Multiplicity of the second Schrödinger eigenvalue on closed surfaces

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Introduction

Some time ago, S. Y. Cheng proved the following :

Theorem 1 *[Che76] Let (S, g) be a compact connected smooth riemannian surface without boundary, and $\triangle = d^*d$ the positive Laplace operator on functions. Then the multiplicity* $\mathrm{mult}(\lambda) = \dim\ker(\triangle - \lambda)$ *of any eigenvalue λ of $\triangle$ is bounded* independently of the metric *by a function of its index* $\mathrm{ind}(\lambda) = \sum_{t<\lambda} \dim\ker(\triangle - t)$ *and the topology of S.*

Recall that the eigenvalues of $\triangle$ on any closed riemannian manifold have finite multiplicity, that they form an infinite discrete subset of $[0, \infty[$, and that on a connected manifold 0 is a simple eigenvalue. The same is true for a Schrödinger operator $H = \triangle + v$, where the "potential" v is (multiplication by) any smooth function v on S : the smallest eigenvalue λ_0 is no longer zero, but remains simple [Bér, Cha].

Thus the second eigenvalue is of index 1, denote it λ_1, and Cheng's result (or rather its immediate adaptation to Schrödinger operators) gave the finiteness of the quantity $\overline{m}_1(S) = \sup_H \dim\ker(H - \lambda_1)$, the supremum of the multiplicity of λ_1 over all Schrödinger operators on a surface S. This fact is by no means evident a priori, and would be false in dimension 3 or higher by [CdV87].

Cheng's bound was quadratic in the Euler characteristic $\chi(S)$, but already gave for S a sphere $\mathbb{S}^2$ the optimal result $\overline{m}_1(\mathbb{S}^2) = 3$, attained for example by the laplacian of the canonical metric.

Then G. Besson [Bes80] improved the bound to one linear in $\chi(S)$, and refined it further to obtain sharp bounds for the projective plane and the torus, namely

$$\overline{m}_1(\mathbb{P}^2) = 5, \qquad \overline{m}_1(\mathbb{T}^2) = 6$$

respectively realized by the round metric and the flat equilateral torus (but not only them). The general bound was $7 - 2\chi(S)$ for an orientable S.

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N. S. Nadirashvili [Nad88] further improved Besson's bound to

$$\overline{m}_1(S) \leq 5 - 2\chi(S) \qquad \text{for } \chi(S) < 0.$$

In the meantime, Y. Colin de Verdière [CdV86, CdV87, CdV88] showed how to find metrics or Schrödinger operators with preassigned eigenvalues in certain degenerating families, giving in particular the beautiful *lower bound*

Theorem 2 *[CdV87] For any S, one has $\overline{m}_1(S) \geq \mathrm{chr}(S) - 1$, where $\mathrm{chr}(S)$ is the maximal n such that one can embed in S the complete graph K_n on n vertices.*

In the same article, he improved the upper bound for the Klein bottle $\mathbb{K}^2 = \mathbb{P}^2 \sharp \mathbb{P}^2$, giving the equality

$$\overline{m}_1(\mathbb{K}^2) = 5 = \mathrm{chr}(\mathbb{K}^2) - 1.$$

These results had the following striking consequence

Corollary 3 *[CdV87] For the four surfaces with $\chi(S) \geq 0$, namely $\mathbb{S}^2, \mathbb{P}^2, \mathbb{T}^2, \mathbb{K}^2$, one has $\overline{m}_1(S) = \mathrm{chr}(S) - 1$.*

For any surface S, the number $\mathrm{chr}(S)$ is known thanks to the work of Ringel and Youngs [RiYo68, Rin] to coincide with the upper bound found by Heawood in the 19th century, and conjectured by him to be sharp, namely

$$\mathrm{chr}(S) = \left\lfloor \frac{1}{2}\left(7 + \sqrt{49 - 24\chi(S)}\right) \right\rfloor \quad ,$$

except for Klein bottle where it is one less, namely 6. As to the notation, $\mathrm{chr}(S)$ stands for "chromatic number of S", since it is known to coincide with the least number of colors sufficient to color any map on S (to be fair, this assertion when $S \neq \mathbb{S}^2$ is reasonably easy [Bol], whereas for $S = \mathbb{S}^2$ it seems not to be so !).

One is now naturally tempted, following Y. Colin de Verdière in [CdV87], to make the

Conjecture *For all S, $\overline{m}_1(S) = \mathrm{chr}(S) - 1$.*

Additional motivation for this conjecture is that it is true for all manifolds of dimensions other than 2, $\overline{m}_1$ being 2 for the circle and ∞ in dimension 3 or higher (loc. cit.). The main purpose of this paper is to improve — asymptotically by a factor of two — Nadirashvili's bound, adding evidence to the conjecture :

Theorem 4 *If $\chi(S) < 0$, $\overline{m}_1(S) \leq 5 - \chi(S)$. In particular, the above conjecture holds for all surfaces with $\chi(S) \geq -3$, the four new cases being $\mathbb{T}^2 \sharp \mathbb{T}^2$ and $\sharp n\mathbb{P}^2$, $n = 3, 4, 5$.*

The values of $5 - \chi(S)$ and $\mathrm{chr}(S) - 1$ for the first few negative $\chi(S)$ are summarized in the array

$5 - \chi(S)$	6	7	8	9	10	11	12	13
$\mathrm{chr}(S) - 1$	6	7	8	8	9	9	9	10

Recall that diffeomorphism types of closed connected surfaces with the connected sum operation $\sharp$ form a commutative semigroup generated by $\mathbb{T}^2$ and $\mathbb{P}^2$, with the relation $\mathbb{T}^2 \sharp \mathbb{P}^2 = \sharp 3\mathbb{P}^2$, and that $\chi(S_1 \sharp S_2) = \chi(S_1) + \chi(S_2) - 2$, with $\chi(\mathbb{P}^2) = 1$. Thus for each even (resp. odd) value of χ not exceeding 2, there are two (resp. one) closed surfaces.

Theorem 4 will be deduced from

Theorem 5 *If $\chi(S) < 0$ and $E \subset C^0(S)$ is a vector space of continuous functions on S such that for all $f \in E-0$, the sets $\{f > 0\} = \{x \in S|f(x) > 0\}$ and $\{f \geq 0\} = \{x \in S|f(x) \geq 0\}$ are connected and nonempty. Then* $\dim(E) \leq 5 - \chi(S)$.

That this implies theorem 4 is an immediate consequence of the following properties of index 1 eigenfunctions f of two-dimensional Schrödinger operators $H = \triangle + v$ (see e.g. [Bér, Cha]).

(1) $S - f^{-1}(0)$ has exactly two connected components, namely $\{\pm f > 0\}$.

(2) If $x \in f^{-1}(0)$, then f is given by $f = \mathrm{Im}(z^n)$ ($n \geq 1$) in a suitable C^1 coordinate chart $z : (S, x) \to (\mathbb{C}, 0)$ centered at x. In particular f changes sign in any neighborhood of x, and $\{f \geq 0\}$ is the closure of $\{f > 0\}$ (hence is connected).

Property (1) is a simple consequence of Courant's celebrated theorem on nodal domains [CoHi], saying more generally (and in any dimension) that $S - f^{-1}(0)$ has at most $k + 1$ connected components, if f is an index k eigenfunction of H.

Property (2) results easily from Carleman's theorem ([Car33] see also [Jos, p. 35]) asserting that eigenfunctions of H cannot vanish to infinite order at any point [1]. Note that the weaker "unique continuation property" of eigenfunctions – they cannot be zero on an open set – already plays a crucial role in the proof of Courant's theorem.

These strong structural properties of two-dimensional Schrödinger eigenfunctions f were also essential in the works on multiplicity mentioned above. They imply that the so-called nodal sets $f^{-1}(0) \subset S$ are homeomorphic to (not necessarily connected) finite graphs, that cut S into two connected components, of which they are the common boundary.

Remark. Yet another reason to believe in Colin de Verdière's conjecture is related to the so-called "two-piece property". Namely T. Banchoff has shown [Ban65] that if S is an embedded polyhedral closed surface in affine m-space, not contained in any hyperplane, and meeting every open (equivalently : closed) affine half-space in a connected set, then $m \leq \mathrm{chr}(S) - 1$. Moreover this bound is optimal, see [Ku80, BaKu97]. Property (1), and more generally the assumption of theorem 5 are weak versions of the two-piece property, viewing S as mapped to the dual E^* of E by the tautological evaluation map. In particular, a "barycenter" is fixed for S (the origin $0 \in E^*$), and one cuts only by hyperplanes containing it. Moreover, the map $S \to E^*$ is a priori only a continuous map, not a polyhedral embedding. Thus theorem 5 implies Banchoff's result for surfaces S with $0 > \chi(S) \geq -3$ (we will also derive bounds for the four surfaces with $\chi \geq 0$, which coincide again with $\mathrm{chr}(S) - 1$ except for the Klein bottle).

Outline of proof of theorem 5

Let us now outline the proof of theorem 5, leaving details to subsequent sections. First consider the space $\mathcal{U} = \mathcal{U}(S)$ of connected and nonempty open subsets $U \subset S$ with connected

[1] Carleman's theorem was later generalized to higher dimensions by Aronszajn [Ar57], but then it no longer implies conjugacy to some harmonic polynomial [BéMe82]

and nonempty complement $S - U$. It is topologized by taking as a basis of open sets the subsets

$$\mathcal{U}_K = \left\{ U \in \mathcal{U} \,\middle|\, U \supset K \right\}$$

for all compact subsets K of S. It is easy to see that this makes the map $f \mapsto \{f > 0\}$ from $E - 0$ to $\mathcal{U}$ continuous. Note that $\mathcal{U}$ is not Hausdorff. For instance the closure of $\{U\}$ is the set of all U' contained in U. In particular $\mathcal{U}$ is a T_0 space : any two distinct points have different closures.

Let $\tilde{\mathcal{P}} \subset \mathcal{U} \times \mathcal{U}$ denote the (closed) subspace of ordered pairs of disjoint elements of $\mathcal{U}$, i.e.

$$\tilde{\mathcal{P}} = \left\{ (U, V) \in \mathcal{U} \times \mathcal{U} \,\middle|\, U \cap V = \emptyset \right\}$$

and let $\mathcal{P} = \tilde{\mathcal{P}}/\mathbb{Z}_2$ be the quotient space of unordered pairs. Then $\tilde{\mathcal{P}} \to \mathcal{P}$ is a double covering. More precisely, any point $x \in S$ yields a trivialisation over the open subset $\mathcal{P}_x = \{\{U, V\} \in \mathcal{P} \,|\, x \in U \cup V\}$, and those subsets obviously cover $\mathcal{P}$.

Observe now that any vector subspace $E \subset C^0(S)$ satisfying the hypothesis of theorem 5 gives rise to a continuous map

$$\mathbb{S}^{\dim E - 1} \cong \mathbb{S}(E) \longrightarrow \tilde{\mathcal{P}}$$

sending f to $(\{f > 0\}, \{f < 0\})$. It is $\mathbb{Z}_2$-equivariant for the antipodal action on the sphere $\mathbb{S}(E)$ of directions in E, and the covering transformation on $\tilde{\mathcal{P}}$. One is thus reduced to establish a "Borsuk-Ulam type" theorem, bounding the possible m's such that there is a continuous $\mathbb{Z}_2$-map $\mathbb{S}^{m-1} \to \tilde{\mathcal{P}}$. Recall that Borsuk-Ulam's theorem says that if there is such a map between two spheres, the dimension of the first is at most that of the second [Mas].

As a warming-up, and also to show that this approach is not doomed to failure, let us sketch how it works in the case $S = \mathbb{S}^2$ (which is not covered by theorem 5, since it assumes negative Euler characteristic for S).

In this case, $\mathcal{U}$ is the space of simply connected domains $U \subsetneq \mathbb{S}^2$, with the non-Hausdorff topology defined above. Then if $F : \mathbb{S}^{m-1} \to \tilde{\mathcal{P}}$ is a continuous $\mathbb{Z}_2$-equivariant map, one can write $F(v) = (D(v), D(-v))$, with $D(v)$, $D(-v)$ disjoint open subsets of $\mathbb{S}^2$ homeomorphic to disks. The "graph" $\bigcup \{v\} \times D(v)$ is open in $\mathbb{S}^{m-1} \times S$, as easily checked from the continuity of $D : \mathbb{S}^{m-1} \to \mathcal{U}$. Now since the $D(v)$ are contractible and $\mathbb{S}^{m-1}$ is finite-dimensional and metric, Bielawski's selection theorem [Bie89] (see also appendix) provides a continuous map $s : \mathbb{S}^{m-1} \to \mathbb{S}^2$ such that $s(v) \in D(v)$, and in particular $s(v) \neq s(-v)$. The normalization of $v \mapsto s(v) - s(-v)$ is then a $\mathbb{Z}_2$-equivariant map $\mathbb{S}^{m-1} \to \mathbb{S}^2$, and the conclusion $m \leq 3$ results from Borsuk-Ulam's theorem. To deal with other surfaces, we will first explain how this latter theorem may be proved.

Let $\widetilde{X} \to X$ be a double covering space, i.e. a locally trivial bundle with discrete two point fibers. It is also a principal bundle with $\mathbb{Z}_2$ structure group, and comparison of local trivializations gives rise to a class $\alpha \in \check{H}^1(X, (\mathbb{Z}_2)_X)$ in Čech cohomology, with coefficients in the sheaf $(\mathbb{Z}_2)_X$ of locally constant $\mathbb{Z}_2$-valued functions over X.

This is the simplest instance of a characteristic class, which means that it enjoys the following naturality property : for any morphism of double covering spaces

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

one has $f^*\alpha_Y = \alpha_X$, with obvious notations. Let us simplify notation by setting $H^*(X) = \check{H}^*(X, (\mathbb{Z}_2)_X)$. This is a (graded) ring for the cup product, and one may consider the vanishing exponent of α_X, i.e. the smallest k such that $\alpha_X^k = 0$, or ∞ if there is no such k.

Borsuk-Ulam's theorem is then an immediate consequence of two basic facts

- For any continuous map $f : X \to Y$, $f^* : H^*(Y) \to H^*(X)$ is a ring homomorphism, hence lowers vanishing exponents.
- $H^*(\mathbb{P}^{m-1}) = \mathbb{Z}_2[\alpha]/(\alpha^m = 0)$ is a truncated polynomial ring, with α the characteristic class of $\mathbb{S}^{m-1} \to \mathbb{P}^{m-1}$. In particular the vanishing exponent of α is m.

This proof is the one found for instance in [Mas] and [Spa].

Unsurprisingly then, our main object of study will be the characteristic class of the double covering $\widetilde{\mathcal{P}} \to \mathcal{P}$, denoted by $\alpha \in H^1(\mathcal{P})$. To state our results it is convenient to adopt the

Definition 6 *A cohomology class $u \in H^*(X)$ is weakly zero, denoted $u \sim 0$, if for any continuous map $f : M \to X$ with M a finite dimensional metric space, one has $f^*u = 0 \in H^*(M)$.*

The above proof of Borsuk-Ulam's theorem then yields the desired criterion

Lemma 7 *If the characteristic class $\alpha \in H^1(\mathcal{P})$ of $\widetilde{\mathcal{P}} \to \mathcal{P}$ satisfies $\alpha^N \sim 0 \in H^N(\mathcal{P})$, and $\mathbb{S}^{m-1} \to \widetilde{\mathcal{P}}$ is a continuous $\mathbb{Z}_2$-equivariant map, then $m \leq N$. In particular $\overline{m}_1(S) \leq N$.*

Remark. For $S = \mathbb{S}^2$, we proved above that $\alpha^3 \sim 0$ in $H^3(\mathcal{P})$, but not that $\alpha^3 = 0$.

We are now reduced to estimating the (weak) vanishing exponent of $\alpha \in H^1(\mathcal{P})$. Note that it is not obviously finite, and on the other hand that even if α^N doesn't vanish it remains possible that $\overline{m_1}(S) \leq N$, as non-vanishing is not necessarily detected by a map $\mathbb{P}^N \to \mathcal{P}$. Fortunately, there is an easy way to prove weak vanishing of products, namely

Lemma 8 *If $u_1, \ldots, u_k \in H^*(X)$ are (Čech) cohomology classes of a topological space X, and $X = \bigcup_1^k X_i$ is a cover by* arbitrary *subspaces such that $u_i|_{X_i} \sim 0 \in H^*(X_i)$ for all i, then $u_1 \ldots u_k \sim 0$ in $H^*(X)$.*

Proof : One may obviously assume that X is metric and $u_i|_{X_i} = 0$. But on metric spaces, Čech cohomology enjoys the following very strong continuity property [God, p. 193], [Bre, p. 73]. If A is any subspace of X, $H^*(A)$ is the direct limit of the $H^*(V)$, with V varying over the open subsets containing A. More precisely the natural map $\lim_{V \supset A} H^*(V) \to H^*(A)$ is an isomorphism [2].

[2] This result is stated in the mentioned references for *sheaf* cohomology, but it coincides with Čech cohomology (with sheaf coefficients) on any paracompact (e.g. metric) space [God, p. 228] [Bre, p.192, cor. 4.12]. Note also that on a paracompact space, Čech cohomology with coefficients in a *pre*sheaf (e.g. the constant presheaf $\mathbb{Z}_2$), coincides with Čech cohomology with coefficients in the associated sheaf [Spa, p. 329] (e.g. the locally constant sheaf $(\mathbb{Z}_2)_X$).

But the kernel of $H^*(X) \to H^*(X_i)$ is the image of $H^*(X, X_i) \to H^*(X)$, so that there are relative classes $v_i \in H^*(X, X_i)$ mapping to u_i. Then, thanks to the above "tautness" of the X_i in X [Bre, p. 83], the relative cup product $v_1 ... v_k$ is defined in the group $H^*(X, \cup_i X_i) = 0$ and maps to $u_1 ... u_k$ in $H^*(X)$, hence the conclusion. ∎

Denoting by b_1 the first Betti number with $\mathbb{Z}_2$-coefficients, define for each integer $i \geq 0$

$$\mathcal{P}(i) = \Big\{ \{U, V\} \in \mathcal{P} \;\Big|\; b_1(U) + b_1(V) = i \Big\}.$$

We will show that $\mathcal{P}(i)$ is empty for $i > b_1(S)$, so that we have a finite partition $\mathcal{P} = \bigsqcup_{0 \leq i \leq b_1(S)} \mathcal{P}(i)$. We will also see that the $\cup_{j \leq i} \mathcal{P}(j)$ are closed subsets of $\mathcal{P}$ (prop. 13), so that the $\mathcal{P}(i)$ themselves are locally closed, and may be thought of as yielding a "stratification" of $\mathcal{P}$. Our results concerning this stratification are summarized in the

Theorem 9 **(1)** *If $i > 0$, the double covering $\widetilde{\mathcal{P}}(i) \to \mathcal{P}(i)$ is trivial, and thus $\alpha|_{\mathcal{P}(i)} = 0$ in $H^*(\mathcal{P}(i))$, except when $i = 2$ and S is a torus or Klein bottle, in which case $\alpha^2|_{\mathcal{P}(2)} \sim 0$.*

(2) *$\alpha^p|_{\mathcal{P}(0)} \sim 0$ with $p = 3$ if S is orientable and $p = 4$ otherwise.*

(3) *$\alpha^4|_{\mathcal{P}(0) \cup \mathcal{P}(1)} \sim 0$ if $\chi(S) < 0$.*

Together with lemma 8 above, (1) and (3) imply theorem 5 (and thus theorem 4) by noting that $4 + (b_1(S) - 1) = 5 - \chi(S)$:

Corollary 10 *If $\chi(S) < 0$, $\overline{m}_1(S) \leq 5 - \chi(S)$.*

Remark. This yields also the (already known) equalities $\overline{m}_1(S) = \mathrm{chr}(S) - 1$ if $\chi(S) \geq 0$ and S is not Klein's bottle, that is, for $S = \mathbb{S}^2, \mathbb{P}^2, \mathbb{T}^2$. For Klein's bottle, one obtains only $\overline{m}_1(\mathbb{K}^2) \leq 7$. Of course, these are also bounds on $\dim(E)$, if $E \subset C^0(S)$ is a vector subspace enjoying the "weak two-piece property" of theorem 5.

The rest of the paper is devoted to the proof of theorem 9, mainly following the order (1), (2), (3). There is an appendix where a selection theorem used in the proof is stated and a simple corollary is proved.

Triviality of the covering over most strata

Let us say that two open subsets U, U' of S are isotopic if there is a diffeomorphism $f : U \to U'$ such that the embeddings $U \hookrightarrow S$ and $U \xrightarrow{f} U' \hookrightarrow S$ are isotopic. This is clearly an equivalence relation. Also, we will say that pairs (U, V), (U', V') of disjoint open subsets S are isotopic if $U \cup V$ and $U' \cup V'$ are, in such a way that the implied diffeomorphism f between them sends U to U' and V to V'.

Unless otherwise stated, coefficients of homology and cohomology will be understood as $\mathbb{Z}_2$. Accordingly, b_1 will denote the first Betti number with $\mathbb{Z}_2$ coefficients.

Lemma 11 *If $U \in \mathcal{U}$, the map $H_1(U) \to H_1(S)$ is injective. Also, U has a "core", that is a compact connected surface with boundary $K \subset U$, such that $U - \mathrm{int}(K)$ is a collar neighborhood of infinity in U, disjoint union of a finite number of half-open annuli $\simeq \mathbb{S}^1 \times [0, 1[$. In particular U is isotopic within itself to $\mathrm{int}(K) = K - \partial K$.*

Proof : By the homology exact sequence of (S, U), the kernel of $H_1(U) \to H_1(S)$ is also the cokernel of $H_2(S) \to H_2(S, U)$, hence of $H^0(S) \to \check{H}^0(S - U)$ by Alexander duality with $\mathbb{Z}_2$ coefficients. But this last map is surjective, since $S - U$ is connected, hence the map $H_1(U) \to H_1(S)$ is injective.

Then, $\dim H_1(U)$ being finite, $\pi_1(U)$ is a finitely generated (free) group, and it is well known that the connected open surface U is then diffeomorphic to the interior of a compact connected surface with boundary $\widehat{U}$. Removing from $\widehat{U}$ an open collar neighborhood of its boundary gives the required K. ∎

Lemma 12 *If* $(U, V) \in \widetilde{\mathcal{P}}$, $b_1(U) + b_1(V) \leq b_1(S)$. *Thus* $\mathcal{P} = \bigsqcup_{0 \leq i \leq b_1(S)} \mathcal{P}(i)$.

Proof : It is enough to observe that since $U \cap V = \emptyset$ by definition of $\mathcal{P}$, the images of the maps $H_1(U) \to H_1(S)$, $H_1(V) \to H_1(S)$ (injective by the previous lemma), are orthogonal for the (mod 2) intersection form, which is non-degenerate on $H_1(S)$. ∎

Proposition 13 *If* $U \in \mathcal{U}$ *and* $U' \in \mathcal{U}$ *is sufficiently close to* U, *then* $b_1(U') \geq b_1(U)$, *and in case of equality,* U, U' *are isotopic.*

Proof : If $U' \in \mathcal{U}$ contains a core K of U (in particular if U' is sufficiently close to U in the space $\mathcal{U}$), the composite map $H_1(K) \to H_1(U') \to H_1(S)$ is injective, since it coincides with the composite map $H_1(K) \xrightarrow{\sim} H_1(U) \to H_1(S)$. Hence $H_1(K) \to H_1(U')$ is injective, and $b_1(U) = b_1(K) \leq b_1(U')$, as claimed.

In case of equality, $H_1(K) \to H_1(U')$ is an isomorphism, as well as $H_p(K) \to H_p(U')$, for all p. Thus $H_*(U', K) = 0$, hence $H_*(U' - \mathrm{int}(K), \partial K) = 0$ by excision, and this easily implies that K is also a core of U'. Indeed, each connected component of $U' - K$ is a surface V with compact boundary and zero mod 2 homology relative to boundary. Thus $H_0(\partial V) = H_0(V) = \mathbb{Z}_2$, i.e. $\partial V \simeq \mathbb{S}^1$, next $H_1(V) = H_1(\partial V) = \mathbb{Z}_2$, i.e. V is homotopically a circle, and since $H_2(V, \partial V) = 0$, V is not compact, hence is the half-open annulus. As the interior of K is isotopic to both U and U' this proves the proposition. ∎

Corollary 14 *On each* $\widetilde{\mathcal{P}}(i)$, $i \geq 0$, *the isotopy equivalence relation* $\sim$ *has open classes.*

Proof : If $(U, V) \in \widetilde{\mathcal{P}}(i)$, and $(U', V') \in \widetilde{\mathcal{P}}(i)$ is near to (U, V) in the induced topology of $\mathcal{U} \times \mathcal{U}$, one has $b_1(U) = b_1(U')$ and $b_1(V) = b_1(V')$, since the right-hand sides are bigger and have the same sum i as the left-hand sides. Then U, U' (resp. V, V') have a common core K (resp. L), and this implies that (U, V), (U', V') are both isotopic to $(\mathrm{int}(K), \mathrm{int}(L))$. ∎

Proposition 15 *If* $(U, V) \in \widetilde{\mathcal{P}}(i)$, *with* $i \geq 0$, *then* (U, V) *is not isotopic to* (V, U), *except in the following cases :*

- U *and* V *are disks (hence* $i = 0$*).*
- U *and* V *are annuli (hence* $i = 2$*), and* S *is a torus or Klein bottle.*

Proof : Assume that (U,V) is isotopic to (V,U), with U, V not disks (they are of course homeomorphic). Then, by extension of isotopies, there exists a diffeomorphism ϕ of S, isotopic to the identity, such that $\phi(K)=L$ and $\phi(L)=K$, where K, L are respective cores of U, V. Consider then $C=\partial K\cup\partial L$. It is a disjoint union of simple two-sided closed curves in S, and no component of $S-C$ is a disk. Indeed, otherwise it would be a component of $S-K$ or $S-L$, hence equal to it (both are connected), and then $L\subset S-K$ or $K\subset S-L$ would be a disk, contradicting our assumption. Now the conclusion comes from the following slight variation (in the non-orientable situation) of theorem 1.2 in [Iva, p. 10]

Theorem 16 *Let $C\subset S$ be a finite disjoint union of simple two-sided closed curves and $\phi\in\mathrm{Diff}(S)_0$ a diffeomorphism isotopic to the identity (or more generally trivially acting on degree one integral homology) such that $\phi(C)=C$. Then if none of the components of $S-C$ is a disk, ϕ doesn't permute them nor the components of C, unless S is a torus or Klein bottle, and C a collection of "meridians".*

Proof : (after [Iva, p. 10]) Consider the finite graph G (maybe with loops and multiple edges) whose vertices are the connected components of $S-C$ and the edges are the connected components of C, joining the (not necessarily distinct) adjacent components of $S-C$. Clearly, ϕ induces an automorphism $\overline{\phi}$ of G. Then, the components of C being two-sided, C has an open tubular neighborhood T diffeomorphic to $C\times\,]-1,1[$. Consider now the continuous map $\nu: S\to G$, which is the map

$$T\cong C\times\,]-1,1[\,\to\pi_0(C)\times\,]-1,1[\,\cong G-\mathrm{vertices}(G)$$

on T, and shrinks each component of $S-T$ to the corresponding vertex of G ($\pi_0(C)$ stands for the set of connected components of C, i.e. the edge set of G). The induced map $\nu_*: H_1(S,\mathbb{Z})\to H_1(G,\mathbb{Z})$ is surjective and $\nu\circ\overline{\phi}$ is homotopic to $\phi\circ\nu$. In particular $\overline{\phi}_*$ is the identity on $H_1(G,\mathbb{Z})$, since ϕ_* is on $H_1(S,\mathbb{Z})$.

Any degree one vertex of G represents a component S' of $S-C$ which cannot be planar, since it has only one boundary component and is not a disk. Consequently S' has a non-zero intersection form on $H_1(S',\mathbb{Z}_2)$, and thus cannot be moved into its complement by ϕ. This shows that $\overline{\phi}$ has to fix all degree one vertices (if any).

But graph automorphisms inducing the identity in integral homology can only rotate each simple circuit (subspace homeomorphic to $\mathbb{S}^1$) in the graph, since such (oriented) circuits are canonical representatives of their homology classes. Thus the union G' of all simple circuits is invariant under $\overline{\phi}$. By contracting each component of G' to a point, one obtains a tree and an automorphism of it which fixes all degree one vertices, and thus must be the identity. Hence $\overline{\phi}$ fixes each edge not in G'.

Furthermore, two distinct simple circuits c_1, c_2 with nonempty intersection are both fixed. Indeed, this is clear if $c_1\cap c_2$ is connected, and otherwise one easily finds in $c_1\cup c_2$ a simple circuit $c_3\neq c_2$ such that $c_2\cap c_3$ is connected and nonempty. Then c_2, c_3 are fixed, and so is c_1 since it meets c_2. Similarly, a simple circuit meeting an edge not in G' is fixed.

This shows that if $\overline{\phi}$ is not the identity, G is a simple circuit, rotated by $\overline{\phi}$. But then each component of $S-C$ is planar, hence an annulus, and S is a torus or a Klein bottle. ■■

Corollary 17 *If $i > 0$, the double covering $\tilde{\mathcal{P}}(i) \to \mathcal{P}(i)$ is trivial, except when $i = 2$ and S is a torus or Klein bottle.*

Proof : Indeed, one then has on the total space $\tilde{\mathcal{P}}(i)$ of the double covering an equivalence relation (isotopy), invariant under the covering involution, all of whose classes are open and meet each fiber in at most one point. This allows us to construct easily a continuous section : the quotient equivalence relation on $\mathcal{P}(i)$ has open classes, over each of which the covering has a section (any of the *two* classes over it), hence the whole covering is trivial. ■

Vanishing exponents over $\mathcal{P}(0)$ and $\mathcal{P}(2)$

Proposition 18 *$\alpha^p|_{\mathcal{P}(0)} \sim 0$, with $p = 3$ if S is orientable, $p = 4$ otherwise.*

Proof : One has to show that if Σ is a finite dimensional metric space and $f : \Sigma \to \mathcal{P}(0)$ a continuous map, $f^*\alpha^p = 0 \in H^p(\Sigma)$. Denote by $\tilde{\Sigma} \to \Sigma$ the pulled-back double covering, $v \to -v$ the involution on $\tilde{\Sigma}$, and $F : \tilde{\Sigma} \to \tilde{\mathcal{P}}(0)$ the $\mathbb{Z}_2$-map covering f.

By definition, F takes the form $F(v) = (D(v), D(-v))$, with $D(v)$, $D(-v)$ disjoint open subsets of S homeomorphic to disks, for all $v \in \tilde{\Sigma}$. Moreover, the "graph" $\bigcup\{v\} \times D(v)$ of D is open in $\tilde{\Sigma} \times S$, as this is equivalent to the continuity of $D : \tilde{\Sigma} \to \mathcal{U}$. Now since the $D(v)$ are contractible and $\tilde{\Sigma}$ is finite-dimensional and metric, Bielawski's selection theorem [Bie89] (see appendix for the statement) provides a continuous selection map $s : \tilde{\Sigma} \to S$ such that $s(v) \in D(v)$. In particular $s(v) \neq s(-v)$, and the double covering $\tilde{\Sigma} \to \Sigma$ is pulled back from the covering of pairs of distinct points on S, i.e.

$$S \times S - \mathrm{Diag} \to (S \times S - \mathrm{Diag})/\mathbb{Z}_2.$$

But S may be embedded in $\mathbb{R}^p$ with $p = 3$ if S is orientable, $p = 4$ otherwise[3]. By norming $v \mapsto s(v) - s(-v) \in \mathbb{R}^p - 0$, one obtains a $\mathbb{Z}_2$-equivariant map $\tilde{\Sigma} \to \mathbb{S}^{p-1}$, and the desired conclusion $f^*\alpha^p = 0$ follows. ■

Together with corollary 17, this already provides us with some optimal estimates for $\overline{m}_1(S)$:

Corollary 19 *If $\chi(S) \neq 0$, $\alpha^m \sim 0 \in H^*(\mathcal{P})$, with $m = 5 - \chi(S)$ if S is orientable, and $m = 6 - \chi(S)$ otherwise. This is optimal for $S = \mathbb{S}^2$, $\mathbb{P}^2$, and $\mathbb{T}^2\sharp\mathbb{T}^2$.*

Proof : One has $\alpha^p|_{\mathcal{P}(0)} \sim 0$ by the above proposition ($p = 3$ or 4), and for $i > 0$, $\alpha|_{\mathcal{P}(i)} = 0$ by corollary 17 (note that $b_1(\mathbb{P}^2) = 1$). The conclusion is then only a matter of rewriting $p + b_1(S)$ as $p + 2 - \chi(S)$. ■

The following proposition analyses $\mathcal{P}(2)$ when S is a torus or Klein bottle.

Proposition 20 *Let $S = \mathbb{T}^2$ or $\mathbb{K}^2$. Then $\alpha^2|_{\mathcal{P}(2)} \sim 0$.*

[3] Alternatively, one may argue that $(S \times S - \mathrm{Diag})/\mathbb{Z}_2$ being an open 4-manifold, has the homotopy type of a 3-dimensional polyhedron.

Proof: Consider as above a $\mathbb{Z}_2$-map $F : \widetilde{\Sigma} \to \widetilde{\mathcal{P}}(2)$, with $\widetilde{\Sigma} \to \Sigma$ a double covering of the finite dimensional metric space Σ.

By corollary 14 and proposition 15, one may assume that F has image contained in a single isotopy class of pairs of disjoint open annuli, $F(v) = (A(v), A(-v))$, both annuli being isotopic to $q^{-1}(]0,1/2[)$ for some fibration $q : S \to \mathbb{R}/\mathbb{Z}$ with $\mathbb{S}^1$ fibers (trivial if $S = \mathbb{T}^2$, twisted if $S = \mathbb{K}^2$).

Now, in the space $\mathcal{S}$ of (smooth, unoriented, unparametrized) simple closed curves in S, the subset $\mathcal{S}_B$ of "core curves" of any open annulus $B \subset S$ is open and contractible, as results from [Gra89], for example. Moreover, the subset $\bigcup_v \{v\} \times \mathcal{S}_{A(v)} \subset \widetilde{\Sigma} \times \mathcal{S}$ is readily seen to be open, since $\bigcup_v \{v\} \times A(v)$ is open in $\widetilde{\Sigma} \times S$.

Bielawski's selection theorem then provides a continuous selection $v \mapsto C(v) \in \mathcal{S}_{A(v)}$, so that in particular $C(v) \cap C(-v) = \emptyset$. Furthermore, all $C(v)$ lie in the same component (isotopy class) $\mathcal{C}$ of $\mathcal{S}$, namely the one containing the fibers of the fibration $q : S \to \mathbb{R}/\mathbb{Z}$. The map $v \mapsto (C(v), C(-v))$ is then continuous and equivariant from $\widetilde{\Sigma}$ to the space $\widetilde{\mathcal{C}}_2 \subset \mathcal{C} \times \mathcal{C}$ of ordered pairs of disjoint simple closed curves in $\mathcal{C}$ with the obvious involution and quotient space $\mathcal{C}_2$ of unordered pairs.

It is now enough to prove that the characteristic class of $\widetilde{\mathcal{C}}_2 \to \mathcal{C}_2$, still denoted α, has zero square. This will result from the

Lemma 21 *There is a continuous function $g : \mathcal{C} \to \mathbb{R}/\mathbb{Z} \simeq \mathbb{S}^1$ such that whenever C, C' are disjoint members of $\mathcal{C}$, $g(C)$ and $g(C')$ are distinct.*

Proof: $\mathcal{C}$ is the space of curves isotopic to a fiber of the fibration $q : S \to \mathbb{R}/\mathbb{Z}$. Consider the pull-back $\hat{q}$ of q by the covering map $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$. It is a (trivial) fibration $\hat{q} : \widehat{S} \to \mathbb{R}$, and $\widehat{S} \simeq \mathbb{R} \times \mathbb{R}/\mathbb{Z} \to S$ is a regular covering of S, with deck transformation group generated by $(x, y) \mapsto T(x, y) = (x + 1, \varepsilon y)$, where $\varepsilon = \pm 1$ according as $S = \mathbb{T}^2$ or $\mathbb{K}^2$. Since the image of $\pi_1(\widehat{S}) \to \pi_1(S)$ is generated by the homotopy class of a curve $C \in \mathcal{C}$, the induced covering $\widehat{C} \to C$ over C is trivial, so that

$$\widehat{C} = \cup_{k \in \mathbb{Z}} \widehat{C}_k = \cup_{k \in \mathbb{Z}} T^k(\widehat{C}_0).$$

Now orient each lift $\widehat{C}_k \subset \mathbb{R} \times \mathbb{R}/\mathbb{Z}$ of C by its (degree 1) projection on $\mathbb{R}/\mathbb{Z}$, and set

$$g(C) = \int_{\widehat{C}_k} x\, dy \mod \mathbb{Z}.$$

As the notation suggests, $g(C)$ doesn't depend on the chosen lift $\widehat{C}_k$. Moreover, if C, C' are disjoint members of $\mathcal{C}$, $g(C) - g(C')$ coincides mod $\mathbb{Z}$ with the area of one of the two (annular) components of $S - (C \cup C')$, for the area $|dxdy|$ on S. Hence $g(C) \neq g(C')$. ∎

We can now conclude the proof of proposition, by noting that g gives rise to a continuous $\mathbb{Z}_2$-equivariant map $\widetilde{\mathcal{C}}_2 \to \mathbb{S}^1 \times \mathbb{S}^1 - \mathrm{Diag} \to \mathbb{S}^1$, yielding $\alpha^2 = 0$ for the characteristic class α of $\widetilde{\mathcal{C}}_2 \to \mathcal{C}_2$. ∎

Corollary 22 *If $\chi(S) = 0$, $\alpha^6 \sim 0$ for $S = \mathbb{T}^2$, and $\alpha^7 \sim 0$ for $S = \mathbb{K}^2$. This is optimal for $S = \mathbb{T}^2$.*

Remark. One may expect $\alpha^5 \sim 0$ for Klein's bottle.

Vanishing exponent over $\mathcal{P}(0) \cup \mathcal{P}(1)$

Proposition 23 *If* $\chi(S) < 0$, $\alpha^4|_{\mathcal{P}(0)\cup\mathcal{P}(1)} \sim 0$.

Remark. If S is orientable, we already know this, since $\alpha^3 \sim 0$ on $\mathcal{P}(0)$ and $\alpha \sim 0$ on $\mathcal{P}(1)$.

Proof: Let $f : \Sigma \to \mathcal{P}(0) \cup \mathcal{P}(1)$ be a continuous map, with Σ a finite dimensional metric space, and $\Sigma_i = f^{-1}(\mathcal{P}(i))$, $i = 0, 1$. Denote $\tilde{\Sigma} \to \Sigma$ the pulled-back double covering, and similarly $\tilde{\Sigma}_i \to \Sigma_i$. Finally, let F denote the $\mathbb{Z}_2$-map covering f, with $F(v) = (U(v), U(-v))$.

On the open subset $\tilde{\Sigma}_1 \subset \tilde{\Sigma}$, $b_1(U(v)) + b_1(U(-v)) = 1$, so that one component of $F(v)$ is a disk, and the other is an annulus or Möbius band. Let $\tilde{\Sigma}_1 = \Sigma_1^+ \cup \Sigma_1^-$, with $\Sigma_1^\pm$ the open subset where $b_1(U(\pm v)) = 1$, so that $\Sigma_1^- = -\Sigma_1^+$.

On the closed subset $\tilde{\Sigma}_0 \cup \Sigma_1^- \subset \tilde{\Sigma}$, $U(v)$ is a disk, and there is a continuous selection $v \mapsto s(v) \in U(v)$, $v \in \tilde{\Sigma}_0 \cup \Sigma_1^-$. Assuming as we may $S \subset \mathbb{R}^4$, one can extend s continuously to $s : \tilde{\Sigma} \to \mathbb{R}^4$ by Urysohn's theorem. For an open neighborhood V of $\tilde{\Sigma}_0 \cup \Sigma_1^-$, $s(V)$ is contained in a tubular neighborood of S, and one may assume that $s(V) \subset S$. Then, by continuity of F, $s(v) \in U(v)$ for $v \in V_0$, a smaller open neighborhood of $\tilde{\Sigma}_0 \cup \Sigma_1^-$. This yields a continuous selection $v \mapsto s(v) \in U(v)$ for $v \in V_0$.

On the open subset Σ_1^+, there is (by Bielawski's theorem again) a continuous choice of a smooth core curve $C(v) \subset U(v)$, $v \in \Sigma_1^+$. Now for v in the intersection $W = V_0 \cap \Sigma_1^+$, we have a point $s(v) \in U(v)$ and a core curve $C(v) \subset U(v)$, both continuous in $v \in W$.

Denoting by Λ the space of continuous paths $\lambda : [0,1] \to S$, and by $\mathcal{C}$ the space of non-separating smooth simple closed curves in S, the subspace of $S \times \mathcal{C} \times \Lambda$ comprising the (x, C, λ) such that $\lambda(0) = x$ and $\lambda(1) \in C$ is easily seen to be a locally trivial bundle over $S \times \mathcal{C}$, with moreover locally contractible metrizable fibers. Hence so is its pull-back over $W = V_0 \cap \Sigma_1^+$ by $v \mapsto (s(v), C(v))$, i.e. the space $\mathcal{W} \subset W \times \Lambda$ of pairs (v, λ) such that $\lambda(0) = s(v)$ and $\lambda(1) \in C(v)$.

Now the subspace $\mathcal{W}' \subset \mathcal{W}$ defined by the further condition $\lambda([0,1]) \subset U(v)$ is open, with contractible fibers over W as readily checked. An easy corollary of Bielawski's theorem (see appendix) then yields a continuous section $v \mapsto \lambda_v \in \mathcal{W}'$, $v \in W$, that is a continuous choice of a path $\lambda_v : [0,1] \to U(v)$ from $s(v)$ to $C(v)$ in the annulus or Möbius band $U(v)$. It is now easy to modify s in order to achieve $s(v) \in C(v)$ over $V_0 \cap V_1$, where $V_1 \subset \Sigma_1^+$ is open and $V_0 \cup V_1 = \tilde{\Sigma}$. Namely, using a continuous $\rho : \tilde{\Sigma} \to [0,1]$, with $\mathrm{supp}(\rho) \subset \Sigma_1^+$ and $\mathrm{supp}(1-\rho) \subset V_0$, one takes $\lambda_v(\rho(v))$ in place of $s(v)$ when $v \in V_0 \cap \Sigma_1^+$.

We will now construct from these data a space Y endowed with a double covering $\tilde{Y} \to Y$ such that $\tilde{\Sigma} \to \Sigma$ is induced by a map $\Sigma \to Y$, and $\alpha_Y^4 = 0$, thereby concluding the proof.

Namely let Y_0 be the space $(S \times S - \mathrm{Diag})/\mathbb{Z}_2$ of unordered pairs of distinct points on S, and

$$Y_1 = \{(x, C) \in S \times \mathcal{C} \mid x \notin C\}$$

$$Y_{01} = \{(x, y, C) \in S \times S \times \mathcal{C} \mid x \notin C,\, y \in C\}.$$

There are obvious maps $Y_{01} \to Y_0$ and $Y_{01} \to Y_1$, forgetting C and y respectively, and the above data yield commuting maps

$$\begin{array}{ccc} N_0 & \longrightarrow & Y_0 \\ \uparrow & & \uparrow \\ N_0 \cap N_1 & \longrightarrow & Y_{01} \\ \downarrow & & \downarrow \\ N_1 & \longrightarrow & Y_1 \end{array}$$

with $\Sigma = N_0 \cup N_1$ an open cover, $N_0 \supset \Sigma_0$, $N_1 \subset \Sigma_1$. There is a similar commuting diagram of double covering spaces and equivariant maps, with easily defined double coverings for the Y_* (they are trivial for Y_1 and Y_{01}).

Let Y be obtained by gluing to $Y_{01} \times [0,1]$ the spaces Y_0 and Y_1, on $Y_{01} \times 0$ and $Y_{01} \times 1$ respectively. It is the "double mapping cylinder" or "homotopy pushout" of the two maps $Y_{01} \to Y_0$, $Y_{01} \to Y_1$. The same construction with double covering spaces and equivariant maps yields the double covering $\widetilde{Y} \to Y$. Then, using a partition of unity for N_0, N_1, the above diagram gives rise to a continuous map $\Sigma \to Y$, covered by an equivariant $\widetilde{\Sigma} \to \widetilde{Y}$. We now show that $\alpha_Y^4 = 0$.

If $h : Y \to [0,1]$ denotes the natural projection, the inclusions $Y_0 \to h^{-1}([0,1[)$, $Y_1 \to h^{-1}(]0,1])$ and $Y_{01} \to h^{-1}(1/2) \subset h^{-1}(]0,1[)$ are homotopy equivalences, hence one may write a Mayer-Vietoris exact sequence

$$... \to H^3(Y_{01}) \to H^4(Y) \to H^4(Y_0) \oplus H^4(Y_1) \to ...$$

But, recalling that $\alpha_Y^4 | Y_0 = 0$ and $\alpha_Y | Y_1 = 0$, we see that $\alpha_Y^4 \in H^4(Y)$ lies in the image of $H^3(Y_{01})$.

We finally show that this group vanishes. Indeed Y_{01} is the space of $(x, y, C) \in S \times S \times \mathcal{C}$ with $x \notin C$ and $y \in C$, and may be viewed as a fiber bundle over $\mathcal{C}$, with fibers $(S - C) \times C$. But each component of $\mathcal{C}$ is contractible since $\chi(S) < 0$ and curves $C \in \mathcal{C}$ are homotopically non trivial in S (they are non-separating)[4]. Hence all components of Y_{01} have the homotopy type of $(S - C) \times C$, itself homotopy equivalent to a 2-complex, so that $H^3(Y_{01}) = 0$ as claimed. ∎

Appendix. Bielawski's selection theorem.

Theorem 24 *[Bie89] Let X be a finite dimensional paracompact space, $n = \dim X$, Y a topological space, and $V \subset X \times Y$ an open subset such that for all $x \in X$, $V(x) \subset Y$ is nonempty and $(n-1)$-connected, i.e. any continuous map $\mathbb{S}^{i-1} \to V(x)$ extends to $\mathbb{B}^i \to V(x)$, $i = 1, ..., n$.*

Then V has a section (selection), i.e. a continuous map $s : X \to Y$ such that for all $x \in X$, $s(x) \in V(x)$.

If moreover X, Y are metrizable and Y is locally $(n-1)$-connected (e.g. locally contractible), any partial section of $V \to X$ over a closed subset $A \subset X$ may be extended to a global section.

[4]By [Gra89] (see also [Ang91, HaSc94]), one even has an explicit curve shortening flow on $\mathcal{C}$ converging to the unique geodesic in each component of $\mathcal{C}$, for any chosen negatively curved riemannian metric on S.

Here $\dim X \leq n$ means that any open cover of X admits an open refinement in which no point belongs to more than $n+1$ sets.

Very roughly, the proof proceeds by using paracompactness to map X to n-dimensional polyhedra (nerves of suitably refined open covers), high connectedness of the $V(x)$ (and openness of V) to map those polyhedra into V. In the relative case, the additional hypotheses are used to modify V over $X - A$ so that application of the absolute case over $X - A$ yields the sought extension.

Corollary 25 *Let $\pi : E \to X$ be a locally trivial fibration with metrizable finite dimensional base X and metrizable locally contractible fibers, and $V \subset E$ an open subset. If $V(x) = \pi^{-1}(x) \cap V$ is nonempty and contractible for all $x \in X$, then $\pi|_V : V \to X$ has a (continuous) section.*

Proof : There is a locally finite cover of X by trivializing open subset X_i, $i \in I$ and a "shrinked" cover by closed subsets $A_i \subset X_i$. Then, using a well-ordering of I, the theorem allows us to construct inductively a partial section s_i of $V \to X$ over $\cup_{j \leq i} A_j$, extending s_j for all $j < i$ (note that any union of A_* subsets is closed, by local finiteness). ■

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