Area of vector fields on the sphere and related problems

by

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This article is divided into two parts. The first one is an announcement of a result regarding unit vector fields on the 2-sphere [2] and the second one is a list of related open problems. Indeed, one of the intentions of the Ecole-CIMPA at El Oued was to provide advanced students with the opportunity to discover new areas and open problems on which they could set to work with a reasonnable hope to make significant progress. Our opinion is that the study of volume of vector fields is such an area and that is the reason why we have included this list of open questions.

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1 Vector fields with least area

A vector field V on a manifold M is a section of its tangent bundle TM. If (M, g) is a Riemannian manifold, its tangent bundle is endowed with a natural Riemannian metric called the *Sasaki metric* (TM, g^{Sas}) which is defined by the following formula

$$\forall X, Y \in TTM, \quad g^{Sas}(X, Y) = g(d\pi(X), d\pi(Y)) + g(K(X), K(Y))$$

where $\pi : TM \longrightarrow M$ is the natural projection and $K : TTM \longrightarrow TM$ the connector of the Levi-Civita connection ∇ of g ($\nabla V = K \circ dV$). It ensues that the horizontal and vertical distributions are orthogonal, moreover the metric on each of these distributions is just the "pull-back" of g by π and K.

Suppose now that (M, g) is compact. The *volume of the vector field* $V : M \longrightarrow TM$ is the volume of the image submanifold $V(M) \subset TM$ computed with the metric induced by the

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[2] Proofs will appear elsewhere.

Sasaki metric g^{Sas}. It is denoted by $Vol(V)$. Historically the first studied question regarding the volume was the one of the infimum: given a compact, oriented Riemannian manifold without boundary, is there a vector field of least volume ? To avoid the trivial answer $V \equiv 0$ it is usually assumed that the infimum is taken among *unit* vector fields, that is vector fields $V : M \longrightarrow TM$ such that

$$\forall x \in M, \quad g_x(V_x, V_x) = 1.$$

From now on, we will always assume that this condition holds, in particular since M has no boundary, the infimum question makes sense if and only if the Euler number $\chi(M)$ vanishes.

Since $V : M \to T^1M$ is a section, that is –roughly speaking– a graph, we have

$$Vol(M) \leq Vol(V)$$

and a direct computation shows that equality holds if and only if the field V is parallel, i. e. $\nabla V = 0$. Therefore the problem is actually interesting when M does not admit any parallel field. A natural example of such a situation is given by standard odd-dimensional spheres $\mathbb{S}^{2m+1}$, $m \geq 1$.

The problem of finding the infimum of the volume turns out to be very difficult, even for the usual spheres $\mathbb{S}^{2m+1}$! In fact the only spheres for which the problem is solved are $\mathbb{S}^1$ and $\mathbb{S}^3$. The case of $\mathbb{S}^1$ is straightforward since there is only two unit vector fields and both are parallel. The case of $\mathbb{S}^3$ is far more involved, the space of unit vector fields is homeomorphic to the space of maps from $\mathbb{S}^3$ to $\mathbb{S}^2$ and none of them is parallel. In a pionnering work, H. Gluck and W. Ziller proved, using a calibration argument, that Hopf vector fields are the unique minimizers of the volume [7] (a Hopf vector field is any vector field tangent to the fibers of a Hopf fibration $\mathbb{S}^1 \longrightarrow \mathbb{S}^{2m+1} \longrightarrow \mathbb{C}P^m$). For the other odd-dimensional spheres what is known is that Hopf vector fields do *not* achieve the minimum [8], [5], [1]. In fact it is conjectured [8] that the infimum of the volume is achieved by singular vector fields that we called, *Pontrjagin fields*. These fields are obtained by parallel translation of a given unit vector $v \in T_p\mathbb{S}^{2m+1}$ along geodesics radiating from the point p. The antipodal point $-p$ is singular with index $0 = \chi(\mathbb{S}^{2m+1})$ and so can be removed, precisely, P can be approximated by a family of non singular unit vector fields $(P_\epsilon)_{\epsilon \in]0,1]}$ and such that

$$\lim_{\epsilon \longrightarrow 0} Vol(P_\epsilon) = Vol(P).$$

One unpleasant fact regarding the volume problem for unit vector fields is that the topological condition $\chi(M) = 0$ eliminates many interesting manifolds, almost all surfaces for instance. Fortunately there are many ways to bypass this restriction: for example by considering unit vector fields with isolated singularities or by considering a new class of unit vector fields that we have called *unit vector fields without boundary*.

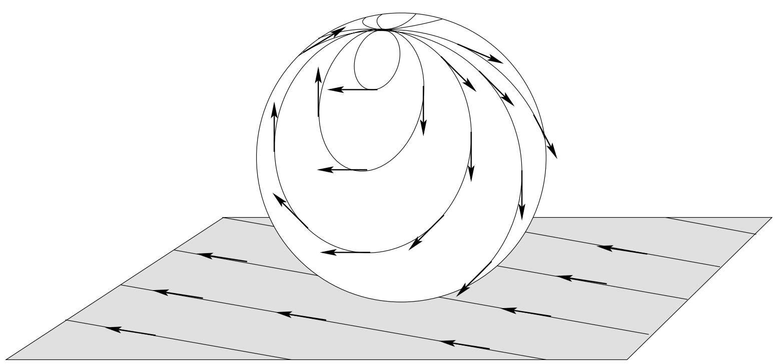
Definition 1.1 *Let $\mathcal{U}$ be a dense open set of M and $V : \mathcal{U} \to T^1M$ be a smooth vector field over $\mathcal{U}$. The field V is said to be without boundary if the adherence $\overline{V(\mathcal{U})}$ of its image in T^1M is a (topological) submanifold without boundary. This submanifold is denoted by $\overline{V}$.*

It is easily seen that every Riemannian oriented manifold admits unit vector fields without boundary and so we can re-set the question of the infimum of the volume for this large family of manifolds. Coming back to our spheres $\mathbb{S}^n$ it is obvious that Hopf vector fields have no boundary since they are defined everywhere. For Pontrjagin fields P it turns out that they also have no boundary. In fact $\overline{P}$ is a submanifold without boundary which is smooth if $n = 2$ and smooth everywhere except at a singular point if $n > 2$, precisely, $\overline{P}$ is homeomorphic to $\mathbb{R}P^n$ with an $\mathbb{R}P^{n-2}$ smashed to a point [8].

Note that if $P : \mathbb{S}^n \setminus \{pt\} \to T^1\mathbb{S}^n$ is a Pontrjagin field then its restriction $V = P_{|\mathcal{U}}$ to any dense open set $\mathcal{U}$ of $\mathbb{S}^n \setminus \{pt\}$ is a vector field without boundary which have same adherence and same volume. Nevertheless, strictly speaking, V is not a Pontrjagin field, that is why it is convenient to enlarge a little bit our definition of a Pontrjagin field. We say that $V : \mathcal{U} \to T^1\mathbb{S}^n$ is a Pontrjagin field in the generalized sense if $\overline{V} = \overline{P}$ for a certain Pontrjagin field P. In particular V and P coincide in a dense open set of $\mathbb{S}^n$.

Theorem 1.2 *Vector fields without boundary of least area on $\mathbb{S}^2$ are Pontrjagin fields (in the generalized sense) and no others.*

The following picture shows a Pontrjagin field on $\mathbb{S}^2$, its integral lines are mapped by the stereographic projection onto straight lines. Note that the index of the unique singularity of P is 2 since $\chi(\mathbb{S}^2) = 2$ and so the singularity can not be removed.



2 Related problems

Problem 1. Here is a question that naturally follows our theorem: what about the other compact orientable Riemannian surfaces of constant curvature ? The genus one case is trivial since any flat torus $T^2 = \mathbb{C}/\Lambda$ admits parallel unit vector fields. The interesting –and most likely difficult– case is the one of the 2-dimensional space forms of negative Gaussian curvature that is genus $g > 1$ Riemannian surfaces which are quotients of the hyperbolic disk $\mathbb{H}^2$ by groups of isometries Γ acting freely and properly discontinuously on $\mathbb{H}^2$.

There is a related problem which could be a bit easier. A fundamental result, due to F. Brito, P. Chacon and A. Naveira [3], states that if V is a unit vector field on a compact Riemannian oriented manifold M of dimension $n = 2m + 1$ then

$$Vol(V) \geq \int_M \left(1 + \sum_{i=1}^{m} \frac{C_m^i}{C_{2m}^{2i}} |\sigma_{2i}(V^\perp)|\right) dvol_M$$

where $\sigma_{2i}(V^\perp)$ is the $2i$-th elementary function of the second fundamental form of the distribution $V^\perp$ (which is not necessarily integrable). When $m \geq 2$, equality holds if and only if V is totally geodesic and $V^\perp$ integrable and umbilic. This inequality can be simplified if M has constant sectional curvature c, in that case there is an integral formula for the symmetric functions σ_i, precisely (see [4])

$$\int_M \sigma_i(V^\perp) dvol_M = \begin{cases} C^m_{i/2} c^{i/2} vol(M) & \text{if } i \text{ even,} \\ 0 & \text{if } i \text{ odd} \end{cases}$$

and so the above above inequality gives a lower bound for the volume in terms of c and $Vol(M)$. In particular, for the standard sphere $M = \mathbb{S}^{2m+1}$, it yields:

$$Vol(V) \geq Vol(R).$$

where R is a radial field, i. e. any unit vector field which is tangent to geodesics radiating from a given point $p \in \mathbb{S}^{2m+1}$. This field has two singular antipodal points, the adherence of its image in the unit tangent bundle is a submanifold with two boundary components, namely the two fibers over the singular points. Therefore R is not a unit vector field without boundary.

Although very important, the result of Brito, Chacon and Naveira does not solve the problem of the infimum of the volume because there is no known globally defined family of unit vector fields R_ϵ of $\mathbb{S}^{2m+1}$ such that

$$\lim_{\epsilon \to 0} Vol(R_\epsilon) = Vol(R).$$

Note also that their result gives no information for even-dimensional manifolds, in fact, the technic used in the proof totally fails in these dimensions. An important topic will be the search of a similar result for even-dimensional manifolds, and in particular, in the 2-dimensional case.

Problem 2. What is the influence of normalization ? Instead of choosing a unit vector field, we could also consider vector fields of constant norm, or equivalently, we could perform a dilatation of the metric $g_k = \frac{1}{k} g$ while keeping the length $\|V\|_{g_k}$ equals to one. Of course if V is a unit vector field for (M, g) then $V^k = k^{-\frac{1}{2}} V$ is a unit vector field for (M, g_k) and a direct computation shows that

$$\begin{aligned} Vol_{g_k}(V^k) &= \int_M \sqrt{det\left(\frac{1}{k} Id + A\right)} dvol_g \\ &= \int_M \sqrt{\frac{1}{k^n} + \frac{1}{k^{n-1}}\sigma_1(A) + ... + \frac{1}{k}\sigma_{n-1}(A) + \sigma_n(A) dvol_g} \end{aligned}$$

where A denotes $(\nabla V)^T \circ \nabla V$ and the σ_j are the elementary symmetric functions of the eigenvalues of A. Note that $\|V\|_g = 1$ implies $\sigma_n(A) = 0$. If k is large, the behaviour of the volume is ruled by the *twisting* of V:

$$Tw(V) = \int_M \sqrt{\sigma_{n-1}((\nabla V)^T \circ \nabla V)} dvol_g.$$

It is shown in [1] that Hopf vector fields achieve the minimum of the twisting in odd-dimensional spheres giving an evidence that Hopf vector field could achieve the minimum of the volume for large value of k. Naturally the proof totally collapses for even-dimensional spheres since in that case there is no Hopf vector fields... The question is to find an analogous result on even-dimensional spheres and in particular when the dimension is 2.

If k is small, the behaviour of the volume is ruled by the energy

$$E(V) = \frac{1}{2}\int_M \big(n + \sigma_1((\nabla V)^T \circ \nabla V)\big)\, dvol_g = \frac{1}{2}\int_M \big(n + \|\nabla V\|^2\big)\, dvol_g.$$

It is shown in [2] that the energy of the radial vector field is the infimum of the energy of smooth unit vector fields in odd-dimensional spheres. The result is based on the contruction of a family of unit vector fields R_ϵ of $\mathbb{S}^{2m+1}$ such that

$$\lim_{\epsilon \to 0} E(R_\epsilon) = E(R).$$

In this construction, the fact that the dimension is odd is also crucial.

Problem 3. What is the "volume cost" of singularities ? Suppose first that V is a unit vector fields without boundary of $\mathbb{S}^2$ with a unique singular point $p \in \mathbb{S}^2$ and let D^2 be a small 2-disk of $\mathbb{S}^2$ centered at p. Since the Euler number $\chi(\mathbb{S}^2)$ is 2 the index of p must also be 2, hence the image $V(\partial D)$ is a closed curve that surrounds twice the fiber over p and the adherence

$$\overline{V(D^2 \setminus \{p\})}$$

is a Moebius band μ^2. It ensues that $\overline{V}$ is a Moebius band glued along its boundary to the image $V(\mathbb{S}^2 \setminus D^2)$ which is topologically a 2-disk, thus $\overline{V}$ is homeomorphic to $\mathbb{R}P^2$. Assume now that V has several singularities, since $\overline{V}$ is a manifold, their indexes must be 2,0 or -2. The essential singularities are the ones with indexes ± 2 and from the above discussion, each of them creates a Moebius band. So, if V has N essential singularities and if $D_1, ..., D_N$ are N disjoint small disks centered at the singularities, $\overline{V}$ is homeomorphic to

$$\big(\mathbb{S}^2 \setminus (D_1 \cup ... \cup D_N)\big) \cup \underbrace{\mu^2 \ \cup ... \cup \ \mu^2}_{N \text{ times}}$$

and so $\overline{V}$ is a connected sum of a projective space and several tori

$$\overline{V} \simeq \mathbb{R}P^2 \sharp \underbrace{\ T^2 \ \sharp ... \sharp \ T^2}_{\frac{N-1}{2} \text{ times}}$$

(N must be odd for obvious topological reasons).

Our intuition seems to tell us that adding singularities should increase the volume since the topology of the image actually increases with the number of singularities. In an other hand, radial fields on $\mathbb{S}^2$ which have two singularities, have less volume than Pontrjagin fields with only one singularity. Whatever finding the infimum of the volume among vector fields without boundary with exactly N essential singularities is an open problem.

Problem 4. A natural way to find minimizers of the volume is to look for stable critical points of the volume functional (recall that a critical field V is said to be *stable* if the quadratic form $(Hess\ Vol)_V$ is non negative). In [6] it has been computed the condition that a vector field should fulfill in order to be critical and in [5] one can find the second variation of the functional at critical points. In particular, all critical vector fields on a surface are stable.

Up to now, the only examples of critical vector fields defined on an open subset of $\mathbb{S}^2$ are those described above: radial vector fields and Pontrjagin ones. An open question is to know if there are other unit critical vector fields on $\mathbb{S}^2$. The same question of determining all critical unit vector fiels is open for compact surfaces of constant negative curvature. For any even-dimensional sphere the similar problem could be addressed, although in that case it would be natural to determine first if radial and Pontrjagin are the only critical stable unit vector fields.

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