On Homogeneous compact spacetimes

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Homogeneous space: $M = G/H$, G a Lie a connected Lie group, and H a closed subgroup of G.

– We suppose everywhere that G acts faithfully on M, i.e. we can not simplify G/H to a smaller G'/H'

– G acts on the left on M: $(g, xH) \in G \times M \to (gx)H \in M$

– This action preserves some “rigid geometric structure”.

– The homogeneous space is of Riemannian type (resp. Lorentzian...) if the G-action preserves a Riemannian (resp. Lorentz...) metric on M.

Example: $G = SL(2, \mathbf{R})$:

1. $H = SO(2)$, $M = SL(2, \mathbf{R})/SO(2) \to$ Riemannian metric

2. $H = AG$ the affine group (the group of affine transformations of the real line)

$$H = \{\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\}$$

$G/H = S^1 \to$ projective structure

3. $H = \{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\}$

$G/H = \mathbf{R}^2 - \{0\} \rightarrow$ affine structure

4. H discrete → Lorentz structure:

Killing form on the Lie algebra $sl(2, \mathbf{R} = \{A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}\}$:

the quadratic form: $k(A, A) = \det(A) = bc - a^2$

(details latter)

(-exercise: have we here exhibited here all the $SL(2, \mathbf{R})$-homogeneous spaces, i.e. 1, 2, 3 and 4 are all the closed subgroups of $SL(2, \mathbf{R})$ (up to conjugacy)?)

Hierarchy of geometric structures:

1. Riemannian: the most beautiful and rigid structures!

2. Lorentz structure: non-degenerate quadratic form of type $- + \ldots +$: it is the nearest to the Riemannian case.

3...?

• The homogeneous Riemannian problem (trivial for our talk here): $M = G/H$ is of Riemannian type iff, H is **compact**

-G is any Lie group,

If we wish M compact, then G is compact.

• From a dynamical point of view, Riemannian homogeneous space are trivial!

• We know very few about non-Riemannian homogeneous space.

– The interest of the Lorentz case: it is the easiest Non-Riemannian problem.

- The homogeneous compact Lorentz problem: Find G a Lie group, and H a closed Lie subgroup of G, such:

C1. The action of G on G/H preserves a Lorentz metric.

C2. $M = G/H$ is compact.

Fact

1. If H is **discrete**, then:

– C1 is equivalent to that the Lie algebra $\mathcal{G}$ has an $Ad(H)$- Lorentz scalar product.

– C2 means (by definition) that H is a co-compact lattice in G.

Explanation: Left translate to G a Lorentz scalar product on $\mathcal{G}$ which is $Ad(H)$-invariant. The Lorentz metric on G is: G-left-invariant, and H-right invariant. Therefore, it passes to a G-invariant Lorentz metric on G/H.

(exercise: Where have we used discreetness of H?)

Fundamental Example 1 $G = SL(2, \mathbf{R})$

▷ The Killing form k on the Lie algebra $sl(2, R)$ has signature $-++$ (may be $--+$, in this case, consider $-k$)

– It is Ad-invariant, in particular $Ad(H)$-invariant for any co-compact lattice.

Thus *For any co-compact lattice $H \subset SL(2, \mathbf{R})$, $SL(2, \mathbf{R}/H$ is a homogenous compact spacetime*

- Alternative explanation of the Fact above:

Right translate k , and get a right invariant Lorentz metric on G. Thus, it passes to right quotients G/H.

-G acts by the left G/H

– This action is isometric, since, also the left action on G on itself preserves the Lorentz metric, because k is bi-invariant.

Remark: $SL(2, \mathbf{R})/H$ is (up to 2-cover) to unit tangent bundle of the hyperbolic surface $\mathbf{H}^2/H$.

Fundamental example 2

A family of sympathetic groups: Warped Heisenberg groups: family of **solvable** groups looking like $SL(2, R)$:

• they admit Lorentz bi-invariant Lorentz metrics, i.e. their Lie algebra admit Ad-invariant Lorentz scalar products ($\neq$ the Killing form, which is degenerate).

• they have co-compact lattices!

▷ G is a warped Heisenberg group, H a lattice

▷ As in the case of $SL(2, \mathbf{R})$,

$M = G/H$ is a compact homogeneous spacetime, where H is a cocompact lattice

(co-compact is superfluous, since any lattice in a solvable Lie group is co-compact).

• **The simplest example** of warped Heisenberg groups: dimension 4 (known as oscillator group, Diamond group...)

The semi-direct product $G = S^1 \ltimes Heis$

$Heis$ = Heisenberg group (of dimension 3):

$$Heis = \{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, x, y, z \in \mathbf{R} \}$$

$Heis$ is characterized essentially, by the existence of a non-split exact sequence:

$1 \to \mathbf{R} \to Heis \to \mathbf{R}^2 \to 1$

$G = S^1 \ltimes Heis$ is defined by:

– S^1 acts trivially on the center $\mathbf{R}$, and acts by rotation on $\mathbf{R}^2$

• Also G is characterized by being a non-trivial central extension of Ec by S^1,

Ec = group of Euclidean isometries of the plane.

$1 \to S^1 \to G \to Ec \to 1$

Generalization: canonical warped Heisenberg groups

• Recall the construction of Heisenberg algebras: $\mathcal{HE}_d$ ($\dim = 2d + 1$)

$\mathbf{R} \bigoplus \mathbf{C}^d$, with basis $Z, e_1, \ldots, e_d$

The only non-vanishing brackets are: $[e_k, ie_k] = Z$. (here $i = \sqrt{-1}$)

Equivalently,

$[X, Y] = \omega(X, Y)Z$.

ω symplectic form, i.e. $\omega(X, Y) =< X, iY >_0$,

$<, >_0$ hermitian product

- Canonical warped Heisenberg algebras

Add an exterior element t, such that:

$[t, e_k] = ie_k, [t, ie_k] = -e_k$, and $[t, Z] = 0$

- Scalar product

$<, >$

Endow $\mathbf{C}^d$ with its hermitian $<, >_0$

Decree $\mathbf{C}^d$ is orthogonal to $\{t, Z\}$.

$< t, t >=< Z, Z >= 0$ and $< t, Z >= 1$.

- $<, >$ is a Lorentz scalar product which is $Ad(\mathcal{HE}_d^t)$-invariant! i.e. $\forall u \in \mathcal{HE}_d^t\ u$, ad_u is antisymmetric with respect to $<, >$

exercise: Why this doesn't work for the Heisenberg algebras themselves?

- Consider $\tilde{G} = \tilde{He}_d^t$ the simply connected Lie group generated by $\mathcal{HE}_d^t$
- $\tilde{He}_d^t$ is a semi-direct product of $\mathbf{R}$ by He_d:

- The action of $\mathbf{R}$ on the center is trivial.
- The action on $\mathbf{C}^d$ is via multiplication by $\exp is$.
- This is in fact an action of S^1

- Consider $G = He_d^t = \tilde{He}_d^t/\mathbf{Z} = S^1 \ltimes He_d$
- Any lattice in the Heisenberg group He_d is a lattice in He_d^t.

- example of a lattice in He_1:

$$Heis_{\mathbf{Z}} = \{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, x, y, z \in \mathbf{Z}\}$$

General construction of warped Heisenberg groups:

- Consider a semi-direct product $\mathbf{R} \ltimes He_d$, where $\mathbf{R}$ acts on $\mathbf{C}^d$ via:

$s \to \exp(2\pi s A) \in U(d)$ such that:

- ▶ $\exp(2\pi A) = 1$,
- ▶ A diagonalizable, $\lambda_1, \dots, \lambda_d \in \mathbf{Z}$
- ▶ The λ_i have the same sign.

Thus, the action of $\mathbf{R}$ factors via an action of S^1

– The semi-direct product $G = S^1 \ltimes He_d$ is called a warped Heisenberg group.

– The conditions guaranty that G has a bi-invariant Lorentz metric.

– G has lattices

– Any quotient G/H, H a lattice, is a compact homogeneous spacetime.

Some remarks:

1. recall that the Lorentz scalar product was defined by: $\mathbf{C}^d$ is endowed with its hermitian metric$<, >_0$ and is orthogonal to the plane $\{t, Z\}$, $< t, t >=< Z, Z >= 0$ and $< t, Z >= 1$.

– One may take $< t, t >=$ Constant $\neq 0$, and multiply the other products by any constant ($\neq 0$), and gets another bi-invariant Lorentz metric.

– However, up to automorphism, there exists only one bi-invariant Lorentz metric on a warped Heisenberg group..

-In particular a metric is isometric to any multiple of it. This follows from existence of homotheties. (This is true for $\mathbf{R}^n$ but not for $SL(2, \mathbf{R})$.

— Warped Heisenberg groups are (locally) symmetric Lorentz spaces of non reductive type.

– They have non-reductive holonomy, i.e. they have a codimension 1 **parallel** foliation which has no supplementary parallel direction field.

– The Ricci curvature of a warped Heisenberg group equals its Killing form (up to constant).

Historical comments

• He_1^t the 4-dimensional Heisenberg group is known as:

– Diamond group, in Representation Theory

– Oscillator group in Representation Theory and quantum mechanics

• The bi-invariant Lorentz metrics were known to medina-Revoy, and "partially" to Zimmer and Gromov.

– This seems folkloric in relativistic literature: some gravitational plane waves spacetimes...

– Witten-Nappi: "a WZW model based on a non semi-simple group" (1993)

Justification of the name "oscillator group":

The Lie algebra $\mathcal{HE}_1^t$ has the following representation in the algebra of operators of the Hilbert space $E = L^2(\mathbf{R})$:

$Z \to 1 =$ (Identity)

$X \to q$ (position)

$Y \to p$ (impulsion)

$t \to p^2 + q^2$ (energy),

where the operators q and p are given by:

$q(f) = xf \ (f \in L^2(\mathbf{R}))$

$p(f) = \frac{\partial f}{\partial x}$

— To show that this gives a homomorphism, one verifies in particular: $[q, p] = 1$, which is the Heisenberg uncertainty principle.

$p^2 + q^2$ is the energy of the harmonic oscillator, which explains the origin of the terminology.

Another examples and constructions

- Products

N Lorentz homogeneous, L Riemannian homogeneous $\Longrightarrow M = N \times L$ is Lorentz homogeneous.

- Local products:

$\tilde{N}$ Lorentz homogeneous, $\tilde{L}$ Riemannian homogeneous, $M = \tilde{N} \times \tilde{L} / \Gamma$, where Γ is a **non split** subgroup of $Isom(\tilde{L}) \times Isom(\tilde{L})$

If the centralizer of Γ acts transitively on $\tilde{N} \times \tilde{L}$, then M is homogeneous.

– Examples exist

- Counter-examples: Nil and SOL-manifolds.

In dimension 3, the nilpotent group, $Heis$ and the solvable SOL admit left invariant complete flat Lorentz metric. Their compact quotients are flat Lorentz manifolds (if fact any compact Lorentz flat 3-manifold, is of this type or is a torus, up to a finite cover).

-Nil and SOL manifolds are homogeneous but not Lorentz homogeneous.

Classification of compact homogeneous spacetimes: Part I: structure of their isometry group

Theorem 1 *Let $M = G/H$ be a compact homogeneous Lorentz manifold.*

Then, up to compact objects: G is $SL(2, \mathbf{R})$ or a warped Heisenberg group.

More precisely: there is a subgroup $S \subset G$, such that:

- *S is normal, and the Lie algebra of S is a factor in $\mathcal{G}$*
- *S is co-compact in G (i.e. G/S is compact)*
- *S is isomorphic to $PSL_k(2, \mathbf{R})$ the k-folded cover of $PSL(2, \mathbf{R})$, or*

-S is a warped Heisenberg group.

- *S acts on M locally freely (i.e. stabilizer in S are discrete)*

Corollary The stabilizer H is **"almost discrete":** its connected component is compact. *(This is non-obvious a priori, and false for non-compact homogeneous spacetimes, and for general homogeneous pseudo-Riemannian manifolds, even compact)*

Classification of compact homogeneous spacetimes: Part II: their geometric structure

Theorem 2 *Up to compact objects, they are isometric to S/H, where H is a co-compact lattice (in particular discrete) in S, which is $PSL(2, \mathbf{R})$ or a warped Heisenberg group.*

- Roughly, M is a a "local product" modeled on $S \times \tilde{L}$, where $\tilde{L}$ is a homogeneous Riemannian manifold

—– Details:

- *The case $S = PSL_k(2, \mathbf{R})$ (due to Gromov)*

- $M = S \times \tilde{L}/H$:

- $\tilde{L}$ is a compact homogeneous Riemannian manifold

- There is H_0 a lattice in S, such that H is the graph of a homomorphism $\rho : H_0 \to Isom(\tilde{L})$

- The centralizer of $\rho(H_0)$ acts transitively on $\tilde{L}$.

- The metric on $S \times \tilde{L}$ is: $c.Killing \bigotimes r_{\tilde{L}}$, c constant, $r_{\tilde{L}}$ the Riemannian metric of $\tilde{L}$

- Conversely, with these data, one constructs a compact homogeneous spacetime.

- *S a warped Heisenberg group: a little bit complicated description...*

Generalization: non-homogeneous case

The motivation of the study of compact homogeneous spacetimes is the general question

When is the isometry group of a **compact** *Lorentz manifold is* **non-compact***?*

Theorem 3 *(Zimmer, Gromov, Adams-Stuck, Zeghib) Recall that the Lie algebra of a compact Riemannian manifold is a sum of an abelian Lie algebra with a semi-simple Lie algebra of compact type. (i.e. the Lie algebra of a compact semi-simple Lie group).*

– In the Lorentz case, the new factor that might occur, is a subalgebra of $\mathcal{S}$*, where* $\mathcal{S}$ *is the Lie algebra of* $SL(2, \mathbf{R})$ *or a warped Heisenberg group.*

More details

(Killing algebra of $M =$ the Lie algebra of its isometry group)

Theorem 4 *(Adams-Stuck, Z.) The Killing Lie algebra of a compact Lorentz manifold is isomorphic to a direct sum*

$$\mathcal{K} + \mathbf{R}^k + \mathcal{S},$$

where $\mathcal{K}$ *is the Lie algebra of a compact semi-simple Lie group,* $k \geq 0$ *is an integer and* $\mathcal{S}$ *is trivial or isomorphic to:*

▷ *a Heisenberg algebra (of some dimension),*

▷ *a warped Heisenberg algebra, or*

▷ $sl(2, \mathbf{R})$.

Conversely, any such algebra is isomorphic to the Lie algebra of the isometry group of some compact Lorentz manifold.

Comments about the non-compact case

▷ There are generalizations to Lorentz non-compact manifolds (Kowalsky, Adams-Stuck):

– The question is to understand non-proper Lorentz isometric actions.

– Non-compact Lorentz manifolds are more interesting in Physics than the compact ones.

– Those with non-proper isometry groups are strongly non-Riemannian and may appear to better satisfy relativistic requirements.

– However, it seems that most of the physical spacetimes (except those of constant curvature) have proper isometry group.

– In particular, if they are homogeneous, then they are of the Riemannian type.

– Relativity is not as revolutionary as we think!

Ingredients of the proof of Theorem 3

Consider the L^2 bilinear form on the Lie algebra $\mathcal{G}$:

$\kappa(X,Y) = \int_M < X(x), Y(x) >_x dx$

(X, Y are Killing fields, $<,>$ the Lorentz metric)

• 1. κ is a **bi-invariant** quadratic form on $\mathcal{G}$ (general fact)

• 2. However, κ might be trivial

• 3. The point is to show that it is sufficiently non-trivial...

Details:

1). Let G be a Lie group acting on M, $\mathcal{G}$ its Lie algebra:

Action means: a homomorphism (of Lie brackets)

$X \in \mathcal{G} \to \bar{X} \in$ Vector-fields on M.

To $Y \in \mathcal{G}$, is associated

- ϕ^t a one-parameter subgroup of G, and

- $\overline{\phi}^t$ a one parameter group of diffeomorphisms on M.

• Naturally:

$$\overline{\phi}^t_* \overline{X} = \overline{Ad(\phi^t)X}$$

If G preserves a volume dx and a q-covariant tensor T, then,

$$\kappa^T(X_1, \ldots, X_q) = \int_M T(\overline{X_1}(x), \ldots, \overline{X_q}(x))dx$$

determines a tensor on $\mathcal{G}$ which is **bi-invariant**.

2). Unfortunately κ^T is generally trivial:

$G = SO(n), n > 2$, T any left invariant quadratic form (degenerate or not, positive or not...), but κ^T is a multiple of the Killing form, by simplicity.

- In particular, it may happen that $\kappa^T = 0$ for T Lorentz.

3). Now, G is **non-compact** and acts isometrically on a compact (or just finite volume) Lorentz manifold M,

Consider the associated κ:

Major first step (of the proofs of Theorems 1 and 3):

κ is sufficiently non-trivial: It satisfies a condition (*)

(*) means that κ is between a Lorentz and a Euclidean product!

• Theorems 1 and 3 follow from an “Algebraic Lemma” classifying” those Lie algebras admitting Ad-invariant scalar product satisfying (*).

– Similar to the lemma saying that a Lie algebra with an $Ad-$invariant positive scalar product is Abelian + compact.

Behind (*) is the following :

Fundamental non-degeneracy Fact *M compact Lorentz manifold,* $\phi^t \subset Isom(M)$ *a one parameter group with infinitesimal generator (the Killing filed)* X.

Suppose ϕ^t *is non-precompact, i.e non-equicontinuous, i.e. the closure of* $\{\phi^t, t \in \mathbf{R}\}$ *in* $Isom(M)$ *is not compact.*

Then, X *is everywhere non-timelike:* $< X(x, X(x) >\geq 0$, $\forall x$.

Corollary: Condition (*): *Let* $\mathcal{P}$ *a linear subspace of* $\mathcal{G}$ *containing a dense set of non-precompact Killing fields.*

Then, $\kappa|\mathcal{P} \geq 0$, *and dim* $Ker(\kappa|\mathcal{P}) \leq 1$

Proof of the Fundamental non-degeneracy Fact: It is based on 2 uniformity facts:

Fact 1 *Let* $\{\phi^t\} \subset Isom(M)$ *be a one parameter group of isometries. If for some* $t_i \to \infty$, $\{\phi^{t_i}\}$ *is precompact (i.e. equicontinuous), then* $\{\phi^t\}$ *is precompact.*

Fact 2 *If for some* $x_i \in M, t_i \to \infty$, $\{D_{x_i}\phi^t\}$ *is equicontinuous (i.e.* $\| D_{x_i}\phi^t \|$ *and* $\| (D_{x_i}\phi^t)^{-1} \|$ *bounded), then* $\{\phi^{t_i}\}$ *is equicontinuous (and therefore by the fact above* $\{\phi^t\}$ *is equicontinuous)*

— Facts 1 and 2 are true in affine dynamics, i.e. for $\{\phi^t\} \subset \mathrm{Affin}(M, \nabla)$, ∇ a connection.

We need a third fact special to the Lorentz case:

Fact 3 *If a Killing field X is somewhere timelike (i.e. $< X(x_0), X(x_0) >< 0$), then it generates an equicontinuous flow $\{\phi^t\}$.*

Sketchs of proofs:

— Fact 3: Let U be a neighborhood of x_0 where X is timelike. By Poincaré recurrence Lemma, there exist x_i near x_0, $t_i \to \infty$, such that $\phi^{t_i} x_i$ is near x_0.

Now, near x_0, $D_{x_i}\phi^{t_i}$ behave as Riemanniann isometries.

Apply Fact 2, and then Fact 1 to deduce that $\{\phi^t\}$ is equicontinuous.

— Fact 1:

$L = \overline{\{\phi^t\}} \subset Isom(M)$

L abelian $\Longrightarrow L =$ a cylinder $T^k \times \mathbf{R}^d$ (where T^k is a torus).

L has a dense one parameter group ($\{\phi^t\}$ itself), i.e. a dense geodesic (when L is seen as a cylinder) $\Longrightarrow L = T^k$, or $L = \mathbf{R}$.

Now, if there $\exists$ $\{\phi^{t_i}\}$ equicontinuous, then $L \neq \mathbf{R}$, thus $L = T^k$, i.e. $\{\phi^t\}$ is equicontinuous.

— Fact 2: By definition of its Lie group structure, $Isom(M)$ acts properly (and freely) on the frame bundle $P(M)$.