Actions of discrete groups on stationary Lorentz manifolds

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Introduction

Global invariants of Lorentz metrics

M a differentiable manifold (today everywhere compact)

$Diff^k(M)$ acts on

$Rie^{k-1}(M)$ (resp. $Lor^{k-1}(M)$) = space of C^{k-1} Riemannian (resp. Lorentz) metrics on M.

Endow them with the Banach topology (or Frechet for $k = \infty$)

It is known that $Diff(M)$ acts **properly** on $Rie(M)$

i.e. The quotient $X = Riem(M)/Diff(M)$ is **Hausdorff** = modular space of M.

- A function on $F : g \in X \to F(g) \in \mathbb{R}$ is a Riemannian invariant: diameter, volume, integral curvature, injectivity radius...

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(SUPER-) QUESTION: **When is the $Diff(M)$-action on $Lor(M)$ proper?**

Recall G acts properly on X if: $\forall$ $K \subset X$ compact, the set (of return times)

$$G_K = \{g \in G, gK \cap K \neq \emptyset\}$$

is compact

– Gromov: the difficulty in the global studying of Lorentz manifolds lies in the fact that $Lor(M)/Diff(M)$ does not exist (as a Hausdorff space).

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The $\mathit{Diff}(M)$-action on $\mathit{Lor}(M)$ proper $\Longrightarrow \forall\, g \in \mathit{Lor}(M)$,
Stabilizer(g) is compact,
But Stabilizer(g) = Isom(g)

Question

When is the isometry group of a **compact** *Lorentz manifold* **non-compact**?

In the non-compact case:

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Cases

$G = Isom(M, g)$

G^0 its identity component (i.e the connected component of 1)

Cases:

G^0 non-compact (strongest hypothesis)

G^0 compact and non-trivial

G^0 trivial

$\Gamma = G/G^0$ the "'**discrete part** of G"

Γ acts by conjugacy: $\Gamma \to Aut(G^0) \to Out(G^0)$

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Paradigmatic example: Flat Lorentz tori

q a Lorentz form on $\mathbb{R}^n$
$\rightarrow$ a Lorentz flat torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$.

The linear isometry group of $(\mathbb{T}^n, q)$:

$$\mathrm{O}(q, \mathbb{Z}) = GL(n, \mathbb{Z}) \cap \mathrm{O}(q)$$

Full isometry group: the semi-direct product: $\mathrm{O}(q, \mathbb{Z}) \ltimes \mathbb{T}^n$

For generic q, $\mathrm{O}(q, \mathbb{Z})$ is trivial.

q rational $\iff q(x) = \alpha(\sum a_{ij}x_ix_j)$, and a_{ij} are rational numbers,

Harich-Chandra-Borel theorem ($n \geq 3$)
$\mathrm{O}(q, \mathbb{Z})$ is *big* in $\mathrm{O}(q)$;

It is a lattice in $\mathrm{O}(q)$.
$\mathrm{O}(q, \mathbb{Z})$ is a "standard" arithmetic (real) hyperbolic group

For $q_0 = -x_1^2 + x_2^2 + \ldots + x_n^2$: $\mathrm{O}(q, \mathbb{Z})$ has finite covolume

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Recall: $PSL(2,\mathbb{R}) \to SO(1,2)$

(Action of $SL(2,\mathbb{R})$ on polynomials of degree 2)

$SL(2,\mathbb{Z}) \to O_{\mathbb{Z}}(1,2)$

Some elements of $O_{\mathbb{Z}}(1,2)$

Hyperbolic:

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \to \ldots$$

Parabolic (unipotent):

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \to \ldots$$

Dimension 2

$q_0 = x^2 - y^2$

$$SO(1,1) = \{ \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \}$$

$SO_{\mathbb{Z}}(1,1) = \{1\}$?

(Avez: observed that Anosov diffeomorphisms on the 2-torus preserve Lorentz metrics ?)

$A \in SL(2, \mathbb{Z})$ hyperbolic, e.g.

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

x^u and x^s coordinates along eigen-directions

$q = x^u x^s$.

A preserves q

$\mathrm{Isom}(\mathbb{T}^2, q) =$ (essentially) $\mathbb{Z} \ltimes \mathbb{T}^2$, $\mathbb{Z}$ generated by A.

A preserves some rational $q = ax^2 + cxy + by^2$ (with all coefficients $\neq 0$)

An arithmetico-dynamical Remark

$A \in O(q, \mathbb{Z})$

A hyperbolic means it has an eigenvalue of norm $\neq 1$

Thus Spectrum $(A) = \{\lambda, \lambda^{-1}, \sigma_1, \ldots \sigma_k\}$, algebraic integers,

λ real and > 1

$\sigma_i \in S^1$

The corresponding diffeomorphism on $\mathbb{T}^n$ is partially hyperbolic with one dimensionnal stable and unstable foliation.

These foliations may be minimal (all leaves dense)

In this case, λ is a Salem number,

Conversely, any Salem number occur as a leading eignevalue for some $A \in O(q, \mathbb{Z})$ for some q.

Connected examples

Suspension $\mathbb{T}^3_A$

The suspension of A gives a flat Lorentz manifold endowed with an isometric flow which is Anosov (chaotic)

$\mathbb{T}^3_A = SOL/\Gamma$,

SOL: the 3-dimensional unimodular solvable non-nilpotent group.

(Compare with Bianchi)

Non-suspension examples

Instead of SOL

take $G = SL(2, \mathbb{R})$,

$M = SL(2, \mathbb{R})/\Gamma$, Γ a co-compact lattice

The G action on G/Γ preserves a Lorentz metric,

This metric has constant negative curvature (locally AdS)

Explanation: at the origin $1 \in G/\Gamma$, take the Killing form

$\kappa : \mathcal{G} \times \mathcal{G} \to \mathbb{R}$ (it has a Lorentz signature).

Another examples: Oscillator (or Warped Heisenberg) groups

There is a (family of) groups G, solvable but looking like $SL(2,\mathbb{R})$:

- they are solvable, so their Killing form is degenerate
- they have a bi-invariant Lorentz form on their Lie algebra
- they have lattices

Results

Hypotheses

M compact Lorentz

G acts isometrically

G^0 compact

G non-compact

$\Gamma = G/G^0$ acts on $Aut(G^0)$.

A geometric hypothesis: The G^0 action is not everywhere non-timelike: there is x_0 such that $G^0 x_0$ is timelike (the induced metric is Lorentz).

Example, strong situation: M is stationary: there is an everywhere timelike Killing field.

Algebraic hypothesis

Essentially: the conjugacy action of Γ on G^0 is not equicontinuous

Fact (non-trivial): The algebraic and geometric hypotheses are equivalent.

First formulation of results, corollaries

Up to finite cover for M and finite index subgroup for G (everywhere),

G^0 has a toral Γ-invariant factor $\mathbb{T}$ (of some dimension d)

• The action of Γ on $\mathbb{T}$ preserves some Lorentz form q and $\Gamma = O(q, \mathbb{Z})$.

• The action of $\mathbb{T}$ on M is **everywhere free**

• The orbits are all timelike: the identification of any orbit $\mathbb{T}x$ with $\mathbb{T}$ gives a Γ-invariant Lorentz form q_x (on $\mathbb{T}$)

Corollary

If a Lorentz manifold has a non compact isometry group and a somewhere timelike Killing field, then M is stationary.

Simply connected case

Corollary

A compact simply connected STATIONARY Lorentz manifold has compact isometry group.

(this will become from the next precise theorem)

D'Ambra Theorem: A compact simply connected ANALYTIC Lorentz manifold has compact isometry group.

Here the metric is C^2

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Challenge: Generalize D'Ambra Theorem to the smooth case

- Why it is important to deal with the non-analytic case?
- reminiscent to the case of codimension 1 foliations: they may exist on the smooth case but not the analytic one (Heafliger).

Why simply connected manifolds?
- Because it is generally thought that dynamics, at least in a rigid geometric background, is encoded in the fundamental group.

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Precise statements

Theorem

$\mathrm{Iso}_0(M, \mathbf{g})$ *contains a torus* $\mathbb{T} = \mathbb{T}^d$, *endowed with a Lorentz form* q, *such that* Γ *is a subgroup of* $\mathrm{O}(q, \mathbb{Z})$.

There is a new Lorentz metric $\mathbf{g}^{\text{new}}$ *on* M *having a larger isometry group than the original* $\mathbf{g}$, *such that* $\Gamma = \mathrm{O}(q, \mathbb{Z})$.

Geometrically:

- M is metric direct product $\mathbb{T} \times N$, where N is a compact Riemannian manifold,

- or M is an amalgamated metric product $\mathbb{T} \times_{S^1} L$, where L is a lightlike manifold with an isometric S^1-action.

The last possibility holds when Γ is a parabolic subgroup of $\mathrm{O}(q)$.

Remarks

- Having this description of g^{new}, one can understand g: the metric on the $\mathbb{T}$ orbits varies in the modular space of Γ-invariant Lorentz metrics on $\mathbb{T}$.

- The difference between the direct product and amalgamated case lies in the fact that the orthogonal distribution of $\mathbb{T}$ is integrable and has closed leaves.

- The statement is optimal: giving data: Γ, N..., one constructs M.

- Consideration of finite covers is necessary...

Amalgamated products

The connected case

Theorem

(Zimmer, Gromov, Adams-Stuck, Zeghib) Let G be a **connected** *non-compact Lie group acting isometricaly on a compact Lorentz manifold.*

Then the Lie algebra $\mathcal{G}$ is isomorphic to a direct sum

$$\mathcal{K} + \mathbb{R}^k + \mathcal{S},$$

where $\mathcal{K}$ is the Lie algebra of a compact semi-simple Lie group, $k \geq 0$ is an integer and $\mathcal{S}$ is trivial or isomorphic to:

▷ *a Heisenberg algebra (of some dimension),*

▷ *a warped Heisenberg algebra, or*

▷ $sl(2, \mathbb{R})$.

Conversely, any such algebra is isomorphic to the Lie algebra

Geometric structure

In particular if the G-orbits are somewhere timelike, the the factor $\mathcal{S}$ is non-trivial, and we have a (local) warped product...

Linear Dynamics

Recurrence vs homogeinity: A Gauß map

G acts on (M, g), g a pseudo-Riemannian metric

Each orbit $G.x$ is a G-homogeneous pseudo-Riemannian space : G/H.
- So the metric is left invariant by G
- and also right invariant by H.
In particular if G/H is compact, the metric is bi-invariant by a big subgroup, essentially bi-invariant,

Zimmer-Gromov ... Philosophy: Since M is compact, $G.x$ looks like a compact space:... the metric is essentially bi-invariant

A Gauß map $Ga : M \to Sym(\mathcal{G})$,

$Ga(x)$ is the quadratic form on $\mathcal{G}$ obtained via $\mathcal{G} \to T_x(Gx)$, the derivative at 1 of the map $G \to Gx$

$Ga(U, V) = g_x(\bar{U}(x), \bar{V}(x))$
$= g_x(\frac{\partial}{\partial t}(\exp tU)(x), \frac{\partial}{\partial t}(\exp tV)(x))$
$\bar{U}$ the vector field on M associated to U
$(\bar{U}(x) = \frac{\partial}{\partial t}(\exp tU)(x))$

Equivariance

$Ga(g.x) = g.Ga(x)$

The system $X = Ga(M) \subset Sym(\mathcal{G})$ is a factor of M.

Opposition:

(G, M) a conservative (general) G-dynamical system

(G, X) a dissipative (linear) dynamical system

Goal: The action on X is trivial!

Interpretation: the metric on orbits is bi-invariant.

What is special for linear systems

"Furstenberg lemma", Illustration (in a radically simple situation)

Let

$$H = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

act on $\mathbb{R}^2$.

Let $z = (x, y)$.

If z is H–recurrent, then $z = 0$

If z is non-escaping, then $x = 0$, or $y = 0$.

Recall:

- z recurrent, if there is $n_i \to \infty$, and $H^{n_i} z \to Z$
- z is non-escaping if there is K a compact set and $H^{n_i} z \in K$

$$U = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

Any U-recurrent point is fixed.

$$E = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

All points are recurrent...

Furstenberg: A **recurrent** linear dynamical system is made up to elliptic elements only.
- If $X \subset \mathbb{R}^N$ admits a finite G-invariant measure, then G acts on its support via a homomorphsim in a compact group in $GL(N)$.

Warning: One also needs linear actions on projective spaces, and "meromorphic" Gauß maps...

Case of semi-simple groups

G a simple Lie group,

$X \subset Sym(\mathcal{G})$ a compact G-invariant subset $\Longrightarrow$ G acts trivially on X.

(Typical case: $SL(2, \mathbb{R})$)

Embedding theorems (Zimmer...): If G acts on M preserving a pseudo-Riemannian metric of type (p, q), then G embeeds in $O(p, q)$.

In fact, the embedding is made via the adjoint representation $Ad : G \rightarrow GL(\mathcal{G})$.

The standard homogeneous example is G/Γ.

(The general case is a "non-commutative" G/Γ)

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Generalities on Toral actions

Notations: $\Gamma = G/G^0$ acts by automorphism on G^0

The action is non-equicontinuous $\Longrightarrow$ G^0 is not semi-simple (since for a compact semisimple G^0, $Aut(G^0) \cong G$ is compact)

$\mathbb{T}_1$ the toral factor
$\mathbb{T} = \mathbb{T}^k \subset \mathbb{T}_1$ a minimal Γ-invariant sub-torus

$\rho : \Gamma \to Aut(\mathbb{T}^k) = GL(k, \mathbb{Z})$
Γ acts on $Sym(\mathbb{R}^k)$

Almost Lorentz implies Lorentz

Lemma

(Case $\Gamma = \{A^n, n \in \mathbb{Z}\}$)
Let $F = \mathrm{Sym}(\mathcal{E})$, ($\mathcal{E} = \mathbb{R}^k$)

and assume $A = EHU$ non-elliptic (i.e., either H or U is non-trivial).

Suppose there is a Lorentz form q_0 which is A-recurrent, and let $K \subset \mathrm{GL}(\mathcal{E})$ be the torus generated by the powers of E.

Then, $\int_K B^F(q_0)\,\mathrm{d}\mu(B)$ is an A-invariant Lorentz form, where μ is the Haar measure on K.

Remarks:

- This fact is trivial in the case of Euclidean (positive) forms

Proposition

Let $\rho : \Gamma \to GL(\mathcal{E})$ be such that $\rho(a)$ is non-elliptic for any $a \in \Gamma$.
Let $F = \mathrm{Sym}(\mathcal{E})$, and assume that the associated action ρ^F preserves a compact set of F contained in the (open) subset of Lorentz forms, and that ρ^F leaves invariant a finite measure on such compact set. Then, $\rho(\Gamma)$ preserves some Lorentz form.

Corollary

Let Γ be a subgroup of $GL(k, \mathbb{Z})$ which acts on $\mathrm{Sym}(\mathbb{R}^k)$ by preserving a finite measure supported in the open set of Lorentz forms. Then, up to a finite index, Γ preserves a Lorentz form.

Lorentz Dynamics

Goal: uniformity and no-singularity

Prototypes of Lorentz isometries: hyperbolic and parabolic

which qualitative properties unify them?

Let ϕ be a diffeomorphism of a compact manifold M.

Definition

A vector $v \in T_x M$ is called approximately stable if there is a sequence $v_n \in T_x M$ such that:

- $v_n \rightarrow v$

- $D_x \phi^n v_n$ is bounded in TM.

The set of approximately stable vectors in $T_x M$ is denoted $\mathrm{AS}(x, \phi)$

Their union over M is denoted $\mathrm{AS}(\phi)$,

The vector v is called **strongly approximately stable** if $D_x \phi^n v_n \rightarrow 0$.

Similar notations: $\mathrm{SAS}(x, \phi)$ and $\mathrm{SAS}(\phi)$

Theorem (Zeghib)

Let ϕ be an isometry of a compact Lorentz manifold $(M, \mathbf{g})$ such that the powers $\{\phi^n\}_{n\in N}$ of ϕ form an unbounded set (i.e., non precompact in $\mathrm{Iso}(M, \mathbf{g})$). Then:

- $\mathrm{AS}(\phi)$ *is a Lipschitz condimension 1 vector subbundle of TM which is tangent to a condimension 1 foliation of M by geodesic lightlike hypersurfaces;*
- $\mathrm{SAS}(\phi)$ *is a Lipschitz 1-dimensional subbundle of TM contained in* $\mathrm{AS}(\phi)$ *and everywhere lightlike.*