Kaehler manifolds with a big automorphism group

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(joint work with Serge Cantat)

The Theorem

Warning: almost all statements are, up to a finite cover for spaces, and finite index for groups!

Theorem

Let M be a compact Kaehler manifold of dimension n.
Let Γ be a lattice in a simple Lie group G of real rank $n-1$.
Let Γ acts on M holomorphically. Then, either
1) The action extends to an action of the full Lie group G.
2) or M is birational to a complex torus.

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More precisely, M is a Kummer variety: it is obtained form an abelian orbifold A/F by blow ups and resolution of singularities.

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   - Discrete groups: lattices in higher rank groups
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Introduction

Meeting of two worlds

The discrete factor of the automorphism group

Let M be a complex manifold.

$Aut(M)$ the group of holomorphic diffeomorphism

– If M is compact, then $Aut(M)$ is a complex Lie group (of finite dimension).

– The Lie algebra of $Aut^0(M)$ is the space of holomorphic fields on M.

- If M is Kaehler, the dynamics of $Aut^0(M)$ is poor

Explanations:

- Elements of $Aut^0(M)$ have vanishing topological entropy.

– In the projective case, $M \subset \mathbb{C}P^N$,

$Aut^0(M) = \{g \in PGL_{N+1}(\mathbb{C}),\ gM = M\}$

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$\rightarrow$ it is more important to consider the discrete factor $Aut^{\#}(M) = Aut(M)/Aut^0(M)$.

• Which discrete groups are equal to $Aut^{\#}(M)$ for some M?

• For fixed dimension n, find M for which $Aut^{\#}(M)$ is as big as possible?

Margulis super-rigidity

G a semi-simple (real) group (e.g. $G = \mathrm{SL}_n(\mathbb{R})$...)

$\Gamma \subset G$ a lattice: G/Γ has finite volume, e.g. co-compact.

Example $\mathrm{SL}_n(\mathbb{Z})$ is non co-compact lattice of $\mathrm{SL}_n(\mathbb{R})$.

The world of (simple Lie) groups:

$\mathcal{F} = \{O(n,1); SU(n,1);\}$

$\mathcal{R} = \{$the others, e.g., $Sp(n,1), SL_n(\mathbb{R}), SO(p,q), p,q > 1...\}$

A Γ a lattice of G, and $G \in \mathcal{R}$, is **super-rigid**: any $h : \Gamma \to \mathrm{GL}_N(\mathbb{R})$ extends to a homomorphism $G \to \mathrm{GL}_N(\mathbb{R})$, unless it is bounded...

The authors: Margulis if $\mathrm{rk}_{\mathbb{R}} G \geq 2$ (e.g. $\mathrm{SL}_n(\mathbb{R})$, $n \geq 3$...)
Gromov-Shoen: for the rk-one: $Sp(n, 1)$ and the isometry group of the hyperbolic Cayley plane.

Zimmer program

– Super-rigidity solves linear representation theory of Γ

– Zimmer program, a tentative to understand "non-linear representations", i.e. $\Gamma \to \mathrm{Diff}(M)$, where M is compact, i.e. differentiable actions of Γ.

Question

Let Γ be a lattice in a simple Lie group G of real rank ≥ 2.
– Find the minimal dimension d_Γ of compact manifolds on which Γ acts, but not via a finite group.
- Describe all actions at this dimension.

Remark

Zimmer proves a " super-rigidity of cocycles".
– In general, one deals (in the question above) with volume preserving actions.

Example: $\Gamma = \mathrm{SL}_n(\mathbb{Z})$ (and congruence groups)

– The minimal linear representation is the standard one in $\mathrm{SL}_n(\mathbb{R})$, or its dual.

– Γ acts on the (real) torus $M = \mathbb{R}^n/\mathbb{Z}^n$.

Rigidity question (variant): prove that all smooth actions of Γ on the torus are smoothly conjugate to the standard one. (Authors: Zimmer, Margulis, Katok, Spatzier, Hurder, Lewis, Kanai...)

Strategy

Fix a kind of geometric structure, and restrict himself to actions preserving such a structure.

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Conjecture

(???) If Γ a lattice in a higher semi-simple Lie group acts on a compact Kaehler manifold, and a Zariski generic point has a Zariski dense orbit, then M is birationnal to a torus?

Remark: another connection: mapping class groups

$Teic(M)$ space of complex structures up to (smooth) isotopy
$Mod(M) = \mathrm{Diff}(M)/Diff^0(M)$ acts on $Teic(M)$

Sullivan: $Mod(M)$ is an arithmetic group...
$Aut^{\#}(M, c) \cong$ stabilzer of $c \in Teic(M)$ in $Mod(M)$.

Special points: those with a big stabilizer.

More details about the statement

Space of tori

Torus $X = X_\Lambda = \mathbb{C}^n/\Lambda$

Λ a lattice in $\mathbb{C}^n = \mathbb{R}^{2n}$

Space of Lattices: $\mathcal{L} = \mathrm{SL}_{2n}(\mathbb{R})/\mathrm{SL}_{2n}(\mathbb{Z})$

$G = \mathrm{SL}_n(\mathbb{C})$ acts on $\mathcal{L}$.

$Aut^{\#}(X_\Lambda) =$ stabilizer of Λ in $G = \mathrm{SL}_n(\mathbb{C}) =$

$$\Gamma_\Lambda = \{g \in \mathrm{SL}_n(\mathbb{C}),\ g\Lambda = \Lambda\}$$

Remark (dual point of view): the Teichmuller space is $\mathrm{SL}_{2n}(\mathbb{R})/\mathrm{SL}_n(\mathbb{C})$, endowed with the action of the modular group $\mathrm{SL}_{2n}(\mathbb{Z})$.

Generically: $\Gamma_\Lambda = \{1\}$

Our case: classify Λ such that: Γ_Λ is isomorphic to a lattice in a semi-simple Lie group of rang $n-1$

$\iff$ its Zariski closure $G \subset \mathrm{SL}_n(\mathbb{C})$ has real rank $= n-1$

$\iff$ $G = \mathrm{SL}_n(\mathbb{C})$, or G conjugate to the standard copy $SL_n(\mathbb{R}) \subset \mathrm{SL}_n(\mathbb{C})$.

Proposition

1) If $G = \mathrm{SL}_n(\mathbb{C})$*, then* $\Lambda = R^n$, $R = \mathbb{Z} + \sqrt{-d}\mathbb{Z}$*, and* $\Gamma = \mathrm{SL}_n(\mathbb{Z} + \sqrt{-d}\mathbb{Z})$.

2) If $G = \mathrm{SL}_n(\mathbb{R})$*, then, either*
2.1) $\Lambda = \mathbb{Z}^n + \delta\mathbb{Z}^n = (\mathbb{Z} + \delta\mathbb{Z})^n$*, and* $\Gamma \cong \mathrm{SL}_n(\mathbb{Z})$*, or*

2.2) $n = 2d$, $\Lambda = R^n$*, where* R *is lattice in* $\mathbb{R}^4 = \mathbb{C}^2$, $R = H_{a,b}(\mathbb{Z})$ *the ring of integer points of a quaternion algebra over* $\mathbb{Q}$.

$\Gamma = \mathrm{SL}_n(H_{a,b}(\mathbb{Z})) \subset \mathrm{SL}_n(H_{a,b}(\mathbb{R})) = \mathrm{SL}_n(Mat_2(\mathbb{R})) = \mathrm{SL}_{2n}(\mathbb{R})$.

In the first two case: $X = Y^n$, Y an an elliptic curve,

In the last case: $X = Z^n$, $Z = \mathbb{C}^2 / H_{a,b}(\mathbb{Z})$

Z is an abelian surface.

More details

$\mathbf{H}_{a,b}(\mathbb{Q}))$ quaternion algebra over $\mathbb{Q}$ defined by its basis $(1, \mathrm{i}, \mathrm{j}, \mathrm{k})$, with

$$\mathrm{i}^2 = a,\ \mathrm{j}^2 = b,\ \mathrm{ij} = \mathrm{k} = -\mathrm{ji},\ \ (a, b > 0)$$

It embeds into $Mat_2(\mathbb{Q}(\sqrt{a}))$ by mapping i and j to the matrices

$$\begin{pmatrix} \sqrt{a} & 0 \\ 0 & -\sqrt{a} \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ b & 0 \end{pmatrix}.$$

$\mathbf{H}_{a,b}(\mathbb{R}) = \mathbf{H}_{a,b} \otimes_{\mathbb{Q}} \mathbb{R}$

The matrix associated to $q = x + y\mathrm{i} + z\mathrm{j} + t\mathrm{k}$ has determinent $\mathrm{Nrd}(q) = x^2 - ay^2 - bz^2 + abt^2$.

Complex multiplication

$End(X)$ complex endomorphsim ring of X

$End(X) \supset \mathbb{Z}$

- If $\dim X = 1$, then $End(X) = \mathbb{Z}$, or $\mathbb{Z} + \sqrt{-d}\mathbb{Z}$

In the last case: X is of CM type,

Higher dimension ...

Abelian orbifolds with a Lattice action

$Y = X/F$

F abelian finite generated by a rotation $\overrightarrow{z} \to \eta \overrightarrow{z}$

η a root of unity,

$\eta^k = 1$,

$k = 1, 2, 3, 4, 6$

Calabi-Yau: if $\dim Y = k$

Actions of Lie groups

Let M be a connected compact complex manifold of dimension $n \geq 3$. Let H be an almost simple complex Lie group with $\mathrm{rk}_{\mathbb{C}}(H) = n - 1$.
If there exists an injective morphism $H \to \mathrm{Aut}(M)^0$, then M is one of the following:

(1) a projective bundle $\mathbb{P}(E)$ for some rank 2 vector bundle E over $\mathbb{P}^{n-1}(\mathbb{C})$, and then H is isogenous to $\mathrm{PGL}_n(\mathbb{C})$;

(2) a principal torus bundle over $\mathbb{P}^{n-1}(\mathbb{C})$, and H is isogenous to $\mathrm{PGL}_n(\mathbb{C})$;

(3) a product of $\mathbb{P}^{n-1}(\mathbb{C})$ with a curve B of genus $g(B) \geq 2$, and then H is isogenous to $\mathrm{PGL}_n(\mathbb{C})$;

(4) the projective space $\mathbb{P}^n(\mathbb{C})$, and H is isogenous to $\mathrm{PGL}_n(\mathbb{C})$ or to $PSO_5(\mathbb{C})$ when $n = 3$;

(5) a smooth quadric of dimension 3 or 4 and H is isogenous to $\mathrm{SO}_5(\mathbb{C})$ or to $\mathrm{SO}_6(\mathbb{C})$ respectively.

Preliminary ingredients: Kaehler dynamics

Tools

Γ acts on $H^*(M, \mathbb{C})$, by preserving the cup product $H^* \times H^* \to H^*$

$W = H^{1,1}(M, \mathbb{R})$

$\rho : \Gamma \to \mathrm{GL}\,(W)$

– $Aut(M)$, in fact $Aut^{\#}(M) = Aut(M)/Aut^0(M)$, acts on W.

Fundamental Kaehler Fact: The action of $Aut^{\#}(M)$ is virtually faithful: its kernel is finite $\iff$ if an automorphism acts trivially on W, then a power of it belongs a flow.

(authors: Liebermann, Fujiki...)

Cohomological actions

Henceforth, we assume the action on the cohomology faithful.

By Margulis super-rigidity, the ambient Lie group G acts on $H^*(M, \mathbb{R})$ preserving all algebraic structures:

– The Hodge decomposition,

– The cup product

– The Poincaré duality

In particluar,

$\rho : G \to \mathrm{GL}(W)$

ρ preserves a n-linear form $W \times \ldots W \to \mathbb{R}$

The Kaehler cone

W is a ordered linear space:

$\mathcal{K} \subset W$ the space of $\alpha \in W$ having a reprensentative $\omega \in [\alpha]$, which is Kaehler, i.e. $g(u, v) = \omega(u, Jv)$ is positive definite. (so $h = g + \omega$ is a hermitian metric)
$\mathcal{K}$ is a convex non-degenerate cone with a non-empty interior.

The nef cone is the closure $\overline{\mathcal{K}}$

Preserved sub-cones

Proposition

Let Γ be a lattice in a semi-simple group G, and $\rho : G \to \mathrm{GL}(W)$. Assume $\rho(\Gamma)$ preserves a non-degenerate cone $\mathcal{K}$. Then $\rho(G)$ preserves a non-degenerate cone $\mathcal{K}'$.

Remarks:

1) The cone is unique if ρ is irreducible.

2) This is not true if Γ is merely a Zariski dense subgroup.

Surface case,

In dim $= 2$, the cup product is a quadratic form: $b : W \times W \to \mathbb{R}$.

Hodge index theorem (Noether theorem): b has (anti-) Lorentz signature $+ - \ldots -$ (or $+$).

Thus: $\rho : G \to O(1, N)$.

Fact A semi-simple Lie group (with no compact factor) can be embedded in $O(1, N)$ iff it has the form $O(1, m)$.

Higher dimension

$c : W \times \ldots W \to W \to \mathbb{R}$

- Is there a kind of Nother theorem for c?

– Can the "signature" be bounded by means of the dimension?

Case: dimension $= 3$,

– "Trilinear forms are a challenge for mathematics" !!!

Dimension 3

Fact

(Lorentz-like property)
$b : W \times W \to W^*$ *satisfies, if* $E \subset W$ *is isotropic for* c*, then* $\dim E \leq 1$.

This allows one to classify ρ assuming $\mathrm{rk}_{\mathbb{R}}(G) \geq 2$.

(for instance, G can not contain $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$...)

– Proof of the Fact: If $\dim E \neq 0$, then, $E \cap [\omega]^{\perp} \neq 0$, and $q(a, b) = \omega \wedge a \wedge b$ negative definite.

Major steps of the proof

Dichotomy

Lemma

(Up to a finite cover and finite index sub-group) Either
— M is a torus or
— there is a non-trivial analytic Γ- invariant subset $Z \subset M$, $\dim Z > 0$.

Classification of invariant sets

Proposition

Assume G simple of rank $n-1$. Then, a non-trivial Γ-invariant analytic set Z is either finite, or is a finite collection of disjoint rigid $\mathbb{C}P^{n-1}$, i.e. the normal bundle of each component is negative.

Final step: contraction

If the normal bundle of Z is $O(-k)$, then after contraction, one gets an orbifold modeled on $\mathbb{C}^n/R_\alpha$, where

$R_\alpha : z \in \mathbb{C}^n \to \alpha z$, and $\alpha^k = 1$.

Method: We do everything for orbifolds.

– We perform contraction within this category.

– We adapt the dichotomy, if there is non invariant set, then M is a flat orbifold.

The dichotomy in the abelian case

Dinh-Sibony work + Zhang contribution

Instead of $\Gamma \to A$ abelian $\cong \mathbb{Z}^k$

Assume $\forall a \in A - \{1\}$, the spectral radius of $\rho(a)$ is > 1.

This means exactly: $\forall a \in A, \; Entropy_{Top}(a) > 0$.

(entropy-hyperbolic action)

Then, there exists $E \subset W$ a $\rho(A)$- invariant space such that:

– $\dim E = k + 1$

– E is generated by its intersection with the nef cone:

$E = \Sigma_1^{k+1} \mathbb{R} w_i$

– The w_i are eigenvalues of A.

– The action of A on E is faithful and unimodular,

– The product $w_1 \dots w_{k+1} \neq 0$.

Idea and explanation

In particular $k+1 \leq n$.

We are considering here: $k = n-1$.

Proposition (Dinh-Sibony)

Let M be a connected compact Kähler manifold. Let u and v be elements of $\overline{\mathcal{K}(M)}$.

1. *If u and v are not colinear, then $u \wedge v \neq 0$.*
2. *Let $v_1, \ldots, v_l$, $l \leq n-2$, be elements of $\overline{\mathcal{K}(M)}$. If $v_1 \wedge \ldots \wedge v_l \wedge u$ and $v_1 \wedge \ldots \wedge v_l \wedge v$ are non zero eigenvectors with distinct eigenvalues for a cohomological automorphism, then $(v_1 \wedge \ldots \wedge v_l) \wedge (u \wedge v) \neq 0$.*

Characterization of the torus

Proposition

A compact Kahler manifold M is a finite cover of a torus if and only if it has vanishing first and second Chern classes: $c_1(M) = c_2(M) = 0$.

Comments: Yau?

Consider $L = w_1 + \ldots + w_n$

The cup product of n nef elements is ≥ 0

Thus L is big: $L^n > 0$

Fact: M is a torus (up to a cover) $\iff$ L is Kaehler (ample)

Proof

1) Let $u \in H^{n-k,n-k}$, $w_I = w_{i_1} \dots w_k$

Assume u is A-invariant,

then $w_I \wedge u = 0$, because it is not A-invariant!

In particular: $L^{n-1}c_1 = L^{n-2}c_1^2 = L^{n-2}c_2 = 0$

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2) Assume L ample, then c_1 is L-primitive,

by Hodge-Riemann bilinear relations: $L^{n-1}c_1^2 < 0$, unless $c_1 = 0$

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3) "Yau formula", for a Ricci flat Kaehler metric,

$\int c_2 \wedge \omega^{n-2} = c \int \| Rm \|^2 \omega^n$

Basis of big and nef non-ample classes

Demailly- Paum:

– A class ω is Kaehler, iff $\int_Y \omega^p > 0$, for any analytic set of dimension p.

– ω is nef iff $\int_Y \omega^p \geq 0$

Corollary: if ω is big and nef but non-ample, then there is Y, of dimension $0 < p < n$, $\int_Y \omega^p = 0$

Improvement: In fact, there is Z of dimension k, such $\int_Z \omega^k = 0$, and such that any Y as above is contained in Z: Z is the basis (or **support** of ω).

For $L = w_1 + \dots w_n$, $L^p.Y = 0 \iff w_I.Y = 0$, for any $I = (i_1, \dots i_p)$.

Thus any $a(Y)$ is contained in the support of L, for $a \in A$.
Thus the Zariski closure X of $\cup a(Y)$ is non-trivial and A-invariant.

The dichotomy in the lattice case

The cohomological action

– G acts cohomologically (on $H^*(M,\mathbb{R})$)

– Any Cartan subgroup A of G acts with positive entropy...

– One proves the same result as Dinh-Sibony (there is an invariant Kaehler sub-cone).

$A \to L_A$

$gA \to L_{gA} = gL_A$, say $L_0 = L_{A_0}$, then $gL = gL_0$

• $G.L_0 = \{gL_0,\ g \in G\}$ a (smooth connected) family of big non-ample classes

Proposition

Let $\mathcal{B} \in H^{1,1}$ a connected smooth analytic submanifold of big-nef non ample classes. Then, there is Y such that $\int_Y b^d = 0$, for any $b \in \mathcal{B}$

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Proof: there is only a countable set of cohomology classes of analytic sets $[Y] \in H^{d,d}$ $(1 \leq d \leq n)$.

$F_Y = \{b \in \mathcal{B},\ \int_Y b^d = 0\}$ closed subset of $\mathcal{B}$

$\mathcal{B} = \cup_Y F_Y$

By Baire, there is Y such that $\int_Y b^d = 0$ on a open set of $\mathcal{B}$

Conclusion, by analyticity, connectedness!

Application: there exists some p and Y of dimension p, such that $(gL_0)^p.Y = 0$, for any $g \in G$.

Corollary: $gL_0^p.Y = g(L_0^p.g^{-1}Y) = 0$, thus $L_0^p.gY = 0$, for any g.
Therefore, gY contained in the support of L_0:
$\cup_g gY \subset Z_{L_0}$

Thus: $\overline{\cup_g gY}^{Zariski}$ is proper analytic invariant set.

Invariant sets

Remark

It is here that one assumes Γ is a lattice in a simple group (not merely an irreducible lattice in a semi-simple Lie group).

Let $Z \subset M$ be an analytic Γ-invariant set

Fact: Assume Z smooth, then the Γ-action extends to a G-one.

Proof: This is the content of Cantat's Theorem, following Dinh-Sibony work.

The action is trivial on the cohomology, and thus contained in $Aut^0(M)$; the closure of $\rho(\Gamma)$ is $Aut^0(M$ is isomorphic to G.

Minimal homogeneous spaces

Proposition

Let G be a simple Lie group of real rank $n-1$ acting on a compact complex manifold X. Then, $\dim X \geq \mathrm{rk}_{\mathbb{R}}(G)$, *with equality, iff* $G = \mathrm{SL}_n(\mathbb{C})$ *or* $\mathrm{SL}_n(\mathbb{R})$ *and* $X = \mathbb{C}P^{n-1}$, *in particular X is G-homogeneous.*

In our situation, more analysis leads to: Z is a disjoint union of $\mathbb{C}P^{n-1}$

Rigidity

Let D a divisor with a Γ-invariant class $E = [D]$

$\kappa : M \to P^*(H^0(E, M))$ the Kadaira map

$x \to [s_1(x), \ldots, s_l(x)]$, $l = \dim H^0(E)$.

$x \to K(x) =$ Kernel of $s \in H^0 \to s(x)$

($K(x)$ is generically a hyperplane of H^0)

Dynamical contrast

This is a meromorphic Γ-equivariant map.

It sends the Γ-dynamics into that of PGL_N on some projective space.

- The latter has no entropy!

Line bundles on $\mathbb{C}P^{n-1}$

Cases:

$D = \mathbb{C}P^{n-1}$

N the normal line bundle,

$N = [D]_D$

Classification of line bundle on $\mathbb{C}P^{n-1}$: $Pic(\mathbb{C}P^{n-1}) \cong \mathbb{Z})$

– If E is positive, then, N and D are ample, κ is holomorphic, impossible.

– If $E = 0$, then, D is deformed into a Γ-invariant singular foliation.

Its quotient space Q has dimension 1.

Γ acts trivially on Q

Leaves are individually Γ-invariant $\Longrightarrow$ entropy $= 0$.