For the question

Conformal groups of Riemannian manifolds:
$(M, g)$ Riemannian manifold,
A priori, Conf$(M, g)$ does not preserve a metric.

**Lichnerowitch conjecture** solved by **Lelong-Ferrand and Obatta**: This happens only for the usual spheres and Euclidean spaces.

**Remark** There are analogous conjectures in geometric dynamics...

## Examples: general constructions

- Products

$N$ Lorentz, with Isom$(N)$ essential, $\Longrightarrow M = N \times L$ has an essential isometry group.

- Local products:

$\tilde{N}$ Lorentz $M = \tilde{N} \times \tilde{L}/\Gamma$, where $\Gamma$ is a **non split** subgroup of $Isom(\tilde{L}) \times Isom(\tilde{L})$

If the centralizer of $\Gamma$ acts non-properly on $\tilde{N} \times \tilde{L}$, then $M$ is essential.

- Same examples for the homogeneous Lorentz problem.

- The warped product construction (generalization of direct products):

$(L, h)$ Riemannian manifolds

$(N, g)$ Lorentz

$w : L \to R^+$ a (warping) function.

The warped product $M = L \times_w N$, is the topological product $L \times N$, endowed with the metric $h \oplus wg$.

- If $f : N \to N$, is an isometry then, the trivial extension: $\bar{f} : (x, y) \in L \times N \to (x, f(y)) \in L \times N$, is an isometry of $L \times_w N$

(In particular, in the class of Lorentz manifolds with large isometry groups, one can perform warped products by (any) Riemannian manifolds.)

- **Examples: compact spaces: 1. Flat case:**

**Flat tori** $= (R^n, g)/Z^n$,

$g$ a Lorentz scalar product on $R^n$

Isom $(T^n, g) = T^n \rtimes O(g, Z)$,

$O(g, Z) = O(g) \cap GL(n, Z)$,

$O(g) =$ the orthogonal group of $g$ $(\sim O(1, n-1))$

- Dimension 2 (Avez):

$A \in SL(2, Z)$ hyperbolic,

$\omega^u$ (resp. $\omega^s$) linear forms on $R^2$ defining, the stable and unstable foliations of $A$.

$g = \omega^u \omega^s$.

$A$ preserves $g$

Isom$(T^2, g) =$ (essentially) $T^2 \rtimes Z$, $Z$ generated by $A$.

- Dimension $> 2$

Harisch-Chandra Borel: if $g$ is rational, then $O(g, Z)$ is a lattice in $O(g)$. (in particular $O(g, Z)$ is isomorphic to the fundamental group of a finite volume hyperbolic manifold)

- Suspension $T^3_A$

The suspension of $A$ gives a flat manifold with an isometric flow which is Anosov (chaotic)

$T^3_A = SOL/\Gamma$, $SOL$ the 3-dimensional unimodular solvable non-nilpotent group.

## • Examples: compact spaces: 2. Local $AdS_3$ space

$G = SL(2, \mathbf{R})$

$\triangleright$ The Killing form $k$ on the Lie algebra $sl(2, R)$ has signature $-++$ (may be $--+$, in this case, consider $-k$ )

– It is $Ad$-invariant, in particular $Ad(H)$-invariant for any co-compact lattice.

Thus *For any co-compact lattice* $H \subset SL(2, \mathbf{R})$, $SL(2, \mathbf{R}/H$ *is a homogenous compact spacetime*

• Alternative explanation of the Fact above:

Right translate $k$ , and get a right invariant Lorentz metric on $G$. Thus, it passes to right quotients $G/H$.

-$G$ acts by the left $G/H$

– This action is isometric, since, also the left action on $G$ on itself preserves the Lorentz metric, because $k$ is bi-invariant.

Remark: $SL(2, \mathbf{R})/H$ is (up to 2-cover) to unit tangent bundle of the hyperbolic surface $\mathbf{H}^2/H$.

## • Examples: compact case: 3. Warped Heisenberg groups

A family of sympathetic groups: Warped Heisenberg groups: family of **solvable** groups looking like $SL(2, R)$:

• they admit Lorentz bi-invariant Lorentz metrics, i.e. their Lie algebra admit $Ad$-invariant Lorentz scalar products ($\neq$ the Killing form, which is degenerate).

• they have co-compact lattices!

▷ $G$ is a warped Heisenberg group, $H$ a lattice

▷ As in the case of $SL(2, \mathbf{R})$,

$M = G/H$ is a compact homogeneous spacetime, where $H$ is a cocompact lattice

(co-compact is superfluous, since any lattice in a solvable Lie group is co-compact).

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- **The simplest example** of warped Heisenberg groups: dimension 4 (known as oscillator group, Diamond group...)

The semi-direct product $G = S^1 \ltimes Heis$

$Heis$ = Heisenberg group (of dimension 3):

$$Heis = \{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, x, y, z \in \mathbf{R} \}$$

$Heis$ is characterized essentially, by the existence of a non-split exact sequence:

$1 \to \mathbf{R} \to Heis \to \mathbf{R}^2 \to 1$

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$G = S^1 \ltimes Heis$ is defined by:

– $S^1$ acts trivially on the center $\mathbf{R}$, and acts by rotation on $\mathbf{R}^2$

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- Also $G$ is characterized by being a non-trivial central extension of $Ec$ by $S^1$,

$Ec$ = group of Euclidean isometries of the plane.

$1 \to S^1 \to G \to Ec \to 1$

## Generalization: canonical warped Heisenberg groups

• Recall the construction of Heisenberg algebras: $\mathcal{HE}_d$ (dim $= 2d+1$)

$\mathbf{R} \oplus \mathbf{C}^d$, with basis $Z, e_1, \ldots, e_d$

The only non-vanishing brackets are: $[e_k, ie_k] = Z$. (here $i = \sqrt{-1}$)

Equivalently,

$[X, Y] = \omega(X, Y)Z$.

$\omega$ symplectic form, i.e. $\omega(X, Y) =< X, iY >_0$,

$<, >_0$ hermitian product

• Canonical warped Heisenberg algebras

Add an exterior element $t$, such that:

$[t, e_k] = ie_k, [t, ie_k] = -e_k$, and $[t, Z] = 0$

• Scalar product

$<, >$

Endow $\mathbf{C}^d$ with its hermitian $<, >_0$

Decree $\mathbf{C}^d$ is orthogonal to $\{t, Z\}$.

$< t, t >=< Z, Z >= 0$ and $< t, Z >= 1$.

• $<, >$ is a Lorentz scalar product which is $Ad(\mathcal{HE}_d^t)$-invariant! i.e. $\forall u \in \mathcal{HE}_d^t$ $u$, $ad_u$ is antisymmetric with respect to $<, >$

exercise: Why this doesn't work for the Heisenberg algebras themselves?

- Consider $\tilde{G} = \tilde{He}^t_d$ the simply connected Lie group generated by $\mathcal{HE}^t_d$

- $\tilde{He}^t_d$ is a semi-direct product of $\mathbf{R}$ by $He_d$:

- The action of $\mathbf{R}$ on the center is trivial.

- The action on $\mathbf{C}^d$ is via multiplication by $\exp is$.

- This is in fact an action of $S^1$

- Consider $G = He^t_d = \tilde{He}^t_d/\mathbf{Z} = S^1 \ltimes He_d$

- Any lattice in the Heisenberg group $He_d$ is a lattice in $He^t_d$.

- example of a lattice in $He_1$:

$$Heis_{\mathbf{Z}} = \{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, x, y, z \in \mathbf{Z} \}$$

**General construction of warped Heisenberg groups:**

- Consider a semi-direct product $\mathbf{R} \ltimes He_d$, where $\mathbf{R}$ acts on $\mathbf{C}^d$ via:

$s \to \exp(2\pi sA) \in U(d)$ such that:

▸ $\exp(2\pi A) = 1$,

▸ $A$ diagonalizable, $\lambda_1, \ldots, \lambda_d \in \mathbf{Z}$

▸ The $\lambda_i$ have the same sign.

Thus, the action of $\mathbf{R}$ factors via an action of $S^1$

– The semi-direct product $G = S^1 \ltimes He_d$ is called a warped Heisenberg group.

– The conditions guaranty that $G$ has a bi-invariant Lorentz metric.

– $G$ has lattices

– Any quotient $G/H$, $H$ a lattice, is a compact homogeneous spacetime.

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**Some remarks:**

1. recall that the Lorentz scalar product was defined by: $\mathbf{C}^d$ is endowed with its hermitian metric$<,>_0$ and is orthogonal to the plane $\{t, Z\}$, $< t, t >=< Z, Z >= 0$ and $< t, Z >= 1$.

– One may take $< t, t >=$ Constant $\neq 0$, and multiply the other products by any constant ($\neq 0$), and gets another bi-invariant Lorentz metric.

– However, up to automorphism, there exists only one bi-invariant Lorentz metric on a warped Heisenberg group..

-In particular a metric is isometric to any multiple of itself. This follows from existence of homotheties. (This is true for $\mathbf{R}^n$ but not for $SL(2, \mathbf{R})$.

— Warped Heisenberg groups are (locally) symmetric Lorentz spaces of non reductive type.

– They have non-reductive holonomy, i.e. they have a codimension 1 **parallel** foliation which has no supplementary parallel direction field.

– The Ricci curvature of a warped Heisenberg group equals its Killing form (up to constant).

## Historical comments

- $He_1^t$ the 4-dimensional Heisenberg group is known as:

– Diamond group, in Representation Theory

– Oscillator group in Representation Theory and quantum mechanics

- The bi-invariant Lorentz metrics were known to medina-Revoy, and "partially" to Zimmer and Gromov.

– This seems folkloric in relativistic literature: some gravitational plane waves spacetimes...

– Witten-Nappi: "a WZW model based on a non semi-simple group" (1993)

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Justification of the name "oscillator group":

The Lie algebra $\mathcal{HE}_1^t$ has the following representation in the algebra of operators of the Hilbert space $E = L^2(\mathbf{R})$:

$Z \to 1 =$ (Identity)

$X \to q$ (position)

$Y \to p$ (impulsion)

$t \to p^2 + q^2$ (energy),

where the operators $q$ and $p$ are given by:

$q(f) = xf \ (f \in L^2(\mathbf{R}))$

$p(f) = \frac{\partial f}{\partial x}$

— To show that this gives a homomorphism, one verifies in particular: $[q, p] = 1$, which is the Heisenberg uncertainty principle.

$p^2 + q^2$ is the energy of the harmonic oscillator, which explains the origin of the terminology.

Examples 4. Counter-examples:

In dimension 3, the nilpotent group, $Heis$ and the solvable $SOL$ admit left invariant complete flat Lorentz metric. Their compact quotients are flat Lorentz manifolds (if fact any compact Lorentz flat 3-manifold, is of this type or is a torus, up to a finite cover).

-$Nil$ and $SOL$ manifolds are homogeneous but not Lorentz homogeneous.

## Sub-question: **Classification of compact homogeneous spacetimes: Part I: structure of their isometry group**

**Theorem 1** *Let $M = G/H$ be a compact homogeneous Lorentz manifold.*

*Then, up to compact objects: $G$ is $SL(2, \mathbf{R})$ or a warped Heisenberg group.*

*More precisely: there is a subgroup $S \subset G$, such that:*

- *$S$ is normal, and the Lie algebra of $S$ is a factor in $\mathcal{G}$*
- *$S$ is co-compact in $G$ (i.e. $G/S$ is compact)*
- *$S$ is isomorphic to $PSL_k(2, \mathbf{R})$ the $k$-folded cover of $PSL(2, \mathbf{R})$, or*

  *-$S$ is a warped Heisenberg group.*
- *$S$ acts on $M$ locally freely (i.e. stabilizer in $S$ are discrete)*

**Corollary** The stabilizer $H$ is **"almost discrete":** its connected component is compact. *(This is non-obvious a priori, and false for non-compact homogeneous spacetimes, and for general homogeneous pseudo-Riemannian manifolds, even compact)*

Subquestion: **Classification of compact homogeneous spacetimes: Part II: their geometric structure**

**Theorem 2** *Up to compact objects, they are isometric to* $S/H$*, where* $H$ *is a co-compact lattice (in particular discrete) in* $S$*, which is* $PSL(2, \mathbf{R})$ *or a warped Heisenberg group.*

*- Roughly,* $M$ *is a a "local product" modeled on* $S \times \tilde{L}$*, where* $\tilde{L}$ *is a homogeneous Riemannian manifold*

*—- Details:*

- *The case* $S = PSL_k(2, \mathbf{R})$ *(due to Gromov)*

*-* $M = S \times \tilde{L}/H$*:*

*-* $\tilde{L}$ *is a compact homogeneous Riemannian manifold*

*- There is* $H_0$ *a lattice in* $S$*, such that* $H$ *is the graph of a homomorphism* $\rho : H_0 \to Isom(\tilde{L})$

*- The centralizer of* $\rho(H_0)$ *acts transitively on* $\tilde{L}$*.*

*- The metric on* $S \times \tilde{L}$ *is:* $c.Killing \otimes r_{\tilde{L}}$*,* $c$ *constant,* $r_{\tilde{L}}$ *the Riemannian metric of* $\tilde{L}$

*- Conversely, with these data, one constructs a compact homogeneous spacetime.*

- $S$ *a warped Heisenberg group: a little bit complicated description...*

The question (non-homogeneous case)

**Theorem 3** *(Zimmer, Gromov, Adams-Stuck, Zeghib) Recall that the Lie algebra of a compact Riemannian manifold is a sum of an abelian Lie algebra with a semi-simple Lie algebra of compact type. (i.e. the Lie algebra of a compact semi-simple Lie group).*

*– In the Lorentz case, the new factor that might occur, is a subalgebra of $\mathcal{S}$, where $\mathcal{S}$ is the Lie algebra of $SL(2, \mathbf{R})$ or a warped Heisenberg group.*

## More details

(Killing algebra of $M$ = the Lie algebra of its isometry group)

**Theorem 4** *(Adams-Stuck, Z.) The Killing Lie algebra of a compact Lorentz manifold is isomorphic to a direct sum*

$$\mathcal{K} + \mathbf{R}^k + \mathcal{S},$$

*where $\mathcal{K}$ is the Lie algebra of a compact semi-simple Lie group, $k \geq 0$ is an integer and $\mathcal{S}$ is trivial or isomorphic to:*

- ▷ *a Heisenberg algebra (of some dimension),*
- ▷ *a warped Heisenberg algebra, or*
- ▷ *$sl(2, \mathbf{R})$.*

*Conversely, any such algebra is isomorphic to the Lie algebra of the isometry group of some compact Lorentz manifold.*

## Ingredients of the proof of Theorem 3

Consider the $L^2$ bilinear form on the Lie algebra $\mathcal{G}$:

$\kappa(X, Y) = \int_M < X(x), Y(x) >_x dx$

($X, Y$ are Killing fields, $<,>$ the Lorentz metric)

• 1. $\kappa$ is a **bi-invariant** quadratic form on $\mathcal{G}$ (general fact)

• 2. However, $\kappa$ might be trivial

• 3. The point is to show that it is sufficiently non-trivial...

Details:

• 1). Let $G$ be a Lie group acting on $M$, $\mathcal{G}$ its Lie algebra:

Action means: a homomorphism (of Lie brackets)

$X \in \mathcal{G} \to \bar{X} \in$ Vector-fields on $M$.

To $Y \in \mathcal{G}$, is associated

- $\phi^t$ a one-parameter subgroup of $G$, and
- $\overline{\phi}^t$ a one parameter group of diffeomorphisms on $M$.

• Naturally:

$$\overline{\phi}^t_* \overline{X} = \overline{Ad(\phi^t) X}$$

If $G$ preserves a volume $dx$ and a $q$-covariant tensor

$T$, then,

$$\kappa^T(X_1, \ldots, X_q) = \int_M T(\overline{X_1}(x), \ldots, \overline{X_q}(x))dx$$

determines a tensor on $\mathcal{G}$ which is **bi-invariant**.

• 2). Unfortunately $\kappa^T$ is generally trivial:

$G = SO(n), n > 2$, $T$ any left invariant quadratic form (degenerate or not, positive or not...), but $\kappa^T$ is a multiple of the Killing form, by simplicity.

- In particular, it may happen that $\kappa^T = 0$ for $T$ Lorentz.

• 3). Now, $G$ is **non-compact** and acts isometrically on a compact (or just finite volume) Lorentz manifold $M$,

Consider the associated $\kappa$:

Major first step (of the proofs of Theorems 1 and 3):

$\kappa$ is sufficiently non-trivial: It satisfies a condition (*)

(*) means that $\kappa$ is between a Lorentz and a Euclidean product!

• Theorems 1 and 3 follow from an "Algebraic Lemma" classifying" those Lie algebras admitting $Ad$-invariant scalar product satisfying (*).

– Similar to the lemma saying that a Lie algebra with an $Ad-$invariant positive scalar product is Abelian + compact.

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Behind (*) is the following :

**Fundamental non-degeneracy Fact** *$M$ compact Lorentz manifold, $\phi^t \subset Isom(M)$ a one parameter group with infinitesimal generator (the Killing filed) $X$.*

*Suppose $\phi^t$ is non-precompact, i.e non-equicontinuous, i.e. the closure of $\{\phi^t, t \in \mathbf{R}\}$ in $Isom(M)$ is not compact.*

*Then, $X$ is everywhere non-timelike: $< X(x, X(x) > \geq 0$, $\forall x$.*

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**Corollary: Condition (*)**: *Let $\mathcal{P}$ a linear subspace of $\mathcal{G}$ containing a dense set of non-precompact Killing fields.*

*Then, $\kappa|\mathcal{P} \geq 0$, and dim $Ker(\kappa|\mathcal{P}) \leq 1$*

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Proof of the Fundamental non-degeneracy Fact: It is based on 2 uniformity facts:

**Fact 1** *Let $\{\phi^t\} \subset Isom(M)$ be a one parameter group of isometries. If for some $t_i \to \infty$, $\{\phi^{t_i}\}$ is precompact (i.e. equicontinuous), then $\{\phi^t\}$ is precompact.*

**Fact 2** *If for some $x_i \in M, t_i \to \infty$, $\{D_{x_i}\phi^t\}$ is equicontinuous (i.e. $\parallel D_{x_i}\phi^t \parallel$ and $\parallel (D_{x_i}\phi^t)^{-1} \parallel$*

*bounded), then* $\{\phi^{t_i}\}$ *is equicontinuous (and therefore by the fact above* $\{\phi^t\}$ *is equicontinuous)*

— Facts 1 and 2 are true in affine dynamics, i.e. for $\{\phi^t\} \subset \mathrm{Affin}(M, \nabla)$, $\nabla$ a connection.

We need a third fact special to the Lorentz case:

**Fact 3** *If a Killing field* $X$ *is somewhere timelike (i.e.* $< X(x_0), X(x_0) >< 0$*), then it generates an equicontinuous flow* $\{\phi^t\}$*.*

Sketchs of proofs:

— Fact 3: Let $U$ be a neighborhood of $x_0$ where $X$ is timelike. By Poincaré recurrence Lemma, there exist $x_i$ near $x_0$, $t_i \to \infty$, such that $\phi^{t_i} x_i$ is near $x_0$.

Now, near $x_0$, $D_{x_i}\phi^{t_i}$ behave as Riemanniann isometries.

Apply Fact 2, and then Fact 1 to deduce that $\{\phi^t\}$ is equicontinuous.

— Fact 1:

$L = \overline{\{\phi^t\}} \subset Isom(M)$

$L$ abelian $\Longrightarrow$ $L =$ a cylinder $T^k \times \mathbf{R}^d$ (where $T^k$ is a torus).

$L$ has a dense one parameter group ($\{\phi^t\}$ itself), i.e. a dense geodesic (when $L$ is seen as a cylinder) $\Longrightarrow L = T^k$, or $L = \mathbf{R}$.

Now, if there $\exists$ $\{\phi^{t_i}\}$ equicontinuous, then $L \neq \mathbf{R}$, thus $L = T^k$, i.e. $\{\phi^t\}$ is equicontinuous.

— Fact 2: By definition of its Lie group structure, $Isom(M)$ acts properly (and freely) on the frame bundle $P(M)$.

**The non-compact case**

— Trivial counter-example: $G$ a Lie group, endow $\mathcal{G}$ with any Lorentz scalar product, and left translate it on $G$. The left $G$ action is isometric → $G$ is a homogeneous spacetime.

But, the $G$-action is proper (stabilizer are trivial).

The same is true, in general, for Isom($G$) (which might cantain properly $G$)

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The subquestion: Find $H$ closed, **non-compact**, such that $Ad(H)$ preserves a Lorentz scaler product on $\mathcal{G}/\mathcal{H}$?

**Example:** Symmetric spaces (reductive or not)

— Non-reductive Lorentz symmetric spaces: clssified by Cahen-Parker.

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Non-compact spacetimes are more interesting in physics.

**Observation:** Only few classical exact solutions have essential isometry groups.

One may try to prove:

"*a physical solution (i.e. a natural energy-implusion tensor + causality conditions $\Longrightarrow$ the solution has a non-essential isometry group, unless, it is very special (e.g. -Minkowski, dS, AdS...)* "

## Results: Non-comapct case

“First work”: Nadine Kowalsky

Thesis with Zimmer,

Notice in CRAS with 8 Theorems,

Article in Ann. Maths: Proofs of 4 Theorems,

Unfortunately, she prematurely dead...

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Algebraic hypothesis:

$G$ a simple (sometimes semi-simple) connected Lie group, acting isometrically non-properly on a Lorentz manifold $M$.

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**Principal algebraic result:** $G = O(1, n)$ or $O(2, n)$

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**Geometric result** (without proof) $M$ is essentially $dS_n$ or $AdS_n$.

More exactly, this is true up to (a local) warped product.

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Works by S. Adams (also an old student of Zimmer):

– New proof of Kowalsky’s algebraic Theorem.

– New groups, but with a stronger dynamical hypothesis.

D. Witte (also an old student of Zimmer):

Assuming $G = O(1,n)$ or $O(2,n)$, and $G$ acts transitively, i.e.

$M = O(1,n)/H$ or $O(2,n)/H$, then $M = dS_n = O(1,n)/O(1,n-1)$ or $AdS_n = O(2,n)/O(1,n)$

(The result is geometric, but the proof is algebraic)

**Super-question** The $Diff(M)$-action on $Lor(M)$

– Pierre Mounoud (Lafontaine's student)

– Case of compact surfaces: Klein Bottle or a torus

**Theorem 1** For $M$ = Klein bottle, the Diff(M)-action on $Lor(M)$ is proper.

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Torus case: $M$

$\mathcal{F}$ the space of flat metrics on $M$.

– Such a metric is linear (on $\mathbf{R}^2$)

– The $Diff_0(M)$-action on $\mathcal{F}$ is proper

– The $Diff(M)$-action on $\mathcal{F}/Diff_0(M)$ is identified to the action of $SL(2,\mathbf{Z})$ on

$$SL(2,\mathbf{R})/\{\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, t \in \mathbf{R}\}$$

which is "dual" to the action of the geodesic flow on the modular surface $\mathbf{H}^2/SL(2,\mathbf{Z})$.

– This is in particular ergodic

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**Theorem 2** The $Diff(M)$-action on $Lor(M) - \mathcal{F}$ (the space of non-flat metrics) is proper.

**An amazing lemma:** If a Lorentz metric on the torus has curvature constant along one isotropic foliation, then this metric is flat.